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Geodesics of Bounded, Symmetric Domains

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1. Introduction

Let G be a semi-simple, non-compact connected LIE group with closed, connected subgroups G', G_u, K such that

$$K = G' \cap G_u .$$

Let G, G', G'_u, K be the corresponding LIE algebras.

We suppose that G is a complex simple LIE algebra in the sense that there is a linear mapping $\iota \colon \mathsf{G} \to \mathsf{G}$ such that:

$$\iota^2 = -1 . \tag{1.1}$$

$$\mathrm{Ad}\ X \iota = \iota\ \mathrm{Ad}\ X \text{ for all } X \in \mathsf{G} . \tag{1.2}$$

Suppose that:

$$\mathsf{K} \text{ is not semi-simple.} \tag{1.3}$$

$$\mathsf{G}' \text{ and } \mathsf{G}_u \text{ are real forms of } \mathsf{G}\text{; i.e.} \tag{1.4}$$

$$\mathsf{G} = \mathsf{G}' \oplus \iota(\mathsf{G}') = \mathsf{G}_u \oplus \iota(\mathsf{G}_u)$$

$$G_u \text{ is compact.} \tag{1.5}$$

It is known [12] that there is a connected, closed subgroup S of G such that

$$\iota(\mathsf{S}) \subset \mathsf{S} , \tag{1.6}$$

$$S \frown G_u = K = S \frown G' \text{, and:} \tag{1.7}$$

$$\text{The natural map } G_u/K \to G/S \text{ is a diffeomorphism, i.e.} \tag{1.8}$$

$M = G_u/K$ (the space of right cosets) admits a complex analytic structure, and G acts on M as a group of complex analytic transformations.

Let x_0 be the identity coset of M and let $D = G'/K$ be the orbit of G' on x_0. By 1.7, D is an open set of M. As the quotient of a LIE group by a compact subgroup, D has a RIEMANNian metric invariant under the action of G'. Our job here is to describe the geodesics of D starting at x_0 as curves in M.

To make this more precise, let $\pi \colon G \to G/S = M$ be the natural projection.

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[1]) This work was performed at Lincoln Laboratory, Massachusetts Institute of Technology, under contract with the U.S. Army, Navy and Air Force. – A short lecture at the International Colloquium on Differential Geometry and Topology (Zurich, June 1960) was based upon this paper.

Let Exp: $G \to G$ be the usual exponential map. [7]. It is known that one can suppose G' and G_u chosen so that there is a subspace $M \subset G_u$ with

$$[K, M] \subset M, \ [M, M] \subset K. \tag{1.9}$$

$$G_u = K \oplus M, \ G' = K \oplus \iota(M). \tag{1.10}$$

Now, we want to find the curves $t \to X(t)$, $0 \leqslant t < \infty$, in M such that

$t \to \pi \operatorname{Exp}(X(t))$ is a geodesic of D starting at x_0.

This will, in particular, enable us to determine

$(\pi \operatorname{Exp})^{-1}(D)$, as a subspace of M.

Our main result can now be stated:

Theorem. *In terms of a linear mapping*

$J: M \to M$ *such that*

$$J^2 = -1, \tag{1.11}$$

$$\operatorname{Ad} X J = J \operatorname{Ad} X \text{ for } X \in K. \tag{1.12}$$

$$[J(X), Y] + [X, J(Y)] = 0 \text{ for } X \text{ and } Y \in M, \tag{1.13}$$

$X(t)$ *is determined as the solution of the differential equation:*

$$\frac{\sinh \operatorname{Ad}(X(t))}{\operatorname{Ad} X(t)} \left(\frac{d}{dt} X(t)\right) = J(\cosh \operatorname{Ad}(X(t))(X)) \tag{1.14}$$

with $J(X) = \dfrac{d}{dt} X(t)$ *at* $t = 0$.

Suppose further that G_u *admits a linear representation* L *by* $N \times N$ *real matrices such that:*

There is an $N \times N$ *real matrix* J_0 *with*

$$J_0^2 = -1. \tag{1.15}$$

$$J_0 L(X) = L(X) J_0 \text{ for } X \in K. \tag{1.16}$$

$$J_0 L(X) = -L(X) J_0 \text{ for } X \in M. \tag{1.17}$$

$$L(J(X)) = J_0 L(X) \text{ for } X \in M. \tag{1.18}$$

Then, if $X(t) = J(Z(t))$ *is a solution of 1.13,*

$$\frac{d}{dt} L(Z(t)) = \cosh(L(2Z(t)))\, L(X), \tag{1.19}$$

and $Z(t) \in H$, *a maximal abelian subalgebra of* M, *for all* t.

Suppose the real-valued linear forms $\varphi_1, \ldots, \varphi_N$ *are the weights of the representation*, i.e.

$$2\pi\sqrt{-1}\,\varphi_j(X),\ 1 \leqslant j \leqslant N$$

are the eigenvalues of $L(X)$, *for* $X \in H$.

