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Autor: Garsia, Adriano M.

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An Imbedding of Closed RIEMANN Surfaces in EUCLIDean Space[1])

By ADRIANO M. GARSIA, Minneapolis, Minnesota (USA)

Introduction

Let Σ be a differentiable surface and let

$$ds^2 = E dx^2 + 2F dx dy + G dy^2$$

be a metric on Σ. Such a metric can be used to introduce a conformal structure on Σ. This is a standard construction. The uniformizers are taken to be the local homeomorphic solutions of the BELTRAMI equation

$$w_{\bar z} = \mu w_z \qquad (z = x + iy) \tag{1.1}$$

where

$$\mu(z) = \frac{\frac{1}{2}(E - G) + iF}{\frac{1}{2}(E + G) + \sqrt{EG - F^2}} . \tag{1.2}$$

It is well known (see for instance [4]), that the conditions

A) $\sup |\mu| < 1$

B) $E, F, G \in C^{n+\alpha}$

are sufficient to guarantee the existence of $C^{n+\alpha+1}$ local homeomorphic solutions of (1.1). Using (1.2) the metric can be written in the form

$$ds^2 = \wedge \, | dz + \mu d\bar z |^2$$

with a suitable $\wedge$, thus any solution of (1.1) reduces the metric to the simple form

$$ds^2 = \wedge \, | dw |^2 .$$

In the following the RIEMANN surface defined by such a construction based on a surface Σ and a metric ds^2 will be denoted by $\Sigma(ds^2)$.

In particular every smooth surface of EUCLIDean space can be made into a RIEMANN surface in a natural way, that is using the metric induced by the surrounding EUCLIDean space to introduce the conformal structure. This ap-

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proach is quite old and was already adopted by BELTRAMI and KLEIN to introduce the theory of functions on arbitrary surfaces.

The question has been open for some time whether or not every conformal type of RIEMANN surface could be obtained by this procedure from the surfaces (in the classical sense) of 3-dimensional space.

If one drops the condition that the surfaces should be "classical" (i.e. at least C^2), then this question (in the compact case) can be given an affirmative answer. This assertion follows from the results on C^1 imbeddings obtained by J. NASH [13] and extended by N. KUIPER [11].

If one drops the dimension restriction on the surrounding space, then the answer is affirmative, even in the non compact case. As a matter of fact, using the results of J. NASH in [14] it can be shown that the C^∞ surfaces already exhaust all conformal types. However, the dimensions required are high: 17 for the compact surfaces, 51 for the non compact ones.

In the genus one case some incomplete results using C^∞ canal surfaces were recently obtained by myself in [7]. Similar results using algebraic canal surfaces were again obtained in [8]. Finally in [9], using a different approach, E. RODEMICH and myself were able to show that there exists in EUCLIDean space a C^∞ model for every conformal type of RIEMANN surface of genus one. This result can be used (as I will show somewhere else) to construct in 3-space a real algebraic model for every RIEMANN surface of genus one.

Some interesting results on C^∞ imbeddings in the higher genus case have been obtained by T. KLOTZ in [10]. This author is almost sucessful in proving that the set of RIEMANN surfaces of a given genus $g \geqslant 2$ which can be imbedded in EUCLIDean space is open[2]) in the TEICHMÜLLER topology. Perhaps we should point out that from some of the results of the present paper one obtains the arguments that are needed to complete her proof.

The main result of the present paper is a proof that there exists in EUCLIDean space a conformally equivalent C^∞ model for every compact RIEMANN surface of genus $g \geqslant 2$. The methods that we have followed are essentially an extension of those in [9]. However, here certain devices introduced by J. NASH in [13], together with some results of L. AHLFORS [2] and L. BERS [3] on spaces of RIEMANN surfaces are quite crucial.

We shall leave for a forthcoming publication the question of real algebraic imbeddings in the higher genus case.

As for the non-compact case, there are some rather interesting results (see [12]), but the imbedding question is still open. However, in this case the answer is quite likely to be negative.

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[2]) Using the results of KUIPER it could be shown that it is dense.

1. Preliminaries on quasiconformal mappings

1.1. Here and in the following we shall make repeated use of some results from the theory of quasiconformal mappings. For an introduction to these methods the reader may refer to [2], [3], here we shall only briefly describe the material that will be needed.

We recall that an object of TEICHMÜLLER space T_g for $g \geqslant 1$ consists of a topologically characterized RIEMANN surface. A way to describe this space for a given g is to choose a fixed RIEMANN surface of genus $g \Sigma_0$ and then consider the set of all couples (Σ, τ) consisting of a RIEMANN surface Σ of genus g and a homeomorphism τ of Σ_0 onto Σ. Then identify any two couples (Σ, τ) and (Σ', τ') such that Σ can be mapped conformally onto Σ' in the homotopy class $\tau' \tau^{-1}$.

TEICHMÜLLER theorem asserts (see for instance [2]) that given two surfaces (Σ, τ) and (Σ', τ') then in the same homotopy class of $\tau' \tau^{-1}$ there exists a unique quasiconformal map $\hat{\tau}$ which enjoys of the following properties: there exists on Σ a unique quadratic differential Φdz^2 defined up to a positive factor and a similarly determined quadratic differential $\Phi' dz'^2$ on Σ' such that if the map $\hat{\tau}$ is expressed in the form $z' = z'(z)$ then whenever $\Phi \neq 0$

$$\sqrt{\Phi'}\, dz' = \sqrt{\Phi}\, dz + k \sqrt{\overline{\Phi}}\, d\bar{z}$$

where $0 \leqslant k < 1$ is constant throughout the surface. Furthermore the zeros of Φ and Φ' correspond together with their multiplicities.

TEICHMÜLLER defines

$$\log \frac{1+k}{1-k}$$

to be the distance between (Σ, τ) and (Σ', τ'). With this T_g becomes a metric space which can be shown to be homeomorphic to the $(6g - 6)$ - dimensional real EUCLIDean space.

