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# Projectively Flat Spaces with Recurrent Curvature

*To the University of Hong Kong on its Golden Jubilee in 1961*

By YUNG-CHOW WONG and KENTARO YANO

## Introduction

Let  $A_N$  be an affinely connected  $N$ -dimensional space with a symmetric connection (i.e. a connection without torsion).  $A_N$  is a *projectively flat space*, or simply, a  $P_N$  if there exists a coordinate system in terms of which the finite equations of the paths are linear.  $A_N$  is of *recurrent curvature*<sup>1)</sup> if the covariant derivative of its curvature tensor is the tensor product of a non-zero covariant vector and the curvature tensor itself.

The purpose of this paper is to determine all the projectively flat spaces with recurrent curvature. For convenience we shall denote such a space by  $P_N^*$ . It is found that the space  $P_N^*$  (or rather its connection) depends on 2 arbitrary functions of one variable or on 3 arbitrary functions of one variable according as its RICCI tensor is symmetric or non-symmetric. The actual construction of the connection of the  $P_N^*$  depends on the solution of a differential equation of the RICCATI type and on the solution of a completely integrable system of differential equations.

Projectively flat space with covariantly constant curvature tensor (i.e. projectively flat symmetric space) is also considered. We prove that it is a well-known type of projectively flat space characterized by its RICCI tensor being symmetric and covariantly constant.

## 1. Preliminaries<sup>2)</sup>

Throughout this paper, each of the indices  $a, h, i, j, \dots$  runs through the range  $1, \dots, N$ ;  $\partial_k$  denotes partial differentiation with respect to the  $k$ th coordinate; and a repeated index implies summation.

Let  $A_N$  be a linearly connected  $N$ -dimensional space with a symmetric connection  $\Gamma_{ji}^h$  (i.e.  $\Gamma_{ji}^h = \Gamma_{ij}^h$ ). The curvature tensor, the RICCI tensor, and the tensor  $P_{ji}$  are defined respectively by:

$$R_{kji}{}^h = \partial_k \Gamma_{ji}^h - \partial_j \Gamma_{ki}^h + \Gamma_{ka}^h \Gamma_{ji}^a - \Gamma_{ja}^h \Gamma_{ki}^a, \quad (1.1)$$

$$R_{ji} = R_{aji}{}^a, \quad (1.2)$$

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<sup>1)</sup> RIEMANNIAN spaces with recurrent curvature have been studied in great detail by H. S. RUSE [2] and A. G. WALKER [3]. Certain classes of non-RIEMANNIAN spaces with recurrent curvature have been studied by Y. C. WONG [4].

<sup>2)</sup> See EISENHART [1], but our notation is slightly different from his.

$$P_{ji} = -\frac{N}{N^2 - 1} R_{ji} - \frac{1}{N^2 - 1} R_{ij}. \quad (1.3)$$

It follows from (1.3) that  $P_{ji}$  is symmetric iff the RICCI tensor  $R_{ji}$  is symmetric.

By definition,  $A_N$  is of *recurrent curvature* if its curvature tensor satisfies the condition

$$\nabla_l R_{kji}{}^h = r_l R_{kji}{}^h, \quad (1.4)$$

where  $\nabla$  denotes the covariant differentiation and  $r_l$  a non-zero covariant vector. If

$$\nabla_l R_{kji}{}^h = 0, \quad (1.5)$$

the space  $A_N$  is said to be *symmetric*.