Then,

$$\frac{d}{dt}\varphi_j(Z(t)) = \cos\left(4\pi\varphi_j(Z(t))\right)\varphi_j(X),\ 1 \leqslant j \leqslant N, \tag{1.20a}$$

hence

$$\cosh\,(4\pi\varphi_j(X))\cos\left(4\pi\varphi_j(Z(t))\right) = 1. \tag{1.20b}$$

$$\textit{If } \varphi_j(Y) = 0, \textit{ then } \varphi_j(Z(t)) = 0 \textit{ for all } t. \tag{1.21}$$

$$\textit{If } \varphi_j(Y) \neq 0, \textit{ then } \varphi_j(Z(t)) \to \pm\tfrac{1}{8} \textit{ as } t \to \infty. \tag{1.22}$$

Qualitatively, at least in the case where G_u admits a representation L with properties 1.15–1.18, these results can be interpreted as follows:

Suppose F is the boundary of D in M. K acts on F. Then, a "general" geodesic of D starting at x_0 tends to one of only a finite number of orbits of K on F. The "exceptions" to this rule can be transformed by operations of K so that their initial vectors lie on one of a finite number of hyperplanes of H.

These facts confirm results (unpublished) of D. Lowdenslager obtained by studying the geodesics in the special examples.

Further he found in the examples that a "general" geodesic tends toward a unique orbit of K on F, which could be identified with the Bergman-Silov boundary [10] of D when it was imbedded as a bounded domain in a complex Euclidean space. These facts have not yet been verified in our general situation, but they seem very plausible. Note at least, by 1.22, that the orbits to which the general geodesics may tend are the orbits of maximal distance in the Riemannian metric on M.

In Section 4 we present some remarks that serve as a complement to Lowdenslager's earlier work [10]. I wish to thank Prof. Lowdenslager for allowing me to use his unpublished results.

2. Differential-geometric generalities

All manifolds, Lie groups, action of Lie groups on manifolds, tensor-fields, maps, etc., will be of differentiability class C^∞. We follow Chevalley [7] for Lie group and differential geometric notations with some of the modifications suggested by Ambrose, Singer and Nomizu [1, 11].

All manifolds will be connected and paracompact. Let M be such a manifold. For $x \in M$, M_x denotes the tangent space to M at x. If $\phi: M \to M'$ is a map of manifolds, $\o_*: M_x \to M'_{\o(x)}$ denotes the linear map $\o$ induces on tangent vectors.

Suppose now that G, G_u, G', S, K, G, G_u, G', S, K $\mathrm{Exp}: \mathsf{G} \to G$, $M = G/S = G_u/K$, $D = G'/K$ and $\pi: G \to M$ have the meaning assigned to them in the Introduction. Let e be the identity element of G. G can be identified with G_e.

For $X \in \mathsf{G}$, we will, with the identification of G_X with G, consider Exp_* as a linear mapping: $\mathsf{G} \to G_{\mathrm{Exp}\, X}$. For $g \in G$, let L_g (resp. R_g) be the diffeomorphism $x \to gx$ (resp. $x \to xg^{-1}$) of G with itself. Then, $L_{\mathrm{Exp}(-X)*}\mathrm{Exp}_*: \mathsf{G} \to \mathsf{G}$ is the map.

$$\frac{1-\exp(-\operatorname{Ad} X)}{\operatorname{Ad} X} \qquad [9, \text{p.} 249] \tag{2.1}$$

For $g \in G$, $\pi_* R_{g^{-1}*}: \mathsf{G} \to M_{\pi(g)}$ (resp. $\pi_* L_{g*}$) has as kernel $\operatorname{Ad} g(\mathsf{S})$ (resp. S).

To show that there is a linear map $J: \mathsf{M} \to \mathsf{M}$ satisfying 1.11–1.13, choose a Y_0 in the center of K. It is known that K acting in M is irreducible, hence $\operatorname{Ad} Y_0$ acting on M can be normalized so as to have as eigenvalues only $\pm\sqrt{-1}$. Then put

$$J = \operatorname{Ad} Y_0.$$

S is usually defined using the root structure of G [12]. But, in this simple case, it can be defined more explicitly as:

$$\mathsf{S} = \{X_1 + Y + \iota(X_2 + J(Y)) : X_1, X_2 \in \mathsf{K},\ Y \in \mathsf{M}\}. \tag{2.2}$$

It is well known [11] that the geodesics of $D = G'/K$ through $\pi(e)$ are of the form

$$t \to \pi(\mathrm{Exp}(\iota(X)t)),\ 0 \leqslant t < \infty, \text{ for } X \in \mathsf{M}. \tag{2.3}$$