1.2. To carry out the construction of a RIEMANN surface sketched in the introduction one needs less than a metric ds^2. As a matter of fact, it is sufficient to have a quadratic form which is determined up to a factor of proportionality, i.e. a conformal metric. Furthermore, some of the smoothness conditions can also be slacked. For instance one can use a quadratic differential Φdz^2 of a RIEMANN surface Σ of genus g and use the singular metric

$$ds^2 = | \sqrt{\Phi}\, dz + k \sqrt{\overline{\Phi}}\, d\bar{z} |^2 \quad \text{for some} \quad 0 \leqslant k < 1$$

to define a new RIEMANN surface $\Sigma_{\Phi, k}$. Perhaps we should point out that such a metric defines $\Sigma_{\Phi, k}$ to be a flat manifold with some mild conical singularities of negative generalized integral curvature (which is always a multiple of π). In view of TEICHMÜLLER theorem every conformal type of RIEMANN surface of genus g can be obtained in this way. And, if Φ is suitably normalized, each topologically characterized RIEMANN surface is obtained only once.

This fact can be used to introduce some very useful coordinate systems in T_g. This is done as follows. Let Σ_0 be a RIEMANN surface of genus $g \geqslant 2$. Let $\Phi_1, \Phi_2, \ldots, \Phi_N$ $N = 3g-3$ be a basis for the quadratic differentials of Σ_0. We can represent each point $\underline{\xi}$ of the open unit ball B in $2N$-dimensional real EUCLIDean space in the form

$$\underline{\xi} = (\xi_1, \xi_2, \ldots, \xi_N)$$

where the ξ_i's are complex numbers satisfying the condition

$$|\underline{\xi}|^2 = \sum_{i=1}^{N} |\xi_i|^2 < 1 . \tag{1.21}$$

For each $\underline{\xi} \in B$ we set

$$\Phi_{\xi} = \sum_{i=1}^{N} \frac{\xi_i}{|\underline{\xi}|} \Phi_i \tag{1.22}$$

and

$$ds_{\xi}^2 = |\sqrt{\Phi_{\xi}}dz + |\underline{\xi}| \sqrt{\overline{\Phi}_{\xi}}d\bar{z}|^2 = |\Phi_{\xi}| \left| dz + |\underline{\xi}| \frac{\overline{\Phi}_{\xi}}{|\Phi_{\xi}|} d\bar{z} \right|^2 . \tag{1.23}$$

It follows from TEICHMÜLLER's results that the mapping of B into T_g which sends $\underline{\xi}$ into $\Sigma_0(ds_{\xi}^2)$ is a homeomorphism.

1.3. Let Σ and Σ' be two closed RIEMANN surfaces of genus $g \geqslant 2$ and let τ be a homeomorphism of Σ onto Σ'. For our purposes we can restrict our considerations to mappings which possess everywhere integrable derivatives that may fail to exist only at a finite number of points. Let then P_0 be a point which is not exceptional for τ and let $P_0' = \tau P_0$. Let D be a uniformizing neighborhood around P_0 and set $D' = \tau D$. We can introduce local uniformizers $w = u + iv$ and $w' = u' + iv'$ in D and D' and express τ in the form

$$w' = w'(w) .$$

The element of area in D' can be written in the form

$$du' \wedge dv' = \frac{1}{-2i} dw' \wedge d\overline{w}' = (|w'_w|^2 - |w'_{\bar{w}}|^2) \frac{1}{-2i} dw \wedge d\bar{w} .$$

Thus the JACOBian of $w'(w)$ is given by the expression

$$|w'_w|^2 - |w'_{\bar{w}}|^2 . \tag{1.31}$$

If the mapping is sense-preserving (in the following this will be tacitly assumed) we shall then have

$$| w'_{\bar{w}} | < | w'_w | . \tag{1.32}$$

To the mapping τ and the coordinates w and w' we can associate the expression

$$\mu(w) = \frac{w'_{\bar{w}}}{w'_w} . \tag{1.33}$$

Observe that $\mu(w)$ is invariant under change of the coordinate w' while under the change $w \to w_0$ we have

$$\mu_0(w_0) = \mu(w) \frac{d\bar{w}}{d\bar{w}_0} / \frac{dw}{dw_0} . \tag{1.34}$$

Consequently the absolute value $| \mu(w) |$ is invariant under the change of coordinate systems, in other words it is a function defined everywhere in Σ except possibly at the exceptional points of τ.

We recall that τ is called quasiconformal if

$$\mu_0 = \sup | \mu(w) | < 1 .$$

The function $| \mu(w) |$ is called the "excentricity" of τ at w; it is a measure of distortion from conformality. In fact, $\mu_0 = 0$ implies that τ is conformal and therefore everywhere regular. The dilatation of τ is defined at every regular point by

$$K(w) = \sup \frac{| dw' |}{| dw |} / \inf \frac{| dw' |}{| dw |} . \tag{1.35}$$

Clearly we have

$$K(w) = \frac{| w'_w | + | w'_{\bar{w}} |}{| w'_w | - | w'_{\bar{w}} |} = \frac{1 + | \mu(w) |}{1 - | \mu(w) |} \tag{1.36}$$

and therefore $K(w)$ is also a function on Σ. The map τ is quasiconformal if and only if

$$K_0 = \sup K(w) < \infty .$$

We observe that in terms of $| \mu(w) |$ we can write the element of area in D' in the form

$$dw' \wedge d\bar{w}' = (1 - | \mu |^2) | w'_w |^2 dw \wedge d\bar{w} . \tag{1.37}$$

Suppose now that we have a surface Σ_0 and two metrics ds^2 and ds'^2. It is easy to see that the identity map on Σ induces a quasiconformal map between $\Sigma_0(ds^2)$ and $\Sigma_0(ds'^2)$. We need an expression for the dilatation of this map. This is easily obtained. If w and w' are local coordinates on $\Sigma_0(ds^2)$

and $\Sigma_0(ds'^2)$ we shall have by definition $ds^2 = \lambda \mid dw \mid^2$ and $ds'^2 = \lambda' \mid dw' \mid^2$; thus we obtain

$$K^2(w) = \frac{\sup \frac{ds'^2}{ds^2}}{\inf \frac{ds'^2}{ds^2}} .$$

1.4. It will be useful in the following to have an estimate for the TEICHMÜLLER distance between two RIEMANN surfaces $\Sigma_0(ds^2)$ and $\Sigma_0(ds'^2)$ from the knowledge of the dilatation

$$K(w) = \sqrt{\sup \frac{ds'^2}{ds^2} \Big/ \inf \frac{ds'^2}{ds^2}} . \tag{1.41}$$

To this end we have the following

Deformation lemma. *If* Φdw^2 *is the quadratic differential associated with the extremal quasiconformal map* τ_0 *of* $\Sigma_0(ds^2)$ *onto* $\Sigma_0(ds'^2)$ *which is homotopic to the identity in* Σ_0, K_0 *is the dilatation of* τ_0 *and* $K(z)$ *is given by* (1.41), *then the following inequality holds:*

$$K_0 \leqslant \frac{\iint_{\Sigma_0} K(z)\, dA_\Phi}{\iint_{\Sigma_0} dA_\Phi} , \tag{1.42}$$

where $dA_\Phi = i/2 \mid \Phi(w) \mid dw \wedge d\bar{w}$ [3]).