$A_N$  is *projectively flat* iff there exists a coordinate system in terms of which the components  $\Gamma_{ji}^h$  have the form

$$\Gamma_{ji}^h = \varphi_j A_i^h + \varphi_i A_j^h, \quad (1.6)$$

where  $A_i^h$  is the KRONECKER delta and  $\varphi_j$  a set of  $N$  functions. The special form (1.6) for  $\Gamma_{ji}^h$  is preserved by affine transformations of coordinates. Expressed in terms of  $P_{ji}$ , a well-known necessary and sufficient condition for  $A_N$  to be projectively flat is

$$\left. \begin{aligned} R_{kji}{}^h + A_k^h P_{ji} - A_j^h P_{ki} - (P_{kj} - P_{jk}) A_i^h &= 0, \\ \nabla_k P_{ji} - \nabla_j P_{ki} &= 0. \end{aligned} \right\} \quad (1.7)$$

For  $N > 2$ , (1.7)<sub>2</sub> is a consequence of (1.7)<sub>1</sub>; for  $N = 2$ , (1.7)<sub>1</sub> is an identity. There exists, for (1.6), a function  $\varphi$  such that  $\varphi_j = \partial_j \varphi$  iff  $P_{ji} = P_{ij}$ . Since on account of (1.7) a projectively flat space with  $P_{ji} = 0$  is flat, we shall always assume that

$$P_{ji} \neq 0. \quad (1.8)$$

## 2. Some necessary conditions

By definition, a  $P_N^*$  is a non-flat projectively flat  $A_N$  with recurrent curvature and therefore it is characterized by the conditions (1.4), (1.7) and (1.8). To find all the  $P_N^*$ 's, we first derive a set of necessary and sufficient conditions and then determine the functions  $\varphi_j$  in (1.6) which satisfy these conditions.

From (1.4) we obtain  $\nabla_l R_{ji} = r_l R_{ji}$ , and consequently

$$\nabla_k P_{ji} = r_k P_{ji} \quad (r_k \neq 0). \quad (2.1)$$

It is easy to see that conversely (1.4) is a consequence of (2.1) and (1.7)<sub>1</sub>.

On account of (2.1), equation (1.7)<sub>2</sub> becomes

$$r_k P_{ji} = r_j P_{ki},$$

which is equivalent to

$$P_{ji} = r_j p_i, \quad (2.2)$$

where  $p_i$  is some non-zero vector (since  $P_{ji} \neq 0$ ).

Using (2.2) in (1.7)<sub>1</sub>, we see that the curvature tensor of a  $P_N^*$  has the form

$$R_{kji}{}^h = -(A_k^h r_j - A_j^h r_k) p_i + (r_k p_j - r_j p_k) A_i^h. \quad (2.3)$$

On account of (2.2), equation (2.1) becomes

$$(\nabla_k r_j) p_i + r_j (\nabla_k p_i) = r_k r_j p_i, \quad (2.4)$$

from which it follows that

$$\begin{aligned} \nabla_j r_i &= s_j r_i, \\ \nabla_j p_i &= t_j p_i, \end{aligned} \quad (2.5)$$

for some vectors  $s_j, t_j$ . These equations show that  $r_i$  and  $p_i$  are of the form

$$\begin{aligned} r_i &= \lambda \xi_i \quad (\xi_i = \partial_i \xi \neq 0), \\ p_i &= \mu \eta_i \quad (\eta_i = \partial_i \eta \neq 0), \end{aligned} \quad (2.6)$$

for some scalars  $\xi, \eta, \lambda, \mu$ .

Substitution of (2.6) in (2.3) and (2.5) gives

$$R_{kji}{}^h = \psi [(A_k^h \xi_j - A_j^h \xi_k) \eta_i - (\xi_k \eta_j - \xi_j \eta_k) A_i^h] \quad (2.7)$$

$$\begin{aligned} \nabla_j \xi_i &= \alpha \xi_j \xi_i, \\ \nabla_j \eta_i &= \beta \eta_j \eta_i, \end{aligned} \quad (2.8)$$

where  $\psi, \alpha, \beta$  are some scalars.