If $\pi(g)$, for $g \in G$, is on this geodesic, then the tangent vector to the geodesic at $\pi(g)$ is:

$$\pi_* R_{g^{-1}*}(\iota(X)). \tag{2.4}$$

Suppose this geodesic is also equal to $t \to \pi\,\mathrm{Exp}\,(X(t))$, for a curve $t \to X(t)$ in M. Then,

$$\pi_*\,\mathrm{Exp}_*\left(\frac{d}{dt}X(t)\right) = \pi_*\,R_{\mathrm{Exp}(-X(t))*}(\iota(x)). \tag{2.5}$$

By 2.1,

$$\pi_*\,L_{\mathrm{Exp}\,X(t)*}\left(\frac{1-\exp(-\operatorname{Ad} X(t))}{\operatorname{Ad} X(t)}\right)\left(\frac{d}{dt}X(t)\right) = \tag{2.6}$$
$$\pi_*\,L_{\mathrm{Exp}\,X(t)*}(\mathrm{Exp}\,\operatorname{Ad}(-X(t))\,(\iota(X)))$$

or

$$\frac{1-\exp(-\operatorname{Ad} X(t))}{\operatorname{Ad} X(t)}\left(\frac{d}{dt} X(t)\right) - \operatorname{Exp} \operatorname{Ad}(-X(t))\,(\iota(X)) \in S \tag{2.7}$$

Now, since $K \subset S$, $\iota(K) \subset S$, and

$[M, M] \subset K$, the condition that $X(t)$ must

satisfy takes the form:

$$\frac{\sinh \operatorname{Ad} X(t)}{\operatorname{Ad} X(t)}\left(\frac{d}{dt} X(t)\right) = J \cosh(\operatorname{Ad} X(t))\,(X). \tag{2.8}$$

3. Proof of the theorem

Now, suppose that G_u has a linear representation L satisfying 1.15–1.19. Let H be a maximal abelian subalgebra of M. We calculate:

$$L([J(X), [J(X), Y]]) = 4\, L(X)^2\, L(Y) \text{ for } X, Y \in H. \tag{3.1}$$

We see from this that:

$$[J(X), [J(X), Y]] \in H \text{ for } X, Y \in H \tag{3.2}$$

and hence that:

If $X \in H$, $t \to X(t)$ satisfies 2.8 if and only if $X(t) \in J(H)$ for all t, and

$$\frac{d}{dt} X(t) = J(\cosh(\operatorname{Ad} X(t))\,(X)). \tag{3.3}$$

Substituting 3.1 in 3.3 gives 1.19. Applying φ_j to both sides of 1.19 gives 1.20a), hence finishes the proof of the theorem since 1.20b)–1.22 follow immediately on solving the ordinary differential equation 1.20a).

Remarks: Unfortunately, only when G_u is a classical group does G_u admit a linear representation of the type we require[2]). Then, the job of explicitly solving 1.14 remains open for the cases of the two exceptional domains.

For the following cases, one can choose the "classical" representation of G_u:

$$G_u/K = SU(n)/SU(p) \times U(n-p), \quad SO(2n)/U(n), \quad SP(n)/U(n).$$

For example, in the first case $SU(n)$ acts as a group of unitary transformations of a complex vector space V (of complex dimension n) with a Hermition form. V is the direct sum of two mutually perpendicular subspaces of dimension p and n–p. K is the subgroup of G_u leaving the subspaces invariant. J_0 is just the transformation which multiplies every element in one subspace by $\sqrt{-1}$, by $-\sqrt{-1}$ in the other.

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[2]) Due to B. Kostant (unpublished).

For the remaining classical case,

$$G_u/K = SO(n) \,/\, SO(2) \times SO(n-2).$$

the "classical" representation of G_u does not suffice. We must use the spinor representation (The fundamental representation g_1, in CARTAN's notation [6]).

The transformation $J: M \to M$ makes M into a complex EUCLIDean space. The solutions of 3.2 determine an imbedding of D in M. If the required representation exists, it is clear from 1.19b) that D is a bounded, open set in M. Then, we have another proof for the classical G_u that G'/K is a bounded domain [4, 8]. Perhaps it is possible to reason directly, in all cases, that all solutions of 3.2 are bounded, hence that G'/K is a bounded domain? This seems, however, to be a difficult problem in ordinary differential equations.

