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Homotopy mod. C of Spaces of Category 2[1])

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The known result of HOPF concerning the cohomology structure of H-spaces may now be restated as follows. An H-space, i.e. a space with a continuous multiplication with unit, has over a field k of characteristic 0 the same cohomology ring as a product of spaces of type (π, n). Denoting by C the class of finite groups, THOM [7] has shown more, namely that an H-space is equivalent mod. C to a product of spaces of type (π, n). On the other hand, in the theory of ECKMANN-HILTON [1], the dual of an H-space is a space of LUSTERNIK-SCHNIRELMANN category ≤ 2. For such spaces we are proving here the dual of the above result of THOM: any finite simply connected CW-complex of category ≤ 2 is equivalent mod. C to an union of spheres with a single common point (here and throughout the paper, C denotes the class of finite groups). The precise result in a slightly more general form is stated in Theorem 2.2.

1. Preliminary lemmas

Let
$$\varphi_k : \pi_r(S^n) \to \pi_r(S^n)$$
be defined by left composition with a map $S^n \to S^n$ of degree k, i.e. $\varphi_k(\gamma) = k\iota \circ \gamma$ for any $\gamma \in \pi_r(S^n)$ ($\iota \in \pi_n(S^n)$) is the class of the identity. Then we have for n even [3]
$$\varphi_k(\gamma) = k\gamma + \frac{k(k-1)}{2}[\iota, \iota] \circ H_0(\gamma) + \frac{(k+1)\,k(k-1)}{3}\big[\iota, [\iota, \iota]\big] \circ H_1(\gamma) \quad (1)$$
where H_0 and H_1 are the generalized HOPF invariants of HILTON. As a consequence of (1)

1.1. $q\gamma = 0$, $\gamma \in \pi_r(S^n)$ *implies* $\varphi_{6q}(\gamma) = 0$ (for n odd it was shown by SERRE that already $\varphi_{2q}(\gamma) = 0$).

Let n be even and let $\pi_{2n-1}(S^n) = Z' + G$, where Z' is infinite cyclic, generated by α and G is finite of order g. Denote by d the classical HOPF invariant of α, i.e. $H_0(\alpha) = d\iota_{2n-1}$, where ι_{2n-1} generates $\pi_{2n-1}(S^{2n-1})$. Let $[\iota, \iota] = s\alpha + \beta$, $\beta \in G$. An easy computation, based on (1), shows that for any $\gamma \in \pi_{2n-1}(S^n)$

1.2. $\varphi_{2mg}(\gamma) = N_0 m\gamma$ *where* $N_0 = 2g + gsd(2mg - 1)$. Moreover,

1.3. if $\gamma = r\alpha + \delta$, $\delta \in G$ *then* $\varphi_{2mg}(\gamma) = N_0 rm\alpha$.

1.4. Lemma. *Let K be a CW-complex, n an even integer and*
$$f : K^{2n-1} \to S^n$$

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[1]) The result of this paper was presented to the International Colloquium on Differential Geometry and Topology, Zurich, June 1960 (in absence of the author, by Professor HILTON).

a map such that $f^*(u) = h \in H^n(K, Z)$ *where* $u \in H^n(S^n, Z)$ *is the fundamental class. If the cup-square* $h \smile h \in H^{2n}(K, Z)$ *is an element of finite order, then there exists a map* $\varphi : S^n \to S^n$ *of degree* $s \neq 0$ *such that* $\varphi \circ f / K^{2n-2}$ *is extendable over* K^{2n}.

Proof. Without loss of generality we may assume, for the sake of convenience, that $\dim K = 2n$. Attach to S^n a $2n$-cell with characteristic map in class α; let Y be the resulting space. Let $c^{2n}(f) \in C^{2n}(K, \pi_{2n-1}(S^n))$ be the obstruction to the extension of f. By 1.3, if $\psi : S^n \to S^n$ has degree $2g$, the obstruction $c^{2n}(\psi \circ f)$ takes on each cell a value which is a multiple of α. Therefore, denoting by $i : S^n \to Y$ the inclusion, $i \circ \psi \circ f$ can be extended to a map

$$F : K \to Y .$$

If $u' \in H^n(Y, Z)$ is the fundamental class, then $u' \smile u' = -da$, where a is the fundamental class of $H^{2n}(Y, Z) \approx Z$ and d is the HOPF invariant of α.

