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# On sphere-bundles over spheres<sup>1)</sup>

By I. M. JAMES, Oxford (England)

## § 1. Introduction

In 1953, J. H. C. WHITEHEAD and myself made an investigation into the homotopy theory of sphere-bundles over spheres (see [6], [7]). The purpose of this note is to add something to that theory, especially in the case when torsion occurs in the homology of the total space. Our main application is

**Theorem 1.1.** *Let  $B$  be an  $(n - 1)$ -sphere bundle over  $S^n$ , where  $n \geq 2$ . If  $B$  is an  $H$ -space then  $B$  is homeomorphic to  $S^3$ , to real projective 3-space, or to  $S^7$ .*

More generally, let  $B$  be a  $q$ -sphere bundle over  $S^n$ , where  $n, q \geq 1$ . If  $B$  admits a cross-section and is an  $H$ -space then  $B$  has the homotopy type of  $S^n \times S^q$ , by<sup>2)</sup> Theorem (1.23) of [6]; and so  $n, q \in \{1, 3, 7\}$ , by ADAMS' theorem [1]. I do not know whether  $B$  is homeomorphic to  $S^n \times S^q$  under these conditions. There always exists a cross-section when  $n \leq q$ . In case  $n > q + 1$  I understand<sup>3)</sup> from Dr. ADAMS that the theory of secondary operations enables it to be shown that  $B$  is not an  $H$ -space unless either (i)  $q = 3$  and  $n = 5$  or  $7$ , or (ii)  $q = 7$  and  $n = 11$  or  $15$ . Examples of (i) are the special unitary group  $SU(3)$  and the symplectic group  $Sp(2)$ . I do not know any examples of (ii) but it might be worthwhile to investigate the quaternionic STIEFEL manifold  $X_{3,2}$  and the octonionic STIEFEL manifold  $Y_{2,2}$  (see [3]).

## § 2. Torsion $q$ -spheres

Let  $X$  be a space and let  $q$  be a positive integer such that  $H_r(X)$  is a torsion group for  $0 < r < q$  and  $H_q(X)$  is cyclic infinite. Then we describe  $X$  as a *torsion  $q$ -sphere* and we define its *spherical index*  $I(X)$  to be the index of  $\theta\pi_q(X)$  in  $H_q(X)$ , where  $\theta$  denotes the HUREWICZ homomorphism. If  $X$  is 1-connected then it follows from the SERRE-HUREWICZ theorem (Theorem 1 on page 271 of [9]) that  $I(X) > 0$ .

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<sup>1)</sup> This paper was presented at the International Colloquium on Differential Geometry and Topology at Zurich, June 1960.

<sup>2)</sup> A direct proof will be found in § 8 below.

<sup>3)</sup> I am most grateful to Dr. ADAMS for this interesting information.

If  $X$  is a torsion  $q$ -sphere then  $SX$ , the suspension of  $X$ , is a torsion  $(q + 1)$ -sphere and  $I(X)$  is a multiple of  $I(SX)$ . We prove

**Theorem 2.1.** *Let  $K$  be a countable connected CW-complex. Suppose that  $K$  is a torsion  $q$ -sphere and also an  $H$ -space. Then  $I(K) = I(SK)$ .*

Let  $K * K$  denote the join of  $K$  with itself and let  $h : K * K \rightarrow SK$  denote the map obtained by the HOPF construction from the multiplication on  $K$ . Then the following diagram is commutative, where  $h_*$  and  $h_\#$  denote induced homomorphisms and  $E$  denotes suspension.

$$\begin{array}{ccccc}
 \pi_q(K) & \xrightarrow{E} & \pi_{q+1}(SK) & \xleftarrow{h_*} & \pi_{q+1}(K * K) \\
 \theta \downarrow & & \theta \downarrow & & \theta \downarrow \\
 H_q(K) & \xrightarrow{E} & H_{q+1}(SK) & \xleftarrow{h_\#} & H_{q+1}(K * K)
 \end{array}$$

By (2.3) of [5] we have

$$\pi_{q+1}(SK) = E\pi_q(K) + h_*\pi_{q+1}(K * K).$$

Also  $H_{q+1}(K * K)$  is a torsion group, by the KÜNNETH formula, and hence  $h_\#$  is trivial. Therefore  $\theta h_* = h_\# \theta = 0$  and consequently

$$\theta \pi_{q+1}(SK) = \theta E\pi_q(K) = E\theta \pi_q(K).$$

Hence (2.1) follows at once.