We now proceed to consider the integrability conditions of the differential equations (2.8). For any covariant vector  $u_i$ , we have the Ricci identity

$$\nabla_k \nabla_j u_i - \nabla_j \nabla_k u_i = -R_{kji}{}^h u_h. \quad (2.9)$$

Using (2.7) in (2.9) and putting  $u_i = \xi_i$ , we get, since  $\xi_i \neq 0$ ,

$$(\xi_k \alpha_j - \xi_j \alpha_k) = -\psi (\xi_k \eta_j - \xi_j \eta_k) \quad (\alpha_j = \partial_j \alpha),$$

i.e.

$$\xi_k (\alpha_j + \psi \eta_j) = \xi_j (\alpha_k + \psi \eta_k)$$

which is equivalent to

$$\alpha_j + \psi \eta_j = \rho \xi_j \quad (2.10)_1$$

for some scalar  $\rho$ . Similarly, using (2.7) in (2.9) and putting  $u_i = \eta_i$ , we get

$$\beta_j - 2\psi \xi_j = \sigma \eta_j \quad (\beta_j = \partial_i \beta) \quad (2.10)_2$$

for some scalar  $\sigma$ .

Equations (2.10) tell us that

*The scalars  $\alpha, \beta, \psi, \rho, \sigma$  can be expressed as functions of  $\xi, \eta$  alone.* (2.11)

On account of (2.11), equations (2.10) can be re-written as

$$\left. \begin{aligned} (\alpha_\xi - \rho) \xi_j + (\alpha_\eta + \psi) \eta_j &= 0 \\ (\beta_\xi - 2\psi) \xi_j + (\beta_\eta - \sigma) \eta_j &= 0 \end{aligned} \right\} \quad (2.12)$$

where  $\alpha_\xi = \frac{\partial \alpha}{\partial \xi}$ ,  $\alpha_\eta = \frac{\partial \alpha}{\partial \eta}$ , etc. Two cases arise according as whether or not  $\eta$  is functionally dependent on  $\xi$ , i.e. whether or not  $\eta_i$  and  $\xi_i$  are proportional. Now, by (2.2) and (2.6),  $\eta_i$  and  $\xi_i$  are proportional or not according as the tensor  $P_{ji}$  (or the RICCI tensor) is symmetric or not. We shall consider these two cases separately in § 3 and § 4 respectively.

### 3. $P_N^*$ with symmetric RICCI tensor

In this case,  $\eta = \eta(\xi)$  so that (2.12) reduce to

$$\begin{aligned} (\alpha_\xi - \rho) + (\alpha_\eta + \psi) \eta' &= 0, \\ (\beta_\xi - 2\psi) + (\beta_\eta - \sigma) \eta' &= 0, \end{aligned}$$

where  $\eta' = d\eta/d\xi$ . Since these equations just determine the unspecified functions  $\rho, \sigma$  of  $\xi$  which appear in (2.10), the *integrability conditions of (2.8) reduce to the mere fact that  $\alpha, \beta$  and  $\psi$  are functions of  $\xi$ .*

On account of  $\eta = \eta(\xi)$ , equations (2.7) and (2.8) become

$$\begin{aligned} R_{kji}{}^h &= \psi \eta' (A_k^h \xi_j - A_j^h \xi_k) \xi_i, \\ \nabla_j \xi_i &= \alpha \xi_j \xi_i, \\ \nabla_j \xi_i &= (\beta \eta' - \eta''/\eta') \xi_j \xi_i. \end{aligned}$$

Hence, we have the first part of the following

**Theorem 3.1.** *For a  $P_N^*$  with symmetric RICCI tensor, there exists a scalar  $\xi$  and functions  $\theta$  and  $\alpha$  of  $\xi$  such that*

$$R_{kji}{}^h = \theta (A_k^h \xi_j - A_j^h \xi_k) \xi_i, \quad (3.1)$$

$$\nabla_j \xi_i = \alpha \xi_j \xi_i, \quad (3.2)$$

where  $\xi_i = \partial_i \xi$ .

[The integrability condition of (3.2) is identically satisfied.]

Conversely, an  $A_N$  satisfying this condition is a  $P_N^*$  (or a symmetric  $P_N$ ) with symmetric Ricci tensor.

