Zeitschrift: Commentarii Mathematici Helvetici

Herausgeber: Schweizerische Mathematische Gesellschaft

Band: 35 (1961)

Artikel: Homotopy mod. C of Spaces of Category 2.

Autor: Berstein, Israel

DOI: https://doi.org/10.5169/seals-27325

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 17.04.2026

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

Homotopy mod. C of Spaces of Category 2[1]

by ISRAEL BERSTEIN, Bucharest

The known result of HOPF concerning the cohomology structure of H-spaces may now be restated as follows. An H-space, i.e. a space with a continuous multiplication with unit, has over a field k of characteristic 0 the same cohomology ring as a product of spaces of type (π, n). Denoting by C the class of finite groups, THOM [7] has shown more, namely that an H-space is equivalent mod. C to a product of spaces of type (π, n). On the other hand, in the theory of ECKMANN-HILTON [1], the dual of an H-space is a space of LUSTERNIK-SCHNIRELMANN category ≤ 2. For such spaces we are proving here the dual of the above result of THOM: any finite simply connected CW-complex of category ≤ 2 is equivalent mod. C to an union of spheres with a single common point (here and throughout the paper, C denotes the class of finite groups). The precise result in a slightly more general form is stated in Theorem 2.2.

1. Preliminary lemmas

Let

$$\varphi_k : \pi_r(S^n) \to \pi_r(S^n)$$

be defined by left composition with a map $S^n \to S^n$ of degree k, i.e. $\varphi_k(\gamma) = k\iota \circ \gamma$ for any $\gamma \in \pi_r(S^n)$ ($\iota \in \pi_n(S^n)$) is the class of the identity. Then we have for n even [3]

$$\varphi_k(\gamma) = k\gamma + \frac{k(k-1)}{2}[\iota, \iota] \circ H_0(\gamma) + \frac{(k+1)k(k-1)}{3}[\iota, [\iota, \iota]] \circ H_1(\gamma) \quad (1)$$

where H_0 and H_1 are the generalized HOPF invariants of HILTON. As a consequence of (1)

1.1. $q\gamma = 0$, $\gamma \in \pi_r(S^n)$ *implies* $\varphi_{6q}(\gamma) = 0$ (for n odd it was shown by SERRE that already $\varphi_{2q}(\gamma) = 0$).

Let n be even and let $\pi_{2n-1}(S^n) = Z' + G$, where Z' is infinite cyclic, generated by α and G is finite of order g. Denote by d the classical HOPF invariant of α, i.e. $H_0(\alpha) = d\iota_{2n-1}$, where ι_{2n-1} generates $\pi_{2n-1}(S^{2n-1})$. Let $[\iota, \iota] = s\alpha + \beta$, $\beta \in G$. An easy computation, based on (1), shows that for any $\gamma \in \pi_{2n-1}(S^n)$

1.2. $\varphi_{2mg}(\gamma) = N_0 m\gamma$ *where* $N_0 = 2g + gsd(2mg - 1)$. Moreover,

1.3. if $\gamma = r\alpha + \delta$, $\delta \in G$ *then* $\varphi_{2mg}(\gamma) = N_0 rm\alpha$.

1.4. **Lemma.** *Let K be a CW-complex, n an even integer and*

$$f : K^{2n-1} \to S^n$$

---

[1] The result of this paper was presented to the International Colloquium on Differential Geometry and Topology, Zurich, June 1960 (in absence of the author, by Professor HILTON).

a map such that $f^*(u) = h \in H^n(K, Z)$ *where* $u \in H^n(S^n, Z)$ *is the fundamental class. If the cup-square* $h \smile h \in H^{2n}(K, Z)$ *is an element of finite order, then there exists a map* $\varphi : S^n \to S^n$ *of degree* $s \neq 0$ *such that* $\varphi \circ f / K^{2n-2}$ *is extendable over* K^{2n}.

Proof. Without loss of generality we may assume, for the sake of convenience, that $\dim K = 2n$. Attach to S^n a $2n$-cell with characteristic map in class α; let Y be the resulting space. Let $c^{2n}(f) \in C^{2n}(K, \pi_{2n-1}(S^n))$ be the obstruction to the extension of f. By 1.3, if $\psi : S^n \to S^n$ has degree $2g$, the obstruction $c^{2n}(\psi \circ f)$ takes on each cell a value which is a multiple of α. Therefore, denoting by $i : S^n \to Y$ the inclusion, $i \circ \psi \circ f$ can be extended to a map

$$F : K \to Y .$$

If $u' \in H^n(Y, Z)$ is the fundamental class, then $u' \smile u' = -da$, where a is the fundamental class of $H^{2n}(Y, Z) \approx Z$ and d is the HOPF invariant of α.