### § 3. The suspension of $B$

Let  $v : S^{n-1} \rightarrow S^q$  be a map such that  $v$  is constant if  $n \leq q$ . Let  $u : S^q \times S^{n-1} \rightarrow S^q$  be a map such that

$$u(x, e) = x, \quad u(e, y) = vy \quad (x \in S^q, y \in S^{n-1}),$$

where  $e$  denotes the basepoint in all cases. Let  $V^n$  denote an  $n$ -element bounded by  $S^{n-1}$  and let  $W^n = V^n - S^{n-1}$ . Let  $B$  be the space obtained from the disjoint union of  $S^q$  and  $S^q \times V^n$  by identifying points of  $S^q \times S^{n-1}$  with their images under  $u$ . The images of  $e \times W^n$  and  $(S^q - e) \times W^n$  under the identification map are open cells which we denote by  $e^n$  and  $e^{n+q}$ , respectively, so that

$$B = S^q \cup e^n \cup e^{n+q}.$$

In this cellular decomposition  $e^n$  is attached by  $\theta \in \pi_{n-1}(S^q)$ , the homo-

topy class of  $v$ , and  $e^{n+a}$  is attached by  $\beta \in \pi_{n+a-1}(S^a \cup e^n)$ , say. It is shown in § 3 of [6] that, with a suitable choice of  $u$ , the total space of any  $q$ -sphere bundle over  $S^n$  can be constructed in this way. In general the construction yields a quasi-fibration of  $(B, S^a)$  over  $S^n$ , as described in § 1 of [5], and it is not difficult to extend the theory of [6] and [7] appropriately.

Now consider the suspended complex

$$SB = S^{a+1} \cup e^{n+1} \cup e^{n+a+1},$$

where  $e^{n+1}$  is attached by  $E\theta \in \pi_n(S^{a+1})$  and  $e^{n+a+1}$  is attached by  $E\beta \in \pi_{n+a}(S^{a+1} \cup e^{n+1})$ . Let  $f: S^a * V^n \rightarrow SB$  be obtained by the HOPF construction from the identification map  $S^a \times V^n \rightarrow B$ . Then  $f$  has degree  $\pm 1$ , and agrees with the map  $g: S^a * S^{n-1} \rightarrow S^{a+1}$  which is obtained by the HOPF construction from  $u$ . Hence  $E\beta = \pm j_*\mu$ , where  $\mu \in \pi_{n+a}(S^{a+1})$  is the class of  $g$  and  $j_*$  denotes the injection. We choose orientations so that

$$E\beta = j_*\mu. \quad (3.1)$$

One application of this relation can be seen in § 4 of [1].

#### § 4. The kernel of $j_*$

Let us recall from [10], [11] and [12] the main facts known about  $\pi_{2n-1}(S^n)$  when  $n$  is even. First we recall FREUDENTHAL's theorem that  $E\pi_{2n-2}(S^{n-1})$  coincides with the kernel of the HOPF invariant. Let  $w_n$  denote the WHITEHEAD product  $[\iota_n, \iota_n]$ , which has HOPF invariant 2.

If  $\alpha \in \pi_{2n-1}(S^n)$  has HOPF invariant  $h$  then

$$\varphi_r\alpha = r\alpha + \frac{1}{2}hr(r-1)w_n, \quad (4.1)$$

where  $\varphi_r$  denotes the endomorphism induced by mapping  $S^n$  to itself with degree  $r$ . By ADAMS' theorem [1],  $\pi_{2n-1}(S^n)$  contains no element of odd HOPF invariant unless  $n = 2, 4$  or  $8$ . If  $n = 2, 4$  or  $8$  then  $w_n = 2\gamma_n + E\beta_n$ , where  $\gamma_n$  denotes the HOPF class and  $\beta_n$  generates the cyclic group  $\pi_{2n-2}(S^{n-1})$ . The order of  $\beta_n$  is 1, 12 or 120 according as  $n = 2, 4$  or  $8$ .

We continue to suppose that  $n$  is even. Let  $Y_m^{n+1}$  denote the complex formed by attaching an  $(n+1)$ -cell to  $S^n$  with a map of degree  $m$ . CHANG [2] has shown that the kernel of the injection

$$j_*: \pi_{2n-1}(S^n) \rightarrow \pi_{2n-1}(Y_m^{n+1})$$

is generated by  $mw_n$  together with the elements of  $\varphi_m\pi_{2n-1}(S^n)$ .