Let $j : Z \to \pi_{2n-1}(S^n)$ map Z onto Z' $(j(1) = \alpha)$; it induces homomorphisms

$$j_* : H^{2n}(K, Z) \to H^{2n}(K, \pi_{2n-1}(S^n)) ,$$
$$j_* : H^{2n}(Y, Z) \to H^{2n}(Y, \pi_{2n-1}(S^n)) .$$

It is easy to check that

$$F^*(j_*(a)) = \gamma^{2n}(\psi \circ f)$$

where $\gamma^{2n}(\psi \circ f)$ is the cohomology class of $c^{2n}(\psi \circ f)$. We further have

$$\begin{aligned} d \cdot \gamma^{2n}(\psi \circ f) = d \cdot F^*(j_*(a)) &= j_*(F^*(d \cdot a)) = -j_*(F^*(u' \smile u')) = \\ &= -j_*(2gh \smile 2gh) = -4g^2 \cdot j_*(h \smile h) . \end{aligned}$$

This proves that $\gamma^{2n}(\psi \circ f)$ is an element of finite order, say m. Let $\chi : S^n \to S^n$ be a map of degree $2mg$. Then 1.2 immediately yields

$$c^{2n}(\chi \circ \psi \circ f) = N_0 m c^{2n}(\psi \circ f)$$

and

$$\gamma^{2n}(\chi \circ \psi \circ f) = N_0 m \gamma^{2n}(\psi \circ f) = 0 .$$

This proves the assertion of the lemma.

1.5. Proposition. *Let* K *be a* q*-dimensional* $(q < \infty)$ CW*-complex,* n *an integer and* h *an arbitrary element of* $H^n(K, Z)$, *such that the cup-square* $h \smile h$ *is an element of finite order. There exist an integer* $N > 0$ *and a map* $f : K \to S^n$ *such that* $f^*(u) = Nh$, *where* u *is the fundamental class of* $H^n(S^n, Z)$.

This proposition was conjectured by SERRE and proved by him for n odd [5, ch. V, Prop. 2]. For n even the proof is practically the same but uses 1.1 and 1.4.

Remark. In view of 1.5 and of [6, II, 2] we may add to [6, II, 4] the following result:

Let V^n be an orientable closed differentiable n-manifold, k an even number and z a class in $H_{n-k}(V^n, Z)$, whose selfintersection is a class of finite order. Then there exists an integer $N > 0$ such that the class Nz can be realized by means of a submanifold whose normal fibre bundle is trivial.

2. The main theorem

The base point of any space will be denoted by $*$. For any spaces $X_1, \ldots, X_m$. $X_1 \vee \ldots \vee X_m$ denotes their union with a single common point $*$. There are obvious retractions $r_j : X_1 \vee \ldots \vee X_m \to X_j$ mapping $X_1, \ldots, X_{j-1}, X_{j+1}, \ldots, X_m$ onto $*$. If $\varphi_j : X_j \to Y_j$ are maps, there is a map

$$\varphi_1 \vee \ldots \vee \varphi_m : X_1 \vee \ldots \vee X_m \to Y_1 \vee \ldots \vee Y_m$$

defined in the obvious way.

In this paper we consider only spaces which have the homotopy type of connected CW-complexes. For such a space, the two following definitions of LUSTERNIK-SCHNIRELMANN category ≤ 2 are equivalent (compare [8, p. 94]).

A) $\operatorname{cat} X \leq 2$ if and only if $X = A_1 \cup A_2$ where A_1 and A_2 are open and contractible in X.