*Proof.* We need only prove the last part of the theorem. It follows from (3.1) that

$$P_{ji} = -\theta \xi_j \xi_i, \quad (3.3)$$

and so  $P_{ji}$  is symmetric. On account of (3.3) and (3.1), condition (1.7)<sub>1</sub> is satisfied. Furthermore, a simple calculation involving (3.3) and (3.2) will show that  $\nabla_k P_{ji} = r_k P_{ji}$ , with

$$r_j = -(\theta_\xi/\theta + 2\alpha)\xi_j \quad \left(\theta_\xi = \frac{d\theta}{d\xi}\right). \quad (3.4)$$

It follows from this that (1.7)<sub>2</sub> is satisfied and that  $\nabla_i R_{kji}^h = r_i R_{kji}^h$ . The  $A_N$  is therefore a  $P_N^*$  or a symmetric  $P_N$  according as  $\theta_\xi/\theta + 2\alpha \neq 0$  or  $= 0$ . Hence our theorem is completely proved.

In order to construct the  $\Gamma_{ji}^h$  of a  $P_N^*$  with symmetric Ricci tensor, we choose a coordinate system in which  $\Gamma_{ji}^h$  have the form

$$\Gamma_{ji}^h = \vartheta_j A_i^h + \vartheta_i A_j^h, \quad (3.5)$$

where  $\vartheta_j$  are  $N$  functions of the coordinates. With respect to the  $\Gamma_{ji}^h$  given by (3.5), the covariant derivation of a covariant vector  $u_i$  is

$$\nabla_j u_i = \partial_j u_i - \vartheta_j u_i - \vartheta_i u_j, \quad (3.6)$$

and the curvature tensor has the components

$$R_{kji}^h = -A_k^h \vartheta_{ji} + A_j^h \vartheta_{ki} + (\vartheta_{kj} - \vartheta_{jk}) A_i^h. \quad (3.7)$$

where

$$\vartheta_{ji} = \partial_j \vartheta_i - \vartheta_j \vartheta_i. \quad (3.8)$$

Now identifying (3.7) with (3.1) and rewriting (3.2) by means of (3.6), we get

$$\begin{aligned} \partial_j \vartheta_i &= \vartheta_j \vartheta_i - \theta(\xi) \xi_j \xi_i, \\ \partial_j \xi_i &= \vartheta_j \xi_i + \vartheta_i \xi_j + \alpha(\xi) \xi_j \xi_i, \end{aligned} \quad (3.9)$$

where  $\xi_i = \partial_i \xi$ .

It is easy to verify that for any functions  $\theta(\xi)$  and  $\alpha(\xi)$  of  $\xi$ , the integrability conditions of the differential equations (3.9) in the  $N+1$  unknown functions  $\vartheta_i, \xi$  are satisfied, on account of the equations (3.9) themselves. Now the solution of a completely integrable system of differential equations contains a finite number of arbitrary constants, while the special form (3.5) for the connection of a  $P_N$  is preserved by affine transformations of coordinates

which depend on a finite number of constants. But neither of these would alter the fact that  $\Gamma_{ji}^h$  as given by (3.5) depends on two arbitrary functions of one variable, namely, the functions  $\theta, \alpha$  of  $\xi$ . Hence (cf. (3.4)) we have

**Theorem 3.2.** *If we take any functions  $\theta, \alpha (\alpha \neq -\theta_\xi/2\theta)$  of one variable  $\xi$ , and any solution  $\varphi_i, \xi$  of the completely integrable system (3.9), then the connection  $\Gamma_{ji}^h = \varphi_j A_i^h + \varphi_i A_j^h$  defines a  $P_N^*$  with symmetric Ricci tensor; and any  $P_N^*$  with symmetric Ricci tensor can be constructed in the way. Thus the most general  $P_N^*$  with symmetric Ricci tensor depends on 2 arbitrary functions of one variable, and the actual construction of the connection of such  $P_N^*$  depends on the solution of a completely integrable system of differential equations.*