We reduce this to

**Theorem 4.2.** *If  $m$  is odd or if  $n \neq 2, 4, 8$  then*

$$j_*^{-1}(0) = m\pi_{2n-1}(S^n).$$

*If  $m$  is even and  $n = 2, 4$  or  $8$  then*

$$j_*^{-1}(0) = 2m\pi_{2n-1}(S^n) + \frac{m}{2} E\pi_{2n-2}(S^{n-1}).$$

Suppose first that  $\pi_{2n-1}(S^n)$  contains no elements of odd HOPF invariant and therefore is generated by  $w_n$  and suspension elements. Then since  $\varphi_m E = mE$  and  $\varphi_m w_n = m^2 w_n$  we conclude from CHANG's result that  $j_*^{-1}(0)$  is generated by  $mw_n$  together with the elements of  $mE\pi_{2n-2}(S^{n-1})$ . This proves (4.2) unless  $n = 2, 4$  or  $8$ . Now suppose that  $n = 2, 4$  or  $8$ . If  $m$  is odd it follows from (4.1), with  $\alpha = \gamma_n$ , and from CHANG's result that  $j_*^{-1}(0)$  is generated by  $mw_n$  and  $m\gamma_n$ . If  $m$  is even it follows similarly that  $j_*^{-1}(0)$  is generated by  $mw_n$  and  $\frac{1}{2}m(2\gamma_n - w_n)$ , and hence is generated by  $mw_n$  and  $\frac{1}{2}mE\beta_n$ . Since  $\pi_{2n-1}(S^n)$  is generated by  $\gamma_n$  and  $E\beta_n$ , this completes the proof of (4.2).

We regard  $Y_m^{n+1}$  as the suspension of  $Y_m^n$  and consider the commutative diagram shown below, where  $i_*$  is the injection.

$$\begin{array}{ccc} \pi_{2n-2}(S^{n-1}) & \xrightarrow{i_*} & \pi_{2n-2}(Y_m^n) \\ E \downarrow & & \downarrow E \\ \pi_{2n-1}(S^n) & \xrightarrow{j_*} & \pi_{2n-1}(Y_m^{n+1}) \end{array}$$

Notice that  $i_* \varphi_m = 0$ , and that

$$m\pi_{2n-1}(S^n) \cap E\pi_{2n-2}(S^{n-1}) = E\varphi_m\pi_{2n-2}(S^{n-1}).$$

Hence and from (4.2) we deduce

**Corollary 4.3.** *Let  $\xi \in \pi_{2n-2}(S^{n-1})$  be an element such that  $Ei_* \xi = 0$ . Suppose that  $m$  is odd or  $m \equiv 0 \pmod{2n}$  or  $n \equiv 2 \pmod{4}$  or  $n \geq 10$ . Then  $i_* \xi = i_* \eta$ , where  $E\eta = 0$ .*

### § 5. The spherical index

Let  $n$  be even and let  $m \neq 0$ . We take  $q = n - 1$  in the construction of § 3, and suppose that  $v$  is a map of degree  $m$ . Consider the composition

$$\pi_{2n-1}(S^n) \xrightarrow{\Delta} \pi_{2n-2}(S^{n-1}) \xrightarrow{E} \pi_{2n-1}(S^n),$$

where  $\Delta$  denotes the transgression operator associated with  $B$ . We prove

**Theorem 5.1.** *The kernel of  $E\Delta$  contains an element of HOPF invariant 1 if either (i)  $m$  is odd, or (ii)  $n = 2$ , or (iii)  $n = 4$  and  $m \equiv 0 \pmod{8}$ , or (iv)  $n = 8$  and  $m \equiv 0 \pmod{16}$ . In all other cases the kernel contains an element of HOPF invariant 2 but does not contain an element of HOPF invariant 1.*

