Convex Immersions of Closed Surfaces in E^3 [1])

Non-orientable closed surfaces in E^3 with minimal total absolute Gauss-curvature

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1. Introduction and theorems

If X, a compact connected closed C^∞-surface with Euler-Poincaré characteristic $\chi(X)$, has a Riemannian metric, and if K is the Gauss-curvature and $d\mu$ is the absolute value of the exterior 2-form which represents the volume, then according to the theorem of Gauss-Bonnet, which holds for orientable as well as non-orientable surfaces,

$$\int_X \frac{K\,d\mu}{2\pi} = \chi(X)\,. \tag{1}$$

The *total absolute Gauss-curvature* of the surface is by definition:

$$\tau(X) = \int_X \frac{|\,K\,d\mu\,|}{2\pi}\,. \tag{2}$$

Clearly

$$\tau(X) \geqslant |\,\chi(X)\,| \tag{3}$$

and equality is obtained from a constant Gauss-curvature metric on X (which is known to exist for any X).

Now let $f: X \to E^3$ be a C^∞-immersion of X in euclidean three space E^3. A C^∞-mapping is called a C^∞-immersion if it has maximal rank, here two, at each point. In X we take the unique Riemannian metric for which f is locally isometric. The integrals (1) and (2) can be split in a contribution from the sets of points for which $K > 0$ and $K < 0$ respectively:

$$\chi(X) = \int \frac{K\,d\mu}{2\pi} = \int_{K>0} \frac{K\,d\mu}{2\pi} + \int_{K<0} \frac{K\,d\mu}{2\pi} \tag{4}$$

$$\tau(X) = \int \frac{|\,K\,d\mu\,|}{2\pi} = \int_{K>0} \frac{K\,d\mu}{2\pi} - \int_{K<0} \frac{K\,d\mu}{2\pi}\,. \tag{5}$$

It can be shown that $K \geqslant 0$ at any point x for which $f(x)$ lies on the

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[1]) Lecture in the International Colloquium on Differential Geometry and Topology, Zürich, June 1960.

convex envelope of $f(X)$, and the contribution of the set of all points with $K \geqslant 0$, in the integral $\int \frac{K d\mu}{2\pi}$ is at least 2.

Then in view of (4) and (5) we have

Theorem 1. Compare [1, 2, 3, 4, 5]. *If* $f: X \to E^3$ *is an immersion of a closed surface in euclidean three space, then*

$$\tau(X) = \int \frac{|K d\mu|}{2\pi} \geqslant 4 - \chi(X) . \tag{6}$$

The *immersion* $f: X^2 \to E^3$ *is called convex,* and the surface $f(X)$ is said to have *minimal total absolute* GAUSS*-curvature* in case the minimum in (6) is attained:

$$\tau(X) = 4 - \chi(X) . \tag{7}$$

For $\chi(X) = 2$ this minimum is attained by the convex surfaces in E^3. In [5] we studied convex immersed surfaces in E^3 and we obtained the following result.

Lemma 1. Let $f: X^2 \to E^3$ be convex. The smallest convex set which contains fX is called: the *convex hull* HfX, with boundary the *convex envelope* ∂HfX. There exist two disjoint open sets U and V in X, such that X is the union of U, V and their common boundary, and such that:

1) The restriction of f to the set U is a homeomorfism onto the complement of a finite number of *plane* closed convex discs $D_1, \ldots D_k$ in ∂HfX.

2) $K \geqslant 0$ for $x \in U$; $K \leqslant 0$ for $x \in V$.

3) Each of the convex discs D_i $i = 1, \ldots k$ contains the image under f of some one-cycle in X that does not bound in X.

From this lemma it is not hard to deduce [5] the

Theorem 2. *No convex immersion in* E^3 *exists for the projective plane* P *or the* KLEIN*-bottle* B.

Hence if f is an immersion of $X = P$ or $X = B$ in E^3, then

$$\int_X \frac{|K d\mu|}{2\pi} > 4 - \chi . \tag{8}$$

Examples of orientable surfaces with $\chi = 2 - 2k$ in E^3 with minimal τ can be obtained in view of the lemma, by starting from a convex surface which has at least two open plane parts in different planes, by deleting $2k$ convex discs suitably chosen in these plane parts, and by gluing k connecting handles on which $K \leqslant 0$ in a suitable manner. By a "*handle*" we mean the differentiable surface which is the product space of a circle and an open interval. An example is the usual picture of the torus in E^3. An example with two handles ($\chi = -2$) is given in fig. 1.