Let $j : Z \to \pi_{2n-1}(S^n)$ map Z onto $Z' \, (j(1) = \alpha)$; it induces homomorphisms

$$j_* : H^{2n}(K, Z) \to H^{2n}(K, \pi_{2n-1}(S^n)) ,$$
$$j_* : H^{2n}(Y, Z) \to H^{2n}(Y, \pi_{2n-1}(S^n)) .$$

It is easy to check that

$$F^*(j_*(a)) = \gamma^{2n}(\psi \circ f)$$

where $\gamma^{2n}(\psi \circ f)$ is the cohomology class of $c^{2n}(\psi \circ f)$. We further have

$$\begin{aligned} d \cdot \gamma^{2n}(\psi \circ f) = d \cdot F^*(j_*(a)) &= j_*(F^*(d \cdot a)) = -j_*(F^*(u' \smile u')) = \\ &= -j_*(2gh \smile 2gh) = -4g^2 \cdot j_*(h \smile h) . \end{aligned}$$

This proves that $\gamma^{2n}(\psi \circ f)$ is an element of finite order, say m. Let $\chi : S^n \to S^n$ be a map of degree $2mg$. Then 1.2 immediately yields

$$c^{2n}(\chi \circ \psi \circ f) = N_0 m c^{2n}(\psi \circ f)$$

and

$$\gamma^{2n}(\chi \circ \psi \circ f) = N_0 m \gamma^{2n}(\psi \circ f) = 0 .$$

This proves the assertion of the lemma.

1.5. Proposition. *Let* K *be a* q*-dimensional* $(q < \infty)$ *CW-complex,* n *an integer and* h *an arbitrary element of* $H^n(K, Z)$, *such that the cup-square* $h \smile h$ *is an element of finite order. There exist an integer* $N > 0$ *and a map* $f : K \to S^n$ *such that* $f^*(u) = Nh$, *where* u *is the fundamental class of* $H^n(S^n, Z)$.

This proposition was conjectured by SERRE and proved by him for n odd [5, ch. V, Prop. 2]. For n even the proof is practically the same but uses 1.1 and 1.4.

Remark. In view of 1.5 and of [6, II, 2] we may add to [6, II, 4] the following result:

Let V^n be an orientable closed differentiable n-manifold, k an even number and z a class in $H_{n-k}(V^n, Z)$, whose selfintersection is a class of finite order. Then there exists an integer $N > 0$ such that the class Nz can be realized by means of a submanifold whose normal fibre bundle is trivial.

2. The main theorem

The base point of any space will be denoted by $*$. For any spaces $X_1, \dots, X_m$. $X_1 \vee \dots \vee X_m$ denotes their union with a single common point $*$. There are obvious retractions $r_j : X_1 \vee \dots \vee X_m \to X_j$ mapping $X_1, \dots, X_{j-1}, X_{j+1}, \dots, X_m$ onto $*$. If $\varphi_j : X_j \to Y_j$ are maps, there is a map

$$\varphi_1 \vee \dots \vee \varphi_m : X_1 \vee \dots \vee X_m \to Y_1 \vee \dots \vee Y_m$$

defined in the obvious way.

In this paper we consider only spaces which have the homotopy type of connected CW-complexes. For such a space, the two following definitions of LUSTERNIK-SCHNIRELMANN category ≤ 2 are equivalent (compare [8, p. 94]).

A) $\operatorname{cat} X \leq 2$ if and only if $X = A_1 \cup A_2$ where A_1 and A_2 are open and contractible in X.

B) $\operatorname{cat} X \leq 2$ if and only if there exists a map

$$\Phi : X \to X \vee X$$

such that $r_j \circ \Phi \simeq \theta_X : X \to X$ $(j = 1, 2)$ where θ_X is the identity map of X and the homotopies are rel. $*$.