As in § 3 let  $\mu$  denote the element of HOPF invariant  $m$  in  $\pi_{2n-1}(S^n)$  which is obtained from  $u$  by the HOPF construction. If  $\alpha \in \pi_{2n-1}(S^n)$  has HOPF invariant  $h$  then

$$E\Delta(\alpha) = \varphi_m(\alpha) - hm\mu, \quad (5.2)$$

by (2.5) of [5]. Hence and from (4.1) it follows that  $E\Delta$  annihilates  $2\mu + (1 - m)w_n$ , which has HOPF invariant 2. Suppose that  $h = 1$ . Then  $n = 2, 4$  or  $8$  and we write

$$\alpha = \gamma_n + sE\beta_n, \quad \mu = m\gamma_n + tE\beta_n,$$

where  $s$  and  $t$  are integers. By (4.1) and (5.2) we have

$$E\Delta(\alpha) = m(s - t)E\beta_n + \frac{1}{2}m(m - 1)E\beta_n.$$

Thus we can choose  $s$  so as to make  $E\Delta(\alpha) = 0$  if, and only if,  $\frac{1}{2}m(m - 1)$  is divisible by the greatest common divisor of  $m$  and the order of  $\beta_n$ . Hence (5.1) follows at once.

The spherical index of  $B$  is defined, since  $m \neq 0$ . By (10.1) of [4] we have

$$I(B) = k | m |, \quad (5.3)$$

where  $k$  denotes the index, in the group of integers, of the image of  $\Delta^{-1}(0)$  under the HOPF invariant. We recall from (3.5) of [4] that

$$E: \pi_{2n-2}(S^{n-1}) \rightarrow \pi_{2n-1}(S^n)$$

is a monomorphism if  $n = 2$  or  $n \equiv 0 \pmod{4}$ , and that in any case  $2E^{-1}(0) = 0$ . Therefore some information about  $k$  can be obtained from (5.1), and by substituting this in (5.3) we obtain

**Corollary 5.4.** *If  $n = 2$  or if  $m$  is odd then  $I(B) = |m|$ . If  $n = 4$  or  $8$  and if  $m$  is divisible by  $2n$  then  $I(B) = |m|$ . If  $n = 4$  or  $8$  and if  $m$  is even but not divisible by  $2n$  then  $I(B) = 2|m|$ . If  $n \equiv 0 \pmod{4}$  and  $n \geq 12$  then  $I(B) = 2|m|$ . If  $n \equiv 2 \pmod{4}$  and  $n \geq 6$  then  $I(B) = 2|m|$  or  $4|m|$ .*

Every case is covered in (5.4). By the methods used in § 7 of [4] it is possible to decide between the alternatives when  $n \equiv 2 \pmod{4}$  and  $n \geq 6$ , but this need not detain us here.

We further deduce

**Corollary 5.5.** *Suppose that  $I(B) = I(SB)$ . Then either (i)  $|m| \leq 2$  and  $n = 2$ , or (ii)  $m$  is odd and  $n = 4$  or  $8$ .*

We can use (4.2) to determine  $I(SB)$ , since it follows from (3.1) that  $I(SB)$  is equal to the least positive multiple of  $\mu$  which is contained in the kernel of  $j_*$ . When  $n = 6$  or  $n \geq 10$  we find that  $I(SB)$  is a divisor of  $m$ . When  $n = 2$  we find that  $I(SB) = 1$  or  $2$  according as  $m$  is odd or even. When  $n = 4$  or  $8$  we find that  $I(SB)$  is not divisible by the greater of the numbers  $4$  and  $(m, 2n)$ . Hence it follows from (5.4) that  $I(SB) \neq I(B)$  unless  $m$  and  $n$  are as in (5.5).

## § 6. Proof of the main theorem

Let  $B$  be constructed as in § 3, with  $q = n - 1$  and  $v$  a map of degree  $m$ . We prove

**Lemma 6.1.** *If  $B$  is an  $H$ -space then either (i)  $n = 2$  and  $m = \pm 1, \pm 2$  or (ii)  $n = 4$  and  $m = \pm 1$ .*

If  $m = 0$  then the cohomology of  $B$  is an exterior algebra on generators of dimension  $n$  and  $n - 1$ . Since one of these numbers is even we have an immediate contradiction with the HOPF-LERAY theorem. Therefore  $m \neq 0$ , and consequently  $n$  is even. If  $n = 2$  then (6.1) follows immediately from (2.1) and (5.5). Let  $n \geq 4$ . Suppose, to obtain a contradiction, that  $m$  is divisible by some odd prime  $p$ . Then  $H^n(B, Z_p)$  contains a non-zero element  $x$ , say, and since  $x$  is primitive, for dimensional reasons, it follows from elementary HOPF algebra theory that  $x^2 \neq 0$ . But  $H^{2n}(B, Z_p) = 0$ , and so we have a contradiction. Since  $m$  is odd, by (2.1) and (5.5), we conclude that  $m = \pm 1$ . Therefore  $B$  is a homotopy  $(2n - 1)$ -sphere and moreover it follows from ADAMS' theorem [1] that  $n = 4$ . This completes the proof of (6.1).