The proof of this lemma can be obtained, after some routine modifications, from the arguments usually given in the proof of the uniqueness part of TEICHMÜLLER's theorem (see for instance [3], pages 111–112). We shall therefore omit it.

For our purposes it will be necessary to free the right hand side of (1.42) from the presence of dA_Φ so as to make the inequality independent of the surface $\Sigma_0(ds^2)$. Such a result can of course be obtained only in a compact portion of TEICHMÜLLER space. However, before obtaining this modification of the lemma we need a few preliminary considerations.

1.5. To prove the result announced in the introduction, we shall start with a closed C^∞ surface Σ_0 of EUCLIDean space of a given genus $g \geqslant 2$ but otherwise quite arbitrary, then we shall construct a family of C^∞, arbitrarily small, deformations of Σ_0 and show that among these there is a conformally equivalent model for every RIEMANN surface of the same genus.

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[3]) In other words dA_Φ denotes the element of area on Σ_0 defined by the quadratic differential Φdw^2. z and w denote local coordinates on Σ_0 and $\Sigma_0(ds^2)$ respectively.

It will be convenient then to carry out all our reasoning on the same fixed surface of three-space, which here and in the following will be referred to as Σ_0. We assume that Σ_0 is C^∞ closed and of genus $g \geqslant 2$. By Σ_0 we shall also denote the RIEMANN surface corresponding to it. We shall introduce once and for all a basis $\Phi_1, \Phi_2, \ldots, \Phi_N$ for the quadratic differentials of Σ_0. And, in the manner described in section (1.2), we shall define in T_g the coordinate system associated with Σ_0 and this choice of $\Phi_1, \Phi_2, \ldots, \Phi_N$.

We shall indicate by U the open unit disk of the z-plane, by U/G, where G is a FUCHSian group, a POINCARÉ model conformally equivalent to Σ_0. We let $a_1, a_2, \ldots, a_g; b_1, b_2, \ldots, b_g$ be a canonical basis for the cycles of Σ_0 consisting of C^∞ curves cutting Σ_0 into a simply connected surface. We denote by λ_i, μ_i the generators of G corresponding to a_i, b_i. We let π denote a fundamental region of G bounded by $4g$ curves $\tilde{a}_i, \tilde{b}_i, \lambda_i \tilde{a}_i^{-1}, \mu_i \tilde{b}_i^{-1}$. We let $\underline{X}(z)$ be a conformal map of U onto Σ_0 and suppose that the vector $\underline{X}(z)$ describes Σ_0 as z describes π in such a way that it maps each $\tilde{a}_i$ onto a_i ($\lambda_i \tilde{a}_i^{-1}$ onto a_i^{-1}) and each $\tilde{b}_i$ onto b_i ($\mu_i \tilde{b}_i^{-1}$ onto b_i^{-1}).

The function $\underline{X}(z)$ will allow us to use a single parametrization for the surface Σ_0. This will be permissible if we restrict ourselves never to cross the curves a_i and b_i. As a matter of fact all our deformations of Σ_0 will be carried out well within π.

We recall (see [3]) that a quantity μ which is bounded away from one and transforms like (1.34) is called a BELTRAMI differential. Every metric ds^2 of Σ_0 can be written in the form

$$ds^2 = \wedge \, | dz + \mu d\bar{z} |^2 \tag{1.51}$$

where $\mu \dfrac{d\bar{z}}{dz}$ is a BELTRAMI differential. Written in terms of z a BELTRAMI differential in Σ_0 is generated by and gives origin to a function $\mu(z)$ in U which under each $\tau \in G$ obeys the transformation law

$$\mu(\tau z) = \mu(z) \frac{d\tau z}{dz} \Big/ \frac{d\overline{\tau z}}{d\bar{z}} \,. \tag{1.52}$$

To such a function $\mu(z)$ we can associate in U the BELTRAMI equation

$$w_{\bar{z}} = \mu w_z \,. \tag{1.53}$$

It is known[4]) that there exist solutions of (1.53) which are homeomorphisms of U onto itself. We shall denote by $w^\mu(z)$ the homeomorphic solution of (1.53) for which

[4]) Here and in the following we shall assume as known certain results concerning solutions of BELTRAMI equations whose proof can be found in [3] and [6].

$$\begin{array}{l} \| \; w^{\mu}(0) = 0 \\ \| \; w^{\mu}(1) = 1 \,. \end{array} \tag{1.54}$$

It can be shown that $w^{\mu}(z)$ gives the POINCARÉ uniformization of the RIEMANN surface $\Sigma_0(|\,dz + \mu d\bar{z}\,|^2)$.

1.6. We are now in a position to present the announced modification of the deformation lemma in a form which will be suitable for our applications.

Let ds^2 and ds'^2 be two metrics on Σ_0 and $K(z)$ be given by (1.41). Let $\underline{\xi}$ and $\underline{\xi}'$ denote the points of B corresponding to the surfaces $\Sigma_0(ds^2)$ and $\Sigma_0(ds'^2)$. Let $\mathfrak{F}$ denote some family of measurable functions defined in U which is compact with respect to a. e. convergence. We can then prove the following

Continuity lemma. *For every* $\mathfrak{F}$, $\varrho < 1$, $1 \leqslant K_0$ *and* $\delta > 0$ *it is possible to find two constants* $\varepsilon_a(\mathfrak{F}, \varrho, K_0, \delta)$ *and* $\varepsilon_k(\mathfrak{F}, \varrho, \delta)$ *such that if*

A) $ds^2 = \Lambda\,|\,dz + \mu d\bar{z}\,|^2$, $\mu \in \mathfrak{F}$ and $|\,\mu\,| \leqslant \varrho$.