B) $\operatorname{cat} X \leq 2$ if and only if there exists a map

$$\Phi : X \to X \vee X$$

such that $r_j \circ \Phi \simeq \theta_X : X \to X$ $(j = 1, 2)$ where θ_X is the identity map of X and the homotopies are rel. $*$.

If $\operatorname{cat} X \leq 2$ define

$$\Phi_m : X \to \underbrace{X \vee \ldots \vee X}_{m\text{-fold}} \tag{2}$$

by

$$\Phi_2 = \Phi\,, \quad \Phi_m = (\underbrace{\theta_X \vee \ldots \vee \theta_X}_{(m-2)\text{-fold}} \vee \Phi) \circ \Phi_{m-1}\,. \tag{2'}$$

If follows readily that

2.1. $r_j \circ \Phi_m \simeq \theta_X\,, \quad j = 1, \ldots, m\,.$

2.2. Theorem. *Let K be a connected and simply connected CW-complex whose homology groupes are finitely generated in each dimension and let C be the class of finite groups. If* $\operatorname{cat} K \leq 2$, *there exists for any integer $r > 1$ a map*

$$f : K^{r+1} \to L$$

where L is an union of spheres, such that

$$f_* : H_i(K^{r+1}, Z) \to H_i(L, Z)$$

and

$$f_{\#} : \pi_i(K^{r+1}) \to \pi_i(L)$$

are C-monomorphisms for $i < r$ and C-epimorphisms for $i \leq r$.

2.3. Corollary. *If* $\dim K < \infty$, *we may choose Y and the map $f : K \to L$ such that*

$$f_* : H_i(K, Z) \to H_i(L, Z)$$

and

$$f_{\#} : \pi_i(K) \to \pi_i(L)$$

are C-isomorphisms in all dimensions.

Remark. If the homology of K is not finitely generated, then Theorem 2.2 is not true. A counter example is provided by a complex $K' = K'(Q, 2)$ such that $H_i(K', Z) = 0$ for $i \neq 0, 2$ and $H_2(K', Z) = Q$ where Q is the group of rationals.

Proof of 2.2. Since all cup-products in K vanish (see [2]) if follows from 1.5 that for any cohomology class $h \in H^i(K^{r+1}, k)$, $i \leq r$, where k is the field of rationals, there exists a map

$$g : K^{r+1} \to S^i$$

and a class $u \in H^i(S^i, k)$ such that

$$g^*(u) = h .$$

Let $h_1, h_2, \ldots, h_m$ be a base of $\sum_{i=1}^{r} H^i(K^{r+1}, k)$. Choose for each j, ($j = 1, \ldots, m$) a map

$$g_j : K^{r+1} \to S_j$$

where S_j is a sphere of the corresponding dimension, such that

$$g_j^*(u_j) = h_j , \qquad u_j \in H^{n_j}(S_j, k) , \qquad n_j = \dim S_j . \tag{3}$$

Let

$$\Phi_m : K \to \underbrace{K \vee \ldots \vee K}_{m\text{-fold}}$$

be as in (2). We may assume that Φ_m is cellular; then Φ_m induces a map

$$\overline{\Phi}_m : K^{r+1} \to K^{r+1} \vee \ldots \vee K^{r+1} .$$

Consider the map

$$g_1 \vee \ldots \vee g_m : K^{r+1} \vee \ldots \vee K^{r+1} \to S_1 \vee \ldots \vee S_m = L .$$

It is easy to check, by means of 2.1 (where all homotopies may be chosen cellular) and (3) that $f : K^{r+1} \to Y$, $f = (g_1 \vee \ldots \vee g_m) \circ \overline{\Phi}_m$ induces isomorphisms

$$f^* : H^i(L, k) \to H^i(K^{r+1}, k)$$

in dimensions $i \leq r$. Applying the known results of SERRE [5, ch. III, Th. 3 and Prop. 1] we obtain the conclusion of the theorem.