Suppose further that  $B$  is an  $(n - 1)$ -sphere bundle over  $S^n$ . If  $n = 2$  then the classification of fibre bundle theory shows that  $B$  is homeomorphic to  $S^3$  if  $m = \pm 1$ , to real projective 3-space if  $m = \pm 2$ . If  $n = 4$  and  $m = \pm 1$  then MILNOR [8] has shown that  $B$  is homeomorphic to  $S^7$ . This completes the proof of (1.1).

### § 7. Homotopy classification

Let  $B$  and  $B'$  be  $(n - 1)$ -sphere bundles over  $S^n$ , where  $n$  is even, such that

$$H_{n-1}(B) \approx Z_m \approx H_{n-1}(B') \quad (m \geq 1).$$

Then  $B$  and  $B'$  can be constructed as described in § 3 by attaching  $(2n - 1)$ -cells to  $Y_m^n$ . Hence  $B$  and  $B'$  have the same homotopy type if, and only if, there exists a cellular homotopy equivalence  $g : Y_m^n \rightarrow Y_m^n$  such that<sup>4)</sup>

$$g_*\beta = \pm \beta', \quad (7.1)$$

where  $\beta, \beta' \in \pi_{2n-2}(Y_m^n)$  denote the attaching classes and  $g_*$  denotes the automorphism induced by  $g$ . Suppose that (7.1) is satisfied where  $g$  is a map of degree  $r$ . Then  $f_*E\beta = \pm E\beta'$ , where  $f_*$  is the automorphism of  $\pi_{2n-1}(Y_m^{n+1})$  induced by the suspension of  $g$ , and hence

$$\varphi_r\mu \equiv \pm \mu' \pmod{j_*^{-1}(0)} \quad (7.2)$$

by (3.1), where  $\mu' \in \pi_{2n-1}(S^n)$  corresponds to  $B'$  as  $\mu$  does to  $B$ . By (5.1) of [7],

$$k_*\beta = w'_n = k_*\beta', \quad (7.3)$$

where  $w'_n \in \pi_{2n-2}(Y_m^n, S^{n-1})$  denotes the relative WHITEHEAD product of  $\iota_{n-1}$  with a generator of  $\pi_n(Y_m^n, S^{n-1})$  and  $k_*$  denotes the injection. Also it follows from (10.1) of [4] that the order of  $w'_n$  is either  $m$  or  $2m$  according as  $\pi_{2n-1}(S^n)$  does or does not contain an element of HOPF invariant 1. The automorphism of  $\pi_{2n-2}(Y_m^n, S^{n-1})$  induced by  $g$  transforms  $w'_n$  into  $r^2w'_n$ . Hence and from (7.1) it follows that

$$\left. \begin{aligned} r^2 &\equiv \pm 1 \pmod{m} \text{ if } n = 2, 4 \text{ or } 8; \\ &\equiv \pm 1 \pmod{2m} \text{ otherwise.} \end{aligned} \right\} \quad (7.4)$$

---

<sup>4)</sup> The signs  $\pm$  in this section are linked together, i.e. they stand for plus in every case or for minus in every case.

Thus (7.2) and (7.4) are satisfied for some value of  $r$  if  $B$  and  $B'$  have the same homotopy type.

Conversely, let  $r$  be an integer such that (7.2) and (7.4) are satisfied. Let  $g : Y_m^n \rightarrow Y_m^n$  be a cellular map of degree  $r$ . Then

$$Eg_*\beta = \pm E\beta', \quad k_*g_*\beta = \pm k_*\beta',$$

by (3.1) and (7.3). Hence  $\pi_{2n-2}(S^{n-1})$  contains an element  $\xi$  such that  $Ei_*\xi=0$  and  $g_*\beta = \pm \beta' + i_*\xi$ . Suppose that either  $m$  is odd or  $m \equiv 0 \pmod{2n}$  or  $n \equiv 2 \pmod{4}$  or  $n \geq 10$ . Then it follows from (4.3) and from (5.3) of [13] that  $\xi \equiv \eta \pmod{i_*^{-1}(0)}$ , where  $\eta$  denotes the WHITEHEAD product of  $\iota_{n-1}$  with some element of  $\pi_n(S^{n-1})$ . By (3.5) of [7] there exists a cellular map  $g' : Y_m^n \rightarrow Y_m^n$ , also of degree  $r$ , such that