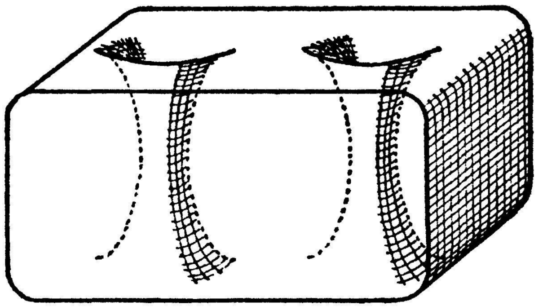


fig. 1

In this paper (§ 3) we give examples of convex immersions in E^3 for all non-orientable closed surfaces X with $\chi(X) \leqslant -2$. This proves

Theorem 3. *For any orientable closed surface, and for any non-orientable closed surface X for which $\chi(X) \leqslant -2$, there exists a convex immersion in E^3.*

We know neither an example, nor non-existence for the case $\chi(X) = -1$. In § 2 we exhibit an immersion of P^2 in E^3, which seems to be less complicated then *Boy's surface* (see HILBERT and COHN-VOSSEN, Anschauliche Geometrie).

N.B. We restrict our considerations to C^∞-immersions of C^∞-manifolds. The case of C^k-immersions with $2 \leqslant k \leqslant \infty$ is not essentially different. It seems likely that analytic convex immersions of closed orientable surfaces can be found. On the other hand we have *no idea about the existence of analytic convex immersions of non-orientable closed surfaces in E^3.*

2. An immersion of the projective plane in E^3

The projective plane P is the quotient of the two-sphere in E^3 with equation say

$$x_1^2 + x_2^2 + x_3^2 = 1$$

by the diametrical equivalence relation:

$$(x_1, x_2, x_3) \sim (x_1, x_2, x_3) \sim (-x_1, -x_2, -x_3)\,.$$

The product $x_1 x_2$ takes on the sphere the same value in diametrical points and it therefore represents a function on P. This function has three critical levels with the critical values -1 (minimum), 0 (saddle point) and 1 (maximum). The maximal and minimal level sets are points. The levelset $x_1 x_2 = 0$ is the union of two differentiable circles (one-spheres) which have in P one point in common. The non-critical levelsets in P are differentiable circles. We exhibit an immersion in the euclidean three-space in which one coordinate is called height, by first of all chosing the height proportional to $x_1 x_2$, and by exhibiting the intersections of the immersed surface with horizontal planes at different heights. The only points at which the surface is assumed to have

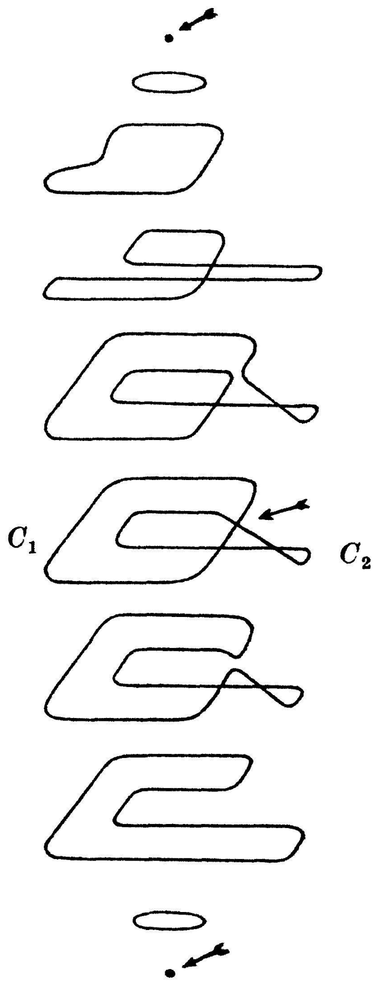


fig. 2

a horizontal tangentplane are the three critical points. See fig. 2, and study in particular a neighborhood of the saddle point and of the corresponding level.