#### 4. $P_N^*$ with non-symmetric Ricci tensors

Let us now return to (2.12). Since  $\eta_i, \xi_i$  are not proportional, (2.12) are equivalent to

$$\begin{aligned} \alpha_\xi &= \varrho, & \beta_\eta &= \sigma, \\ \alpha_\eta + \psi &= 0, & \beta_\xi - 2\psi &= 0. \end{aligned}$$

The first two equations merely determine the unspecified scalars  $\varrho, \sigma$ , which appear in (2.10). Therefore,

$$\alpha_\eta + \psi = 0, \quad \beta_\xi - 2\psi = 0$$

are the integrability conditions of equations (2.8).

From this and (2.7), (2.8) and (2.11), we have the first part of the following

**Theorem 4.1.** *For a  $P_N^*$  with non-symmetric Ricci tensor, there exist functionally independent scalars  $\xi$  and  $\eta$  and functions  $\psi, \alpha, \beta$  of  $\xi$  and  $\eta$  such that*

$$R_{kji}^h = \psi [(A_k^h \xi_j - A_j^h \xi_k) \eta_i - (\xi_k \eta_j - \xi_j \eta_k) A_i^h] \tag{4.1}$$

$$\left. \begin{aligned} \nabla_j \xi_i &= \alpha \xi_j \xi_i, \\ \nabla_j \eta_i &= \beta \eta_j \eta_i, \end{aligned} \right\} \tag{4.2}$$

where  $\xi_i = \partial_i \xi, \eta_i = \partial_i \eta$ .

[The integrability conditions of (4.2) are

$$\alpha_\eta + \psi = 0, \quad \beta_\xi - 2\psi = 0.] \tag{4.3}$$

Conversely, any  $A_N$  which satisfies the above condition is a  $P_N^*$  with non-symmetric Ricci tensor.

*Proof.* We need only prove the last part of this theorem. It follows from (4.1) that

$$P_{ji} = -\psi \xi_j \eta_j, \tag{4.4}$$

and so  $P_{ji}$  is not symmetric. Now (4.4), together with (4.1) and (4.2) show that condition (1.7) for a  $P_N$  is satisfied. Furthermore, a simple calculation involving (4.4) and (4.2) will show that  $\nabla_k P_{ji} = r_k P_{ji}$ , and hence  $\nabla_i R_{kji}^h = r_i R_{kji}^h$ , with

$$r_k = \left( \frac{\psi_\xi}{\psi} + \alpha \right) \xi_k + \left( \frac{\psi_\eta}{\psi} + \beta \right) \eta_k .$$

This vector  $r_k$  is not zero. In fact, since  $\eta_k, \xi_k$  are not proportional,  $r_k = 0$  would imply that

$$(\log \psi)_\xi + \alpha = 0, \quad (\log \psi)_\eta + \beta = 0 .$$

Differentiations of these give

$$\alpha_\eta = \beta_\xi ,$$

from which and (4.3) it follows that  $\psi = 0$ . But this gives a flat space. Hence our theorem is completely proved.

In order to construct the  $\Gamma_{ji}^h$  of a  $P_N^*$  with non-symmetric RICCI tensor, we proceed as in the case of symmetric RICCI tensor, and have (3.5), (3.6), (3.7) and (3.8). Identifying (3.7) with (4.1) and rewriting (4.2) by means of (3.6), we get

$$\left. \begin{aligned} \partial_j \varphi_i &= \varphi_j \varphi_i + \psi(\xi, \eta) \xi_j \eta_i, \\ \partial_j \xi_i &= \varphi_j \xi_i + \varphi_i \xi_j + \alpha(\xi, \eta) \xi_j \eta_i, \\ \partial_j \eta_i &= \varphi_j \eta_i + \varphi_i \eta_j + \beta(\xi, \eta) \xi_j \eta_i, \end{aligned} \right\} \quad (4.4)$$