$$g_*\beta - g'_*\beta = i_*\eta = i_*\xi.$$

Then  $g'$ , like  $g$ , is a homotopy equivalence, because  $r$  is prime to  $m$ . Since  $g'_*\beta = \pm \beta'$  this proves that  $B$  and  $B'$  have the same homotopy type.

As an application we deduce

**Theorem 7.5.** *Let  $n = 6$  or let  $n \geq 10$ . Then  $B$  and  $B'$  have the same homotopy type if, and only if, there exists an integer  $r$ , where  $r^2 \equiv 1 \pmod{2m}$ , such that*

$$rE\mu \equiv E\mu' \pmod{m\pi_{2n}(S^{n+1})}.$$

Suppose that  $B$  and  $B'$  have the same homotopy type. Then (7.2) and (7.4) are satisfied, as we have seen. Since  $m$  is even, by ADAMS' theorem, and since  $-1$  is not a quadratic residue of 4 it follows from (7.4) that  $r^2 \equiv 1 \pmod{2m}$ . By (7.2), therefore,  $\varphi_r\mu \equiv \mu' \pmod{j_*^{-1}(0)}$ , and since  $j_*^{-1}(0) = m\pi_{2n-1}(S^n)$ , by (4.2), it follows by suspension that

$$rE\mu \equiv E\mu' \pmod{m\pi_{2n}(S^{n+1})}.$$

Conversely, suppose that  $r^2 \equiv 1 \pmod{2m}$  and  $rE\mu \equiv E\mu' \pmod{m\pi_{2n}(S^{n+1})}$ . Certainly (7.4) is satisfied. The suspension

$$E : \pi_{2n-1}(S^n) \rightarrow \pi_{2n}(S^{n+1})$$

is an epimorphism, and its kernel is generated by  $w_n$ . Hence it follows that

$$r\mu - \mu' \equiv \frac{1}{2}m(r-1)w_n \pmod{m\pi_{2n-1}(S^n)},$$

and therefore

$$\varphi_r\mu - \mu' \equiv \frac{1}{2}m(r^2-1)w_n \equiv 0 \pmod{m\pi_{2n-1}(S^n)},$$

since  $r^2 - 1$  is even. Hence and from (4.2), (7.2) follows at once. Therefore  $B$  and  $B'$  have the same homotopy type, which completes the proof of (7.5).

It is clear that for some values of  $m$  a classification theorem is obtainable by these methods in case  $n = 4$  or  $8$ , but more elaborate techniques are necessary before all values of  $m$  can be treated. The appropriate theory has been developed by Dr. M. G. BARRATT (unpublished). However, except for these cases (7.5) fills the gap in the classification of sphere-bundles over spheres by homotopy type of the total space as given in [6] and [7].

### § 8. Appendix

Let  $E, F, B$  be spaces, with  $F \subset E$ , and let  $p: E \rightarrow B$  be a map which is constant on  $F$ . Suppose that  $p$  induces an isomorphism

$$p_*: \pi_r(E, F) \rightarrow \pi_r(B) \quad (r = 0, 1, \dots)$$

and that there exists a map  $f: B \rightarrow E$  such that  $pf \simeq 1$ . Then we obtain the direct sum decomposition

$$i_* \oplus f_*: \pi_r(F) \oplus \pi_r(B) \approx \pi_r(E),$$

where  $f_*$  is induced by  $f$  and  $i_*$  is the injection. Hence if  $E$  is an  $H$ -space it follows that

$$h_*: \pi_r(F \times B) \approx \pi_r(E),$$

where  $h: F \times B \rightarrow E$  is the map defined by

$$h(x, y) = x \cdot f(y) \quad (x \in F, y \in B).$$

Suppose in addition that  $E$  and  $F \times B$  have the homotopy type of  $CW$ -complexes. Then  $h$  is a homotopy equivalence, by Theorem 1 of [14]. Hence  $F \times B$  is an  $H$ -space, and hence  $F$  and  $B$  are  $H$ -spaces.

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