Remark on regular homotopy classes of immersions of the two-sphere S^2 *in* E^3. According to SMALE [6] any two immersions of S^2 in E^3 can be connected by a regular homotopy of immersions. This is in particular therefore the case for the following two immersions: (I) the identity map $g(0)$ of an ordinary two-sphere S in E^3 onto itself, and (II) the reflection of S with respect to some plane through the centre. These two immersions have the same image pointset, but with different orientations. H. HOPF suggested to me that an explicit regular homotopy could be obtained via an immersion of P. This is indeed the case:

If f is the immersion of P given above and $\pi : S \to P$ is the covering map obtained from identifying diametrical points of S, then $g(1) = f \circ \pi$ is an immersion of S which can be seen to be regularly homotopic to $g(0)$. (Distinguish two sides on $f(P)$, pull them slightly apart and then pull the whole surface out.) Let $g(t)$, $0 \leqslant t \leqslant 1$, be such a regular homotopy and let τ map any point of S onto its diametrical point. We put

$$h(t) = \begin{cases} g(2t) & 0 \leqslant t \leqslant \frac{1}{2} \\ g(2-2t) \circ \tau & \frac{1}{2} \leqslant t \leqslant 1 \, . \end{cases}$$

Then $h(t)$, $0 \leqslant t \leqslant 1$, is a regular homotopy of S with the same image set S for $t = 0$ as for $t = 1$, but interchanging the orientations.

3. Examples of convex immersions of non-orientable surfaces

A. KLEIN-bottle with handles

In fig. 3 a convex immersion $f: X \to E$ of a KLEIN-bottle with one handle $\chi(X) = -2$ is exhibited. From the picture it is clear that only the part in the convex envelope of $f(X)$ contributes in $\int\limits_{K>0} \frac{K\,d\mu}{2\pi}$ and it yields 2. Then the immersion is convex. Of

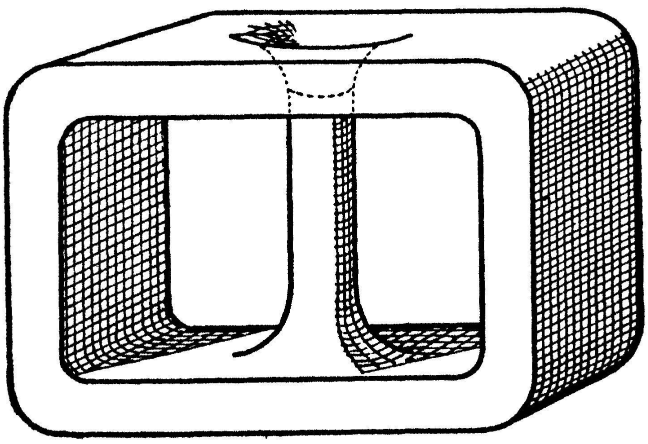


fig. 3

course more handles can be constructed, each with non-positive GAUSS-curvature K, and so a convex immersion of the KLEIN-bottle with $k \geqslant 1$ handles $\{\chi(X) = 0 - 2k\}$ is obtained.

B. Projective plane with two handles

We first give *a general idea* about the construction. We want to obtain a convex immersion of a surface X which is topologically a projective plane with two handles $\chi(X) = -3$. In order to do so we first construct an immersion of a surface Y with $\chi(Y) = -1$, which is a projective plane with one handle, and we obtain this from the immersion $f: P \to E^3$ of the projective plane, given in fig. 2 as follows. The intersection of $f(P)$ with a horizontal plane decreases to a point when the height h increases to the maximal value $h = 1$ as well as when the height decreases to the minimal value -1. We alter the image $f(P)$ such, that for increasing height the differentiably immersed circle at height $h \geqslant 0$, converges to a big convex curve C_1 in the plane at height $h = 1$. Moreover we take care that the surface is at its boundary C_1 tangent to the interior in the plane of C_1.

We apply an analogous alteration to the immersed circles in the planes at height $h < 0$, so that they converge to a big convex curve C_{-1} in the plane $h = -1$. Also at this boundary the surface so obtained must be tangent to the interior in the plane of C_{-1}.