If $\operatorname{cat} X \leq 2$ define

$$\Phi_m : X \to \underbrace{X \vee \dots \vee X}_{m\text{-fold}} \tag{2}$$

by

$$\Phi_2 = \Phi\,, \quad \Phi_m = (\underbrace{\theta_X \vee \dots \vee \theta_X}_{(m-2)\text{-fold}} \vee \Phi) \circ \Phi_{m-1}\,. \tag{2'}$$

If follows readily that

2.1. $r_j \circ \Phi_m \simeq \theta_X\,, \quad j = 1, \dots, m\,.$

2.2. Theorem. *Let K be a connected and simply connected CW-complex whose homology groupes are finitely generated in each dimension and let C be the class of finite groups. If* $\operatorname{cat} K \leq 2$, *there exists for any integer $r > 1$ a map*

$$f : K^{r+1} \to L$$

where L is an union of spheres, such that

$$f_* : H_i(K^{r+1}, Z) \to H_i(L, Z)$$

and

$$f_\# : \pi_i(K^{r+1}) \to \pi_i(L)$$

are C-monomorphisms for $i < r$ *and C-epimorphisms for* $i \leq r$.

2.3. Corollary. *If* $\dim K < \infty$, *we may choose* Y *and the map* $f : K \to L$ *such that*

$$f_* : H_i(K, Z) \to H_i(L, Z)$$

and

$$f_\# : \pi_i(K) \to \pi_i(L)$$

are C-isomorphisms in all dimensions.

Remark. If the homology of K is not finitely generated, then Theorem 2.2 is not true. A counter example is provided by a complex $K' = K'(Q, 2)$ such that $H_i(K', Z) = 0$ for $i \neq 0, 2$ and $H_2(K', Z) = Q$ where Q is the group of rationals.

Proof of 2.2. Since all cup-products in K vanish (see [2]) if follows from 1.5 that for any cohomology class $h \in H^i(K^{r+1}, k)$, $i \leq r$, where k is the field of rationals, there exists a map

$$g : K^{r+1} \to S^i$$

and a class $u \in H^i(S^i, k)$ such that

$$g^*(u) = h .$$

Let $h_1, h_2, \ldots, h_m$ be a base of $\sum_{i=1}^{r} H^i(K^{r+1}, k)$. Choose for each j, $(j = 1, \ldots, m)$ a map

$$g_j : K^{r+1} \to S_j$$

where S_j is a sphere of the corresponding dimension, such that

$$g_j^*(u_j) = h_j , \qquad u_j \in H^{n_j}(S_j, k) , \qquad n_j = \dim S_j . \tag{3}$$

Let

$$\Phi_m : K \to \underbrace{K \vee \ldots \vee K}_{m\text{-fold}}$$

be as in (2). We may assume that Φ_m is cellular; then Φ_m induces a map

$$\overline{\Phi}_m : K^{r+1} \to K^{r+1} \vee \ldots \vee K^{r+1} .$$

Consider the map

$$g_1 \vee \ldots \vee g_m : K^{r+1} \vee \ldots \vee K^{r+1} \to S_1 \vee \ldots \vee S_m = L .$$

It is easy to check, by means of 2.1 (where all homotopies may be chosen cellular) and (3) that $f : K^{r+1} \to Y$, $f = (g_1 \vee \ldots \vee g_m) \circ \overline{\Phi}_m$ induces isomorphisms

$$f^* : H^i(L, k) \to H^i(K^{r+1}, k)$$

in dimensions $i \leq r$. Applying the known results of SERRE [5, ch. III, Th. 3 and Prop. 1] we obtain the conclusion of the theorem.

Remark. Set $K^{r+1} = A$, $L = B$; in order to apply **Serre**'s results, quoted above, we must assume that $\pi_2(A) \to \pi_2(B)$ is an epimorphism. In fact, this restriction may be removed. We may assume that f is an inclusion. Further, passing if necessary to singular polytopes, we may also assume that B is a simplicial complex with the strong (metric) topology and A is a subcomplex (the strong topology and the weak one on a simplicial complex yield spaces of the same homotopy type [4]).

Let Y be the space of paths in B beginning at the base point and ending in A. According to [4], Y is sufficiently smooth in order to admit an universal covering space; as noticed by **Serre** [5, ch. III, Remarque 3], this is sufficient for the validity of his Théorème 3 without the above assumption concerning the second homotopy groups.

3. Concluding remarks

The notion of space of category ≤ 2 may be relativized by introducing spaces of category ≤ 2 mod. C. Namely, with the notations of the beginning of the previous section, $\operatorname{cat} X \leq 2$ (mod. C) if there is a map

$$\Phi : X \to X \vee X$$

such that $r_j \circ \Phi$, $j = 1, 2$ are C-isomorphisms in homology. Obviously

3.1. Remark. *Theorem 2.2 remains true if we replace* $\operatorname{cat} K \leq 2$ *by* $\operatorname{cat} K \leq 2$ (mod. C).