Remark. Set $K^{r+1} = A$, $L = B$; in order to apply SERRE's results, quoted above, we must assume that $\pi_2(A) \to \pi_2(B)$ is an epimorphism. In fact, this restriction may be removed. We may assume that f is an inclusion. Further, passing if necessary to singular polytopes, we may also assume that B is a simplicial complex with the strong (metric) topology and A is a subcomplex (the strong topology and the weak one on a simplicial complex yield spaces of the same homotopy type [4]).

Let Y be the space of paths in B beginning at the base point and ending in A. According to [4], Y is sufficiently smooth in order to admit an universal covering space; as noticed by SERRE [5, ch. III, Remarque 3], this is sufficient for the validity of his Théorème 3 without the above assumption concerning the second homotopy groups.

3. Concluding remarks

The notion of space of category ≤ 2 may be relativized by introducing spaces of category ≤ 2 mod. C. Namely, with the notations of the beginning of the previous section, $\operatorname{cat} X \leq 2$ (mod. C) if there is a map

$$\Phi : X \to X \vee X$$

such that $r_j \circ \Phi$, $j = 1, 2$ are C-isomorphisms in homology. Obviously

3.1. Remark. *Theorem 2.2 remains true if we replace* $\operatorname{cat} K \leq 2$ *by* $\operatorname{cat} K \leq 2$ (mod. C).

In view of Theorem 2.2 all computations mod. C of the homotopy groups of a simply connected space X of category ≤ 2 with finitely generated singular homology groups reduce to similar computations for an union of spheres, a problem solved by HILTON [3]. For, we may replace X by its singular polytope $P(X)$ whose category is also ≤ 2 [2]. It results that for a space X of category ≤ 2 the HUREWICZ homomorphism $\pi_n(X) \to H_n(X)$ is always a C-epimorphism; its kernel consists mod. C of iterated WHITEHEAD products. This means that the homology groups of X (their free part) entirely determine mod. C its homotopy groups. This enables us to prove

3.2. Corollary. *For any two simply connected CW-complexes K and L with finitely generated homology groups in each dimension, the groups $\pi_n(K \vee L)$ are determined* mod. C *by* $\pi_n(K)$, $\pi_n(L)$, $H_*(\Omega K, k)$ *and* $H_*(\Omega L, k)$ (where k is the field of rationals).

Proof. As is well known

$$\pi_n(K \vee L) = \pi_n(K) + \pi_n(L) + \pi_n(K \square L)$$

where $K \square L$ is the space of paths in $K \times L$ beginning in the subspace

$K \times * \cup * \times L$ and ending at the base point. It is easy to see that

$$K \square L = EK \times \Omega L \cup \Omega K \times EL \subset EK \times EL = E(K \times L) ,$$

where ΩK, ΩL are the loop spaces and $EK, EL, E(K \times L)$ are the spaces of paths ending at $*$. Applying the relative Künneth theorem to $(EK, \Omega K) \times (EL, \Omega L)$ we obtain

$$H_n(K \square L, k) = \sum_{\substack{p,q=n-1 \\ p,q>0}} H_p(\Omega K, k) \otimes H_q(\Omega L, k) .$$

Or the other hand, by [4], $K \square L$ has the homotopy type of a CW-complex and by 2.4 below, cat $(K \square L) \leq 2$. Then, as we have remarked, the homology groups $H_*(K \square L, k)$ determine $\pi_n(K \square L)$ and 2.3 is proved.

2.4. Lemma. cat $(K \square L) \leq 2$.

Proof. Let U be a contractible open neighbourhood of $*$ in K and V be such a neighbourhood of $*$ in L. Then $K \square L$ has the homotopy type of the space Z of paths in $K \times L$, beginning in $K \times V \cup U \times L$ and ending at $*$.

$$Z = EK \times E_0L \cup E_0K \times EL$$

where $E_0K \subset EK$ consists of paths beginning in U and $E_0L \subset EL$ of paths beginning in V. It suffices to prove that cat $Z \leq 2$. This is true since Z is the union of the following two open contractible sets

$$A = EK \times E_0L \cup EU \times EL$$
$$B = E_0K \times EL \cup EK \times EV .$$

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