B) There exists a subregion σ of π such that area $\sigma < \varepsilon_a$, $K(z) < K_0$ a. e. in σ, $K(z) < 1 + \varepsilon_k$ a. e. in $\pi - \sigma$, then

$$|\,\underline{\xi}' - \underline{\xi}\,| < \delta\,. \tag{1.61}$$

Proof. Since the TEICHMÜLLER distance and the EUCLIDean distance in B are continuous functions of each other and the lemma is to be proved for a compact $(|\,\xi\,| \leqslant \varrho)$ subset of T_g, we shall replace (1.61) by the inequality

$$K(ds^2, ds'^2) \leqslant 1 + \delta \tag{1.62}$$

where $K(ds^2, ds'^2)$ is to represent the dilatation of the extremal quasiconformal map of $\Sigma_0(ds^2)$ onto $\Sigma_0(ds'^2)$ which is homotopic to the identity in Σ_0.

If we apply the deformation lemma together with B), we obtain

$$K(ds^2, ds'^2) \leqslant \frac{\iint\limits_{\pi-\sigma}(1+\varepsilon_k)\,dA_\Phi + \iint\limits_{\sigma} K_0\,dA_\Phi}{\iint\limits_{\pi} dA_\Phi} \leqslant 1 + \varepsilon_k + K_0 \frac{\iint\limits_{\sigma} dA_\Phi}{\iint\limits_{\pi} dA_\Phi}\,.$$

Thus the lemma will be proved if we can show that the ratio

$$\frac{\iint\limits_{\sigma} dA_\Phi}{\iint\limits_{\pi} dA_\Phi} = \frac{\iint\limits_{\sigma} |\,\Phi(w)\,|\,dw \wedge d\bar{w}}{\iint\limits_{\pi} |\,\Phi(w)\,|\,dw \wedge d\bar{w}} \tag{1.63}$$

can be made arbitrarily small independently of ds^2 as long as $\mu \in \mathfrak{F}$, $|\,\mu\,| \leqslant \varrho$ and the area of σ is sufficiently small.

We shall proceed by contradiction. Suppose that we could find a constant $\theta > 0$, a sequence of surfaces $\Sigma_0(ds_n^2)$ with $ds_n^2 = \Lambda_n \,|\, dz + \mu_n d\bar{z} \,|^2$ and $|\, \mu_n \,| \leqslant \varrho$, $\mu_n \in \mathfrak{F}$, a quadratic differential Φ_n on each $\Sigma_0(ds_n^2)$ and a sequence σ_n of subregions of π such that

1) $\iint\limits_{\pi} |\, \Phi_n \,| \dfrac{dw^n \wedge d\overline{w^n}}{-2i} = 1$ (we have set $w^{\mu_n} = w^n$)

2) $\iint\limits_{\sigma_n} |\, \Phi_n \,| \dfrac{dw^n \wedge d\overline{w^n}}{-2i} \geqslant \theta$

3) area $\sigma_n \to 0$.

Because of our assumptions on $\mathfrak{F}$, we can suppose that $\mu_n \to \mu$ a. e. in U. Since each μ_n satisfies (1.52), this will be the case also for μ. Thus μ defines a BELTRAMI differential on Σ_0, and we can then speak of the RIEMANN surface $\Sigma_0(|\, dz + \mu d\bar{z} \,|^2)$. If we let $w = w^\mu$, in view of the normalization (1.54), the following will be true. The sequence of maps $w^n(z)$ will converge towards $w(z)$ uniformly in $\bar{U}$. We shall have also (if we select a proper subsequence)

$$w_z^n \to w_z \quad \text{(weakly in } L_2(U)) . \tag{1.64}$$

Let G_n be the FUCHSian group corresponding to $\Sigma_0(ds_n^2)$ and w_n and let π_n denote the fundamental region of G_n image of π. Since the convergence of $w_n \to w$ is uniform, we can find[5]) a number $r > 0$ and an integer n_0 such that for $n \geqslant n_0$ and $w_0 \in \pi_n$ the inequality

$$|\, w - w_0 \,| \leqslant r$$

implies that w is in U and is not equivalent to w_0 under G_n.

For such a choice of r and n_0 if $w_0 \in \pi_n$ and $n \geqslant n_0$, we shall have

$$\Phi_n(w_0) = \frac{1}{\pi r^2} \iint\limits_{|w - w_0| \leqslant r} \Phi_n(w) \frac{dw \wedge d\bar{w}}{-2i} .$$

Thus, in view of 1) we obtain

$$|\, \Phi_n(w_0) \,| \leqslant \frac{1}{\pi r^2} \iint\limits_{\pi n} |\, \Phi_n(w) \,| \frac{dw \wedge d\bar{w}}{-2i} = \frac{1}{\pi r^2}$$

and 2) becomes

$$\iint\limits_{\sigma_n} \frac{dw_n \wedge d\bar{w}_n}{-2i} \geqslant \theta \pi r^2. \tag{1.65}$$

Now, it follows from (1.64) and the definition of w_n that the convergence of

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[5]) See [1], section 8.