In view of Theorem 2.2 all computations mod. C of the homotopy groups of a simply connected space X of category ≤ 2 with finitely generated singular homology groups reduce to similar computations for an union of spheres, a problem solved by **Hilton** [3]. For, we may replace X by its singular polytope $P(X)$ whose category is also ≤ 2 [2]. It results that for a space X of category ≤ 2 the **Hurewicz** homomorphism $\pi_n(X) \to H_n(X)$ is always a C-epimorphism; its kernel consists mod. C of iterated **Whitehead** products. This means that the homology groups of X (their free part) entirely determine mod. C its homotopy groups. This enables us to prove

3.2. Corollary. *For any two simply connected CW-complexes K and L with finitely generated homology groups in each dimension, the groups $\pi_n(K \vee L)$ are determined* mod. C *by $\pi_n(K)$, $\pi_n(L)$, $H_*(\Omega K, k)$ and $H_*(\Omega L, k)$* (where k is the field of rationals).

Proof. As is well known

$$\pi_n(K \vee L) = \pi_n(K) + \pi_n(L) + \pi_n(K \square L)$$

where $K \square L$ is the space of paths in $K \times L$ beginning in the subspace

$K \times * \cup * \times L$ and ending at the base point. It is easy to see that

$$K \square L = EK \times \Omega L \cup \Omega K \times EL \subset EK \times EL = E(K \times L),$$

where $\Omega K, \Omega L$ are the loop spaces and $EK, EL, E(K \times L)$ are the spaces of paths ending at $*$. Applying the relative KÜNNETH theorem to $(EK, \Omega K) \times (EL, \Omega L)$ we obtain

$$H_n(K \square L, k) = \sum_{\substack{p,q=n-1 \\ p,q>0}} H_p(\Omega K, k) \otimes H_q(\Omega L, k).$$

Or the other hand, by [4], $K \square L$ has the homotopy type of a CW-complex and by 2.4 below, cat $(K \square L) \leq 2$. Then, as we have remarked, the homology groups $H_*(K \square L, k)$ determine $\pi_n(K \square L)$ and 2.3 is proved.

2.4. Lemma. cat $(K \square L) \leq 2$.

Proof. Let U be a contractible open neighbourhood of $*$ in K and V be such a neighbourhood of $*$ in L. Then $K \square L$ has the homotopy type of the space Z of paths in $K \times L$, beginning in $K \times V \cup U \times L$ and ending at $*$.

$$Z = EK \times E_0L \cup E_0K \times EL$$

where $E_0K \subset EK$ consists of paths beginning in U and $E_0L \subset EL$ of paths beginning in V. It suffices to prove that cat $Z \leq 2$. This is true since Z is the union of the following two open contractible sets

$$A = EK \times E_0L \cup EU \times EL$$
$$B = E_0K \times EL \cup EK \times EV.$$

Institute of Mathematics R.P.R. Academy, Bucharest

BIBLIOGRAPHY

[1] B. ECKMANN et P. J. HILTON, *Groupes d'homotopie et dualité.* C. R. Acad. Sci. Paris 246 (1958), 2444–2447, 2555–2558.

[2] T. GANEA and P. J. HILTON, *On the decomposition of spaces in Cartesian products and unions.* Proc. Cambridge Phil. Soc. 55 (1959), 248–256.

[3] P. J. HILTON, *On the homotopy groups of the union of spheres.* J. London Math. Soc. 30 (1955), 154–174.

[4] J. MILNOR, *On spaces having the homotopy type of a CW-complex.* Trans. Amer. Math. Soc. 90 (1959), 272–280.

[5] J. P. SERRE, *Groupes d'homotopie et classes de groupes abeliens.* Annals of Math. 58 (1953), 258–294.

[6] R. THOM, *Quelques propriétés globales des variétés différentiables.* Comment. Math. Helv. 28 (1954), 17–86.

[7] R. THOM, *L'homologie des espaces fonctionnels.* Colloque de Topologie algébrique tenu à Louvain (1956), 29–39.

[8] G. W. WHITEHEAD, *The homology suspension.* Colloque de Topologie algébrique tenu à Louvain (1956), 89–95.

(Received June 2, 1960)