Next we attach a big outside handle H' to this surface bounded by $C_1 \cup C_2$. We take for this handle the complement with respect to the *plane* interiors of C_1 and C_{-1} of a convex C^∞-surface which contains C_1 and C_{-1} as well as their plane interiors. On this outside handle therefore the GAUSS-curvature is non negative: $K \geqslant 0$. This handle H' will lateron completely take care of the contribution $\displaystyle\int_{K>0} \frac{K\,d\mu}{2\pi}$ in (5). We now have obtained an immersion $g: Y \to E^3$ with $g(H) = H'$ and we concentrate further on the part $Y - H$. We *try* to arrange the immersion such that $K \leqslant 0$ for $x \in Y - H$.

We observe that for ruled surfaces at any rate the GAUSS-curvature is non-positive, and so we immerge large parts of ruled surface, and try to connect these parts with each other and with C_1 and C_{-1} by pieces of surface that have also $K \leqslant 0$.

This however seems to be impossible. But we can succeed in finding an immersion $g: Y \to E^3$ such that $K \leqslant 0$ for $x \in Y - H - D$, where D is a small disc with $K > 0$ and $g(D)$ is a small part of a convex surface (a nap). Assuming this for the moment we can now take out of Y a slightly bigger disc $D' \supset D$, and we can attach one end (component of the boundary) of a handle on which $K < 0$, at the boundary of $f(Y - D^2)$.

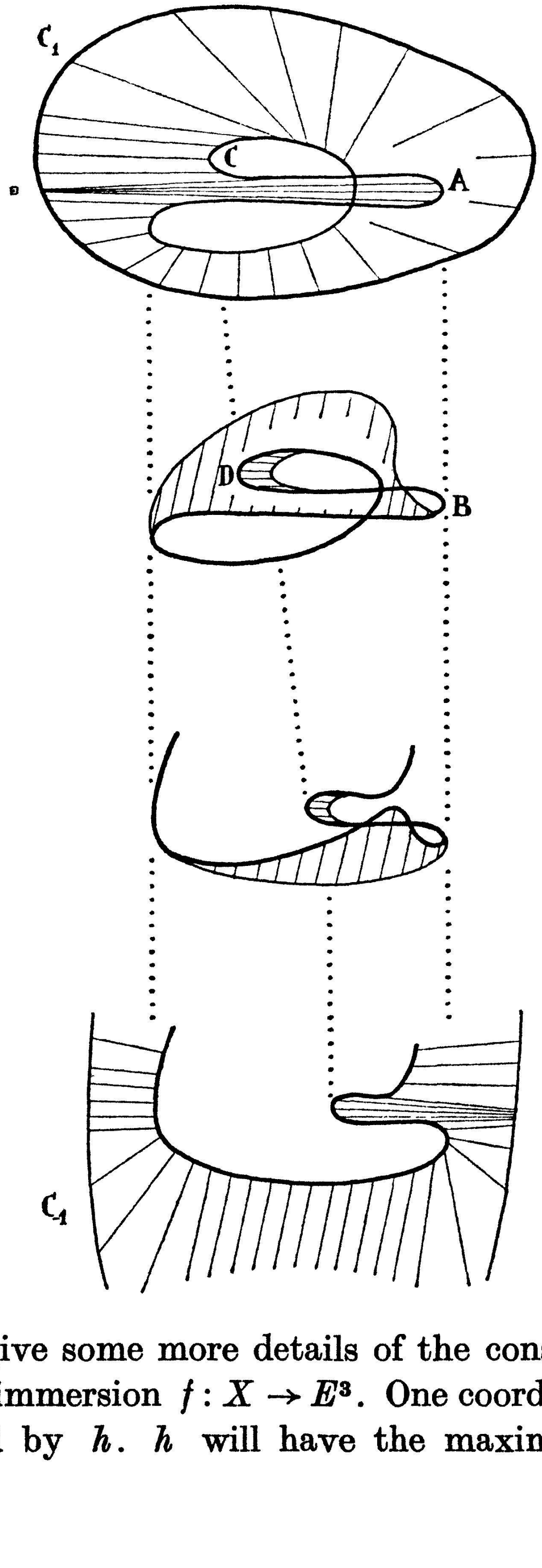
It can be arranged so that the other end of this handle is attached to (the inside of) the big outside handle $f(H)$ along a convex curve contained in a flat (= in a plane) part of $f(H)$. Such a flat part can be assumed to be available on $f(H)$. Finally the interior of this convex curve in this tangent plane is deleted from $f(H)$ and the required immersion is obtained.