$w_z^n \to w_z$ is also strong in $L_2(U)$. In fact, first of all we observe that the integrals

$$\iint_\pi |w_z^n|^2 \frac{dz \wedge d\bar{z}}{-2i} \leqslant \frac{1}{1-\varrho^2} \iint_U \{|w_z^n|^2 - |w_{\bar{z}}^n|^2\} \frac{dz \wedge d\bar{z}}{-2i} = \frac{\pi}{1-\varrho^2}$$

are uniformly bounded. This, together with (1.64) and the fact that $\mu_n \to \mu$ a. e. in U and $|\mu_n| \leqslant \varrho$, implies that

$$|\mu_n|^2 w_z^n \to |\mu|^2 w_z \quad \text{(weakly in } L_2(U)) . \tag{1.66}$$

We also have

$$\iint_U |w_z^n - w_z|^2 dx \wedge dy \leqslant \frac{1}{1-\varrho^2} \iint_U (1 - |\mu_n|^2) |w_z^n - w_z|^2 dx \wedge dy .$$

However,

$$\iint_U (1 - |\mu_n|^2) |w_z^n - w_z|^2 dx \wedge dy = \iint_U \frac{dw^n \wedge d\overline{w^n}}{-2i} -$$
$$- 2 \iint_U (1 - |\mu_n|^2) \operatorname{Re} w_z^n \overline{w}_z dx \wedge dy + \iint_U (1 - |\mu_n|^2) |w_z|^2 dx \wedge dy .$$

Thus in view of (1.64), (1.66) and the properties of w^n, w and μ_n

$$\lim_{n \to \infty} \iint_U |w_z^n - w_z|^2 dx \wedge dy \leqslant \frac{1}{1-\varrho^2} \{2\pi - 2 \iint_U (1 - |\mu|^2) |w_z|^2 dx \wedge dy\} = 0 . \tag{1.67}$$

Observe then that since $|dw^n \wedge d\overline{w^n}| \leqslant |w_z^n|^2 |dz \wedge d\bar{z}|$ and $|w_z^n|^2 \leqslant \leqslant 2 |w_z^n - w_z|^2 + 2 |w_z|^2$, we have

$$\iint_{\sigma_n} \frac{dw^n \wedge d\overline{w^n}}{-2i} \leqslant 2 \iint_U |w_z^n - w_z|^2 dx \wedge dy + 2 \iint_{\sigma_n} |w_z|^2 dx \wedge dy .$$

But this inequality together with (1.67), 3), and (1.65) leads to a contradiction. Thus the lemma is established.

2. Deformations of an imbedded surface

2.1. It will be convenient in the following that we should introduce in the region π a family of open sets π_n and C^∞ functions $\varphi_\varepsilon(z)$ defined in U and such that the following conditions are satisfied: for each $\varepsilon \leqslant \varepsilon_0$

1) $\bar{\pi}_\varepsilon$ is contained in the interior of π,
2) if $\varepsilon_1 > \varepsilon_2$, then $\pi_{\varepsilon_2} \supset \bar{\pi}_{\varepsilon_1}$,
3) $\varphi_\varepsilon(z) \equiv 1$ for $z \in \pi_\varepsilon$, $\varphi_\varepsilon(z) \equiv 0$ for $z \in \pi - \pi_{\varepsilon/2}$,

4) Both the area of $\pi - \pi_\varepsilon$ and the distance of each point of the boundary of π_ε from the boundary of π are less than ε.

The main tool used here to construct the deformations of an imbedded surface is a saw-tooth shaped C^∞ function. We shall proceed with its definition.

Let us first construct a function $\gamma(x)$ which is C^∞ in [0,2], symmetrical about $x = 1$, and such that

1) $\gamma(x) = 1 - |1 - x|$ for $|x - 1| \geqslant \frac{1}{2}$.
2) $\dot{\gamma}(x)$ decreases monotonically to zero as x increases from $\frac{1}{2}$ to 1.

Then set

$$\gamma_\eta(x) = \begin{cases} 1 - |1 - x| & \text{for } |1 - x| \geqslant \eta/2 \\ 1 - \eta + \eta\gamma\left(1 + \dfrac{x - 1}{\eta}\right) & \text{for } |1 - x| \leqslant \eta/2 \,. \end{cases}$$

Finally, we shall denote our wave function by $\nu_\eta(x)$ and shall define it for all x by requiring that

1) $\nu_\eta(x + 4) = \nu_\eta(x)$
2) $\nu_\eta(x) = \gamma_\eta(x)$ for $0 \leqslant x \leqslant 2$
3) $\nu_\eta(x) = -\gamma_\eta(x - 2)$ for $2 \leqslant x \leqslant 4$.

2.2. The next tool we shall use to carry out our deformations is due to NASH (see [13]). We shall introduce it in the special form needed for our applications.

If each matrix

$$\gamma = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \tag{2.21}$$

is identified with a point of the space of all triplets (E, F, G), the positive definite matrices will fill the interior C of the cone defined by the conditions

$$\left\| \begin{array}{l} EG - F^2 = 0 \\ E, G \geqslant 0 \end{array} \right.$$

The positive definite matrices (2.21) for which $E + G = 2$ fill an ellipse E which lies in a plane p perpendicular to the (E, G)-plane, has a major axis of length $\sqrt{2}$ lying upon the line $E + G = 2$ of the (E, G)-plane and a minor axis of length 1 lying upon the line $E = 1, G = 1$.

We can thus write every positive definite matrix γ in the form

$$\gamma = \frac{E + G}{2}\gamma' \quad \text{with} \quad \gamma' \in E \,.$$

Further, each matrix $\gamma' \in E$ can be written as a linear combination

$$\gamma' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + A \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + B \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We then have $\det \gamma' = 1 - A^2 - B^2 = 1 - \theta^2$ $(0 \leqslant \theta < 1)$ and we can conclude that the matrices of E which have a determinant $> 1 - \theta^2$ lie in the interior of an ellipse E_θ which has the same axes of symmetry as E and has major and minor axes $\theta\sqrt{2}$ and θ respectively.

For each $\theta < 1$ we shall construct a finite number of matrices

$$\gamma_1, \gamma_2, \ldots, \gamma_{n_\theta}$$

lying on the boundary of E (for instance equally spaced) and shall determine some positive functions

$$\varphi_1(\gamma'), \varphi_2(\gamma'), \ldots, \varphi_{n_\theta}(\gamma')$$

defined and C^∞ for γ' in $\bar{E}_\theta$ and such that

$$\sum_{i=1}^{n_\theta} \varphi_i^2(\gamma') \equiv 1, \quad \sum_{i=1}^{n_\theta} \gamma_i \varphi_i^2(\gamma') = \gamma' \quad \text{for all} \quad \gamma' \in \bar{E}_\theta . \tag{2.22}$$

Clearly such functions $\varphi_i(\gamma')$ can be constructed as soon as the convex hull of $\gamma_1, \gamma_2, \ldots, \gamma_{n_\theta}$ contains the ellipse E_θ in its interior.