where  $\xi_i = \partial_i \xi$ ,  $\eta_i = \partial_i \eta$ . From Theorem 4.1 it follows that an  $A_N$  with connection (3.5) is a  $P_N^*$  with non-symmetric RICCI tensor if the functions  $\varphi_j$  together with two functionally independent scalars  $\xi, \eta$  and some functions  $\psi, \alpha, \beta$  of  $\xi, \eta$ , satisfy the differential equations (4.4). We know already (from Theorem 4.1) that the integrability conditions of (4.4)<sub>2,3</sub> are (4.3). It is easy to verify that on account of (4.4), the integrability condition of (4.4)<sub>1</sub> reduces to

$$(\psi_\eta + \psi\beta)(\xi_k \eta_j - \xi_j \eta_k) = 0 .$$

Since  $\eta \neq \eta(\xi)$ , this is equivalent to

$$\psi_\eta + \psi\beta = 0 .$$

Hence combining this with (4.3), we obtain the following complete set of integrability conditions of (4.4):

$$\alpha_\eta = -\psi, \quad \beta_\xi = 2\psi, \quad \psi_\eta + \psi\beta = 0 .$$

Rewrite these as

$$\frac{\partial \alpha}{\partial \eta} = -\psi, \quad \frac{\partial \beta}{\partial \xi} = 2\psi, \quad \frac{\partial \psi}{\partial \eta} = -\psi\beta. \quad (4.5)$$

From the last two equations, we obtain

$$\frac{\partial^2 \beta}{\partial \xi \partial \eta} = 2 \frac{\partial \psi}{\partial \eta} = -2\psi\beta = -\beta \frac{\partial \beta}{\partial \xi},$$

i.e. 
$$\frac{\partial^2 \beta}{\partial \xi \partial \eta} + \frac{1}{2} \frac{\partial \beta^2}{\partial \xi} = 0.$$

Integration of this with respect to  $\xi$  gives

$$\frac{\partial \beta}{\partial \eta} + \frac{1}{2} \beta^2 = f(\eta), \quad (4.6)_1$$

where  $f(\eta)$  is an arbitrary function of  $\eta$ . If  $\beta$  is any function of  $\xi, \eta$  satisfying (4.6)<sub>1</sub>, then in virtue of (4.5),  $\psi$  and  $\alpha$  can be expressed in terms of  $\beta$  as follows:

$$\psi = \frac{1}{2} \frac{\partial \beta}{\partial \xi}, \quad (4.6)_2$$

$$\alpha = -\frac{1}{2} \int \frac{\partial \beta}{\partial \xi} d\eta + h(\xi). \quad (4.6)_3$$

where  $h(\xi)$  is an arbitrary function of  $\xi$ . Equations (4.5) are equivalent to equations (4.6).

Since equation (4.6)<sub>1</sub> is of the RICCATI type, the most general function  $\beta(\xi, \eta)$  satisfying it is of the form

$$\beta(\xi, \eta) = \frac{\beta_1(\eta) + g(\xi)\beta_2(\eta)}{\beta_3(\eta) + g(\xi)\beta_4(\eta)}, \quad (4.7)$$

where  $\beta_1(\eta), \dots, \beta_4(\eta)$  are certain functions of  $\eta$  and  $g(\xi)$  is an arbitrary function of  $\xi$ .

Hence, with an observation similar to that immediately before Theorem 3.2, we have

**Theorem 4.2.** *With any functions  $f(\eta)$  and  $h(\xi)$  of one variable  $\eta$  and  $\xi$  respectively, any solution  $\beta(\xi, \eta)$  [cf. (4.7)] of equation (4.6)<sub>1</sub>, and the functions  $\psi(\xi, \eta)$ ,  $\alpha(\xi, \eta)$  given by (4.6)<sub>2</sub>, (4.6)<sub>3</sub>, the system of differential equations (4.4) is completely integrable. If  $\varphi_i, \xi, \eta$  are any solution of the completely*

integrable system (4.4), then the connection  $\Gamma_{ji}^h = \varphi_j A_i^h + \varphi_i A_j^h$  defines a  $P_N^*$  with non-symmetric Ricci tensor; and any such  $P_N^*$  can be constructed in this way. Thus the most general  $P_N^*$  with non-symmetric Ricci tensor depends on 3 arbitrary functions of one variable, and the actual construction of the connection of such  $P_N^*$  depends on the solution of a differential equation of the RICCATI type and on the solution of a completely integrable system of differential equations.