fig. 4

fig. 6

fig. 8

fig. 9



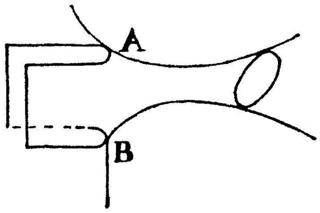


fig. 5

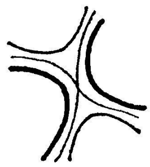


fig. 7

We now give some more details of the construction. We begin with a preliminary C^0-immersion $f: X \to E^3$. One coordinate in E^3 will be called height and denoted by h. h will have the maximum $h = 1$ and the minimum

$h = -1$ on X. The intersection of $f(X)$ with the planes at $h = 1$ and $h = -1$ are big convex curves C_1 and C_{-1}. C_1 and C_{-1} are boundaries of a big C^∞-outside handle fH, $H \in X$ as mentioned above, on which $K > 0$. For later use there will be a hole at a suitable place in fH, obtained by deleting a plane convex dix. One end of a handle will be attached to this hole.

The rest of the surface will consist of six C^∞-parts, four of which are parts of ruled surface and hence have $K \leqslant 0$. The seven parts are attached to each other along their boundaries which are plane curves, such that a C^0-immersion of X results. (Under a more general definition of curvature the C^0-immersion so obtained will be convex.)

We will give the description of the surface so obtained, but remark before that the required convex C^∞-immersion of X in E^3 is then easily seen to be obtainable by smoothing processes in the neighborhoods of the boundaries of the seven parts.

In fig. 4 the ruled surface $f(X - H) \cap (0{,}3 \leqslant h \leqslant 1)$ is given by its orthogonal projection on a horizontal plane. The projections of the rules are indicated. The *upper boundary* (at height 1) is drawn *fat*.

Figures 6, 8 and 9 refer analogously to ruled surfaces at heights:

fig. 6	$0{,}1 \leqslant h \leqslant 0{,}2$
fig. 8	$-0{,}2 \leqslant h \leqslant -0{,}1$
fig. 9	$-1 \leqslant h \leqslant -0{,}2$

In the interval $0{,}2 \leqslant h \leqslant 0{,}3$ the extra handle going to the outside handle is attached (fig. 5), whereas in the remaining part in most points the tangent planes are vertical. This however is not true in the neighborhood of the line CD, where the surface is non vertical, because otherwise we would have to get $K \geqslant 0$ when we want to pull the part of the horizontally immersed circle close to D out to the right at a lower stage.

In the interval $-0{,}1 < h < 0{,}2$ the saddle (fig. 7) occurs in a saddlepoint at height $h = 0$.

We finally remark that a convex immersion of a surface Z with $\chi(Z) = 1 - 2k$ for $k > 2$, can be obtained from the above convex immersion of X by attaching $k - 2$ handles on which $K \leqslant 0$, in a suitable manner.

Remark. Instead of considering the set of *all* immersions of a closed surface X in E^3, one can also ask for lower bounds of $\tau(X, f)$ with f in a given regular homotopy class of immersions. For example if X is the torus and f

belongs to the class of the 2×2-fold covering of the standard imbedding of the torus in E^3, then $\tau(X, f) \geqslant 6$.

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REFERENCES

[1] S. S. CHERN and R. K. LASHOF, *On the total curvature of immersed manifolds.* I. Amer. J. of Math. LXXIX (1957), 306–313.
[2] S. S. CHERN and R. K. LASHOF, *On the total curvature of immersed manifolds.* II. Michigan Math. J. (1958), 5–12.
[3] NICOLAAS H. KUIPER, *Immersions with minimal total absolute curvature.* Colloque C.B.R.M. 1958, p. 75–88.
[4] NICOLAAS H. KUIPER, *Sur les immersions à courbure totale minimale.* Séminaire Ehresmann 1959, 5 p.
[5] NICOLAAS H. KUIPER, *On surfaces in euclidien three-space.* Bull. soc. math. de Belgique (1960), à paraître.
[6] S. SMALE, *A classification of immersions of the two-sphere.* Trans. Amer. Math. Soc. *90* (1959), 281–290.
[7] M. W. HIRSCH, Thesis Chicago (1959).

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