We observe also that since each γ_i is a degenerate matrix, it can be written in the form

$$\gamma_i = \begin{pmatrix} g_{i1}^2 & g_{i1}g_{i2} \\ g_{i1}g_{i2} & g_{i2}^2 \end{pmatrix}. \tag{2.23}$$

For convenience we shall introduce the functions $\psi_i(z)$ defined by setting

$$\psi_i(z) = g_{i1}x + g_{i2}y . \tag{2.24}$$

2.3. Suppose that we are given in U a family of C^∞ regular metrics

$$d\sigma_\xi^2 = (dx\,dy)\gamma(z, \underline{\xi}) \begin{pmatrix} dx \\ dy \end{pmatrix} = E dx^2 + 2F dx dy + G dy^2 \tag{2.31}$$

defined for $z \in U$ and $\underline{\xi} \in B$ whose coefficients together with their derivatives are continuous in $U \times B$. We shall introduce a process, which is defined when $\underline{\xi}$ is restricted to a compact set, that uses these metrics to generate a family of C^∞ deformations of the surface Σ_0.

Let then C denote a compact subset of B. Since $\bar{\pi} \times C$ will also be compact, a constant θ can be found such that for all $(z, \underline{\xi}) \in \bar{\pi} \times C$ we shall have

$$4(EG - F^2) / (E + G)^2 \geqslant 1 - \theta^2 .$$

For such a choice of θ the matrix $\gamma' = \dfrac{2}{E+G}\gamma$ will be contained in E_θ. Thus for $(z, \underline{\xi}) \in \bar{\pi} \times C$, using (2.22), (2.23), (2.24), we obtain an expression for $d\sigma_\xi^2$ as a linear combination of the quadratic forms $(d\psi_i)^2$, namely

$$d\sigma_\xi^2 = \frac{E+G}{2} \sum_{i=1}^{n_\theta} \varphi_i^2(\gamma')(d\psi_i)^2 . \tag{2.32}$$

Since we are going to use always the same coordinate system in U, we can write (2.32) in the more convenient form

$$d\sigma_\xi^2 = \sum_{i=1}^{n_\theta} \alpha_i^2(d\sigma_\xi^2)(d\psi_i)^2 . \tag{2.33}$$

Clearly the functions $\alpha_i(d\sigma_\xi^2) = \sqrt{\dfrac{E+G}{2}}\,\varphi_i(\gamma')$ together with their first derivatives are defined and continuous at least when $(z, \underline{\xi}) \in \bar{\pi} \times C$.

Given $0 < \varepsilon \leqslant \varepsilon_0$, $0 < \eta \leqslant \eta_0$ for each $\underline{\xi} \in C$ we shall construct a sequence of n_θ deformations of the surface Σ_0, according to the following process.

We first set

$$\underline{X}_1 = \underline{X} + \frac{1}{\lambda_1} \varphi_\varepsilon \alpha_1(d\sigma_\xi^2)\, v_\eta(\lambda_1\psi_1)\underline{N} , \tag{2.34}$$

where $\underline{X}(z)$ is the vector introduced in (1.5), $\underline{N}(z)$ denotes the positive unit normal to Σ_0 at $\underline{X}(z)$ and λ_1 is a constant. Since C is compact and (2.34) is a purely normal deformation, there will be a constant $\lambda_1(C)$ such that for $\lambda_1 \geqslant \lambda_1(C)$ the vector $\underline{X}_1$ defines for each $\underline{\xi} \in C$ a deformation of the surface Σ_0 which is C^∞ and deprived of selfintersections.

We then proceed recurrently, having defined $\underline{X}_{i-1}$ $(i = 2, \ldots, n_\theta)$ in such a way that for each $\underline{\xi} \in C$ the deformation of Σ_0 described by $\underline{X}_{i-1}$ is C^∞ and deprived of selfintersections, we introduce the unit vector $\underline{N}_i$ by the condition that it should be orthogonal to $d\underline{X}_{i-1}$ and define

$$\underline{X}_i = \underline{X}_{i-1} + \frac{1}{\lambda_i} \varphi_\varepsilon \alpha_i(d\sigma_\xi^2)\, v_\eta(\lambda_i\psi_i)\underline{N}_i . \tag{2.35}$$

By an elementary but tedious argument, it can be shown each time that, provided we choose $\lambda_i \geqslant \lambda_i^0(C, \lambda_1, \lambda_2, \ldots, \lambda_{i-1})$, the metric $d\underline{X}_i$ is positive definite and the surface described by $\underline{X}_i$ for each $\underline{\xi} \in C$ is C^∞ and deprived of selfintersections.

The surface described by $\underline{X}_{n_\theta}$ shall be denoted by $\Sigma_0[d\sigma_\xi^2]$.

The family of deformations which we obtain by this process of course will depend of the choice of the constants $\varepsilon, \eta, \lambda_1, \lambda_2, \ldots, \lambda_{n_\theta}$. We have introduced these constants to allow ourselves a certain control over the metrics of the resulting surfaces $\Sigma_0[d\sigma^2_\xi]$.

In fact, the metric of $\Sigma_0[d\sigma^2_\xi]$ can be written in the form

$$d\underline{X}^2_{n\theta} = d\underline{X}^2 + \varphi^2_\varepsilon \sum_{i=1}^{n\theta} \alpha_i^2 \dot{\nu}^2_\eta(\lambda_i \psi_i)(d\psi_i)^2 + o(1)\,, \tag{2.36}$$

where $o(1)$ is a term which can be made uniformly small for $\underline{\xi} \in C$ by a suitable choice of the constants λ_i. This is seen as follows.

For each $i = 2, \ldots, n_0$ we can write

$$d\underline{X}_i = d\underline{X}_{i-1} + \varphi_\varepsilon \alpha_i \dot{\nu}_\eta(\lambda_i \psi_i) d\psi_i \underline{N}_i + \frac{\underline{e}_i}{\lambda_i}\,.$$

Thus, using the definition of $\underline{N}_i$

$$d\underline{X}^2_i = d\underline{X}^2_{i-1} + \varphi^2_\varepsilon \alpha_i^2 \dot{\nu}^2_\eta(\lambda_i \psi_i)(d\psi_i)^2 + 0_i(1/\lambda i)\,. \tag{2.37}$$

(2.36) is obtained by a recursive application of (2.37).