4. Remarks on the behavior of the LAPLACian of D on its boundary in M

Suppose first that M is a complex-analytic manifold and that D is an open subset of M with a KAEHLERian metric, i.e. for $x \in D$, M_x has a positive-definite inner product (,). Let F be the boundary of D in M and let $b_0 \in F$. Suppose Δ is the LAPLACian of the metric on D, a second-order, linear differential operator acting on C^∞ real-valued functions in D. Suppose Δ can be extended smoothly to a differential operator in a neighborhood of $D \cup F$. Suppose that the following condition is satisfied:

4.1. For every C^∞ curve $\sigma: [0, 1] \to M$ with $\sigma(t) \in D$ for $0 \leqslant t < 1$ and $\sigma(l) = b_0$ and every C^∞ vector-field v along σ, $v: t \to v(t) \in M_{\sigma(t)}$ for $0 \leqslant t \leqslant 1$, with $v(1) \neq 0$, we have:

$$\lim_{t \to 1} (v(t), v(t)) = \infty$$

Then, one sees easily, using the classical expression for the LAPLACian operator in terms of the metric tensor, that:

4.2. $\Delta = 0$ at b_0, i.e. $\Delta(f)(b_0) = 0$ for every C^∞ real-valued function f defined in a neighborhood of $D \cup F$.

(Condition 4.1 is just a way of making precise the idea that the metric tensor of D approaches ∞ on the boundary.)

Now, return to the case where $M = G_u/K = G/S$, $D = G'/K \subset M$, and the metric on D is the unique one (up to a constant multiple) invariant under G'.

It is well-known that this metric on D has non-positive sectional curvature (since K is a maximal compact subgroup of the simple LIE group G') and that the map $\pi \operatorname{Exp}: \iota(M) \to D$ is a diffeomorphism.

Suppose that $x = \pi \operatorname{Exp}(\iota(X))$, for $X \in H$, is a point of D. Consider the endomorphism of $\iota(M)$:

$$\frac{\sinh(\operatorname{Ad}(\iota(X)))}{\operatorname{Ad}(\iota(X))} \tag{4.3}$$

One sees that it can be put into diagonal form, i.e. there is an orthonormal basis X_i, $1 \leqslant i \leqslant m$ ($= \dim M$) for M such that the (X_i) are eigenvectors for 4.3 with eigenvalues

$$\sinh(\lambda_i(X)) \,/\, \lambda_i(X), \tag{4.4}$$

where λ_i are the real-valued linear forms on H obtained by diagonalizing the adjoint representation of H in G_u. (They are just the roots of G_u restricted to H.) Suppose λ_j, for $1 \leqslant i \leqslant p$, are the non-zero roots.

Define:

$$v_i(x) = \pi_* R_{\operatorname{Exp}(-X)*}(\iota(X_i)) \in D_x, \text{ for } 1 \leqslant i \leqslant m. \tag{4.5}$$

From formula 2.4 of [9], due to Cartan and Helgason, we see that:

$$(v_i(x), v_j(x)) = 0 \text{ if } t \neq j. \tag{4.6}$$

$$(v_i(x), v_i(x)) = (\sinh(\lambda_i(X)) \,/\, \lambda_i(X))^2, \ 1 \leqslant i \leqslant m. \tag{4.7}$$

Now, $x \to v_i(x)$ determines vector fields v_i on all of D. These are precisely the vector fields obtained from the action on D of the one parameter subgroups $s \to \operatorname{Exp}(s\iota(X_i))$ of G'. Since G' acts on all of M, these vector fields can be extended smoothly over all of M. 4.7 then proves the following result:

Proposition 4.1. *Suppose that* $\sigma(t) = \operatorname{Exp}(\iota(X)(t))$, $0 \leqslant t < \infty$, *is a* C^∞ *curve in* D *such that* $\lim_{t \to \infty}(t) \in F$. *Suppose further that* $v: t \to v(t) \in D_{\sigma(t)}$ *is a vector-field along* σ *such that* $\lim_{t \to \infty} v(t)$ *exists and is not zero. If*

$$\lim_{t \to \infty} (\lambda_i(X(t))^2 = \infty, \ 1 \leqslant i \leqslant p, \ \textit{then} \ \lim_{t \to \infty} (v(t), v(t)) = \infty. \tag{4.8}$$

Notice, in particular, that if $X(t) = tX_0$, with X_0 an element of H not lying on any of the hyperplanes $\lambda_i = 0$, $1 \leqslant i \leqslant p$, i.e. if σ is a geodesic of D in general position, then 4.8 is satisfied. By our previous remarks, the points on the boundary towards which these geodesics tend are then points at which the Laplacian of D is zero.

These results tie in with the earlier work of Lowdenslager [10]. The "Bergman-Silov" boundary of D in M is at least contained in the set of points of F where the Laplacian of D vanishes. We have seen that a general geodesic of D tends to a point on F where the Laplacian of D vanishes. If one could prove for all cases that the general geodesics of D tend to a single orbit of K on F, the identification of the BERGMAN-SILOV boundary with the set of points of F where the LAPLACian vanishes would be complete, and independent of LOWDENSLAGER's case-by-case verification [10]. We have not tried very hard to push the method used in Sections 2 and 3 to prove this since A. KORANYI has obtained a proof of this (to be published) using HARISH-CHANDRA's more general approach [8].

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