### 5. Projectively flat symmetric $N$ -space

We now consider the  $A_N$ 's characterized by

$$R_{kji}^h = -A_k^h P_{ji} + A_j^h P_{ki} + (P_{kj} - P_{jk}) A_i^h \quad (5.1)$$

$$\nabla_l R_{kji}^h = 0. \quad (5.2)$$

On account of (5.1), the condition (5.2) is equivalent to

$$\nabla_k P_{ji} = 0. \quad (5.2')$$

Substituting (5.2') into the RICCI identity

$$\nabla_l \nabla_k P_{ji} - \nabla_k \nabla_l P_{ji} = -R_{lkj}^a P_{ai} - R_{lki}^a P_{ja}$$

and taking account of (5.1), we obtain

$$0 = P_{li} P_{kj} - P_{ki} P_{lj} - (P_{lk} - P_{kl}) P_{ji} \\ + P_{jl} P_{kj} - P_{jk} P_{li} - (P_{lk} - P_{kl}) P_{ji},$$

$$\text{i.e.} \quad P_{li} (P_{kj} - P_{jk}) + P_{ki} (P_{jl} - P_{lj}) - 2(P_{lk} - P_{kl}) P_{ji} = 0.$$

Putting  $l = j$ ,  $k = i$ , we get

$$P_{ji} (P_{ij} - P_{ji}) - 2(P_{ji} - P_{ij}) P_{ji} = 0,$$

$$\text{i.e.} \quad 3(P_{ji} - P_{ij}) P_{ji} = 0 \quad (i, j \text{ not summed}).$$

This gives

$$P_{ji} = P_{ij}, \quad (5.3)$$

and consequently, (5.1) becomes

$$R_{kji}^h = -A_k^h P_{ji} + A_j^h P_{ki}. \quad (5.4)$$

Since, conversely, (5.1) and (5.2) are easy consequences of (5.4), (5.3) and (5.2'), we have

**Theorem 5.1.** *An  $A_N$  is a symmetric  $P_N$ , iff*

$$R_{kji}{}^h = -A_k^h P_{ji} + A_j^h P_{ki},$$

$$P_{ji} = P_{ij}, \quad \nabla_k P_{ji} = 0.$$

The properties stated in Theorem 5.1 characterize a well-known type of projectively flat space; in particular, if  $P_{ji}$  is of rank  $n$ , it is RIEMANNIAN and of constant curvature (cf. EISENHART [1], p. 97 and p. 166). We can easily prove that in contrast, a  $P_N$  with recurrent curvature, i.e. a  $P_N^*$ , is never a RIEMANNIAN space.

Obviously, a  $P_N^*$  with a non-symmetric RICCI tensor cannot be RIEMANNIAN. If a  $P_N^*$  with a symmetric RICCI tensor is RIEMANNIAN, let  $g_{ji}$  be a fundamental tensor whose CHRISTOFFEL symbol  $\left\{ \begin{smallmatrix} h \\ ji \end{smallmatrix} \right\}$  is equal to  $\Gamma_{ij}^h$ . Then the covariant components

$$R_{kji}{}^h = R_{kji}{}^a g_{ha}$$

of the curvature tensor satisfy the well-known equation

$$R_{kji}{}^h = R_{ihkj},$$

which on account of (3.1) reduces to

$$g_{kh} \xi_j \xi_i = g_{ij} \xi_h \xi_k.$$

But this contradicts the fact that  $g_{ji}$  is of rank  $N > 1$ . Therefore, no  $P_N^*$  is RIEMANNIAN.

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