For our purposes we need only restrict the term $o(1)$ in (2.36) to satisfy

$$|\, o(1)\,| \leqslant d\underline{X}^2/2\,. \tag{2.38}$$

Thus each λ_i need only be chosen large enough so that

$$|\, 0_i(1/\lambda_i)\,| \leqslant 1/2n_\theta\, d\underline{X}^2\,.$$

Let us now denote by $\chi_{\varepsilon,\eta}$ the subset of π where at least one of the functions φ^2_ε, $\dot{\nu}^2_\eta(\lambda_i \psi_i)$ is not 1. It is clear that by a suitable choice of ε and η we can make the area of $\chi_{\varepsilon,\eta}$ arbitratily small independently of the λ_i's (provided each of these is sufficiently large). In $\pi - \chi_{\varepsilon,\eta}$ we shall have

$$d\underline{X}^2_{\eta\theta} = d\underline{X}^2 + d\sigma^2_\xi + o(1)\,. \tag{2.39}$$

This, together with (2.38), is all that we need for our applications.

2.4. We shall now proceed to show that every conformal type of RIEMANN surface of genus g can be obtained by small deformations of Σ_0. This result will be obtained by means of the deformation process defined in the last section. We need only construct a suitable family of metrics $d\sigma^2_\xi$.

Let $\beta(x)$ denote a real function defined and C^∞ for $x \geqslant 0$ and such that a) $\beta(x) \equiv 0$ for $0 \leqslant x \leqslant \frac{1}{2}$, b) $\beta(x)$ increases to 1 for $\frac{1}{2} \leqslant x \leqslant 1$, c) $\beta(x) \equiv 1$ for $x \geqslant 1$. Let Γ_ξ for each $\underline{\xi}$ denote the set of all points of

U which are zeros of the quadratic differential Φ_{ξ}. We shall set for each $0 < r < 1$

$$\beta_{r,\xi}(z) = \prod_{w \in \Gamma_{\xi}} \beta\left(\left|\frac{z - w}{1 - z\bar{w}}\right| \frac{1}{r}\right). \tag{2.41}$$

Since in every compact subset of U there is only a finite number of zeros of Φ_{ξ}, the function $\beta_{r,\xi}$ is well defined. Furthermore, since the zeros of Φ_{ξ} vary continuously with $\underline{\xi}$, $\beta_{r,\xi}$ and its z-derivatives will be continuous functions of $(z, \underline{\xi})$ in $U \times B$.

The function $\beta_{r,\xi}$ can be different from 1 only within a distance $2r$ from a zero of Φ_{ξ}, and for each $\underline{\xi}$ there can be at most $4g(4g - 4)$ zeros of Φ_{ξ} that are contained in $\bar{\pi}$. Therefore we can make the area of the subset of π where $\beta_{r,\xi} \neq 1$ as small as we please independently of $\underline{\xi}$ by choosing r sufficiently small.

The vector $\underline{X}(z)$, by definition, gives a C^{∞} conformal map of U onto Σ_0, thus for each $z \in U$ we can write

$$d\underline{X}^2 = \Lambda(z) \, | dz |^2 \tag{2.42}$$

where $\Lambda(z)$ is positive and C^{∞} in U.

We shall set

$$d\sigma_{\xi}^2 = \Lambda(z) \left| dz + \beta_{r,\xi} \, | \underline{\xi} | \frac{\bar{\Phi}_{\xi}}{| \Phi_{\xi} |} \, d\bar{z} \right|^2 . \tag{2.43}$$

Observe that we have

$$d\underline{X}^2 (1 - | \underline{\xi} |)^2 \leqslant d\sigma_{\xi}^2 \leqslant d\underline{X}^2 (1 + | \underline{\xi} |)^2 . \tag{2.44}$$

In addition it is easy to see that the metrics $d\sigma_{\xi}^2$ for each choice of $0 < r < 1$ will satisfy the conditions stated at the beginning of last section.

Given a constant $\gamma > 0$ let us consider the family of surfaces

$$\Sigma[\gamma d\sigma_{\xi}^2] \tag{2.45}$$

constructed by means of the deformation process with $\underline{\xi}$ restricted to the compact set $B_{\varrho} = \{| \underline{\xi} | \leqslant \varrho\}$.

For a given choice of $\gamma, \varepsilon, \eta, r$ the constant θ will be determined by the process. The constants $\lambda_1, \lambda_2, \ldots, \lambda_{n\theta}$ should be chosen big enough so that the surfaces (2.45) are deprived of selfintersections and in addition (2.38) is satisfied.

For each $\underline{\xi} \in B_{\varrho}$ there exists a unique $\underline{\xi}'$ such that the RIEMANN surface $\Sigma_0(ds_{\xi'}^2)$ is conformally equivalent to $\Sigma_0[\gamma d\sigma_{\xi}^2]$ by a conformal map homotopic to the identity in Σ_0. The constants $\varrho, \gamma, \varepsilon, \eta, r, \lambda_1, \lambda_2, \ldots, \lambda_{n\theta}$ being fixed, the point $\underline{\xi}'$ will be a function of $\underline{\xi}$, let us denote it by $\underline{\xi}'(\underline{\xi})$.

From the continuity lemma it follows easily that $\underline{\xi}'(\underline{\xi})$ is a continuous function of $\underline{\xi}$. Thus, in order to establish our imbedding theorem we only need to show that for every $\varrho < 1$ and $\delta > 0$ we can choose the constants $\gamma, \varepsilon, \eta, r, \lambda_1, \lambda_2, \ldots, \lambda_{n\theta}$ in such a way that

$$|\underline{\xi}'(\underline{\xi}) - \underline{\xi}| < \delta \tag{2.46}$$

for all $\underline{\xi} \in B_\varrho$.

2.5. We will be able to assure such an inequality as (2.46) by another application of the continuity lemma. To this end we shall obtain some estimates on the dilatation of the identity map between $\Sigma_0(ds_\xi^2)$ and $\Sigma_0[\gamma d\sigma_\xi^2]$.

It will be convenient to decompose the deformation from $\Sigma_0(ds_\xi^2)$ to $\Sigma_0[\gamma d\sigma_\xi^2]$ into two deformations τ_1 and τ_2 defined as follows

$$\Sigma_0(ds_\xi^2) \overset{\tau_1}{\to} \Sigma_0(d\underline{X}^2 + \varphi_\varepsilon^2 \gamma d\sigma_\xi^2) \overset{\tau_2}{\to} \Sigma_0(dX_{n\theta}^2)\ .$$

Let us call $K_1(z)$ and $K_2(z)$ the dilatations of the maps τ_1 and τ_2 respectively. We shall obtain separate estimates for these two functions.

a) *Estimates for* $K_1(z)$. The set $\bar{\pi}$ will be divided into two parts: a set χ where τ_1 may be far from being conformal and a set $\pi - \chi$ where τ_1 can be assured to be close to being conformal. More precisely, χ will be made up of the complement of π_ε in $\bar{\pi}$ (where $\varphi_\varepsilon \not\equiv 1$) and the small regions around the zeros of Φ_ξ where $\beta_{r,\xi} \neq 1$. Because of our previous assumptions we have

$$\text{area}\ \chi < \varepsilon + (4g - 4)\pi r^2\ . \tag{2.51}$$

As for $K_1(z)$ we observe that in $\pi - \chi$ we have (cfr. (2.44))

$$K_1^2(z) \leqslant \left(\frac{1}{(1-\varrho)^2} + \gamma\right) \Big/ \left(\frac{1}{(1+\varrho)^2} + \gamma\right). \tag{2.52}$$

For $z \in \chi$ we can proceed as follows. We know that the map between $\Sigma_0 = \Sigma_0(d\underline{X}^2)$ and $\Sigma_0(ds_\xi^2)$ has maximal dilatation $K_1' = (1 + |\underline{\xi}|)/(1 - |\underline{\xi}|)$. The map between $\Sigma_0(d\underline{X}^2)$ and $\Sigma_0(d\underline{X}^2 + \varphi_\varepsilon^2 \gamma d\sigma_\xi^2)$ has a dilatation K_2' which satisfies (in view of (2.44))

$$(K_2')^2 \leqslant [1 + \gamma(1+\varrho)^2] \,/\, [1 + \gamma(1-\varrho)^2]\ .$$

Thus, in χ we have

$$K_1^2 \leqslant (K_1' K_2')^2 \leqslant \left(\frac{1+\varrho}{1-\varrho}\right)^2 \frac{1 + \gamma(1+\varrho)^2}{1 + \gamma(1-\varrho)^2}\ . \tag{2.53}$$

b) *Estimates on* $K_2(z)$. To estimate $K_2(z)$ we have to compare the metric

$$d\underline{X}^2_{n\theta} = d\underline{X}^2 + \gamma\varphi^2_\varepsilon \sum_{i=1}^{n\theta} \dot{v}^2_\eta(\lambda_i\psi_i)\alpha^2_i(d\sigma^2_\xi)(d\psi_i)^2 + o(1)$$

(cfr. (2.36)) with the metric $d\underline{X}^2 + \gamma\varphi^2_\varepsilon d\sigma^2_\xi$. We again subdivide $\bar{\pi}$ into two parts: the set $\chi_{\varepsilon,\eta}$ of section (2.3) and its complement $\pi - \chi_{\varepsilon,\eta}$.

For $z \in \pi - \chi_{\varepsilon,\eta}$ we have (in view of (2.38) and (2.44))

$$K^2_2(z) \leqslant \frac{1 + 1/2(1 + \gamma[1 - \varrho]^2)}{1 - 1/2(1 + \gamma[1 - \varrho]^2)} . \tag{2.54}$$

Finally, for $z \in \chi_{\varepsilon,\eta}$ we have

$$K^2_2(z) \leqslant \frac{\max \dfrac{dX^2 + \gamma\varphi^2_\varepsilon d\sigma^2_\xi + d\underline{X}^2/2}{d\underline{X}^2 + \gamma\varphi^2_\varepsilon d\sigma^2_\xi}}{\min \dfrac{d\underline{X}^2 - d\underline{X}^2/2}{d\underline{X}^2 + \gamma\varphi^2_\varepsilon d\sigma^2_\xi}} \leqslant 3[1 + \gamma(1 + \varrho)^2] . \tag{2.55}$$

2.6) We shall terminate the proof of the imbedding theorem by giving the order in which the constants $\gamma, \varepsilon, \eta, r, \lambda_1, \lambda_2, \ldots, \lambda_{n\theta}$ have to be chosen.

First of all we will be given a $\varrho < 1$ and a $\delta > 0$ and will be required to choose the yet unspecified constants so that (2.46) is satisfied. As we have already observed, as long as $|\xi| \leqslant \varrho$, (2.46) can be replaced by

$$K(\underline{\xi}'(\underline{\xi}), \underline{\xi}) < 1 + \delta \tag{2.61}$$

where $K(\underline{\xi}'(\underline{\xi}), \underline{\xi})$ denotes the dilatation of the extremal quasiconformal map between $\Sigma_0(ds^2_\xi)$ and $\Sigma_0(ds^2_{\xi'})$.

We then proceed as follows. We find a γ so large that the product of the bounds (2.52) and (2.54) is less than $(1 + \delta/2)^2$. We let K^2_0 be equal to the product of the bounds (2.53) and (2.55). We then choose ε, η, r so small that the inequality

$$K_0 \iint_{\chi \cup \chi_{\varepsilon,\eta}} dA_\Phi \leqslant \delta/2 \iint_K dA_\Phi \tag{2.62}$$

holds for all $|\xi| \leqslant \varrho$. This is possible, as it was shown in the continuity lemma, since the area of $\chi \cup \chi_{\varepsilon,\eta}$ can be made arbitrarily small.

It remains to fix θ and the constants $\lambda_1, \lambda_2, \ldots, \lambda_{n\theta}$. However, these constants may be chosen in any way that satisfies the requirements of the deformation process.

We shall have then $K(z) < 1 + \delta/2$ in $\pi - \chi \cup \chi_{\varepsilon,\eta}$ and $K(z) \leqslant K_0$ in $\chi \cup \chi_{\varepsilon,\eta}$ for all $(\xi) \leqslant \varrho$, and the inequality (2.61) will necessarily follow.

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