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On Geodesic Vector Fields in a Compact Orientable Riemannian Space

By Kentaro Yano and Tadashi Nagano, Tokyo[1])

§ 1. Killing vectors and harmonic vectors

We consider an n-dimensional Riemannian space covered by a system of coordinate neighborhoods (ξ^h) and with a positive definite metric $ds^2 = g_{ji}(\xi)d\xi^j d\xi^i$, where the indices $h, i, j, \ldots$ run over the range $1, 2, \ldots, n$. We denote the Christoffel symbols by

$$\begin{Bmatrix} h \\ j\,i \end{Bmatrix} = \tfrac{1}{2} g^{ha}(\partial_j g_{ia} + \partial_i g_{ja} - \partial_a g_{ji})$$

and the covariant derivative of a tensor, say $T_i{}^h$, by

$$\nabla_j T_i{}^h = \partial_j T_i{}^h + \begin{Bmatrix} h \\ j\,a \end{Bmatrix} T_i{}^a - \begin{Bmatrix} a \\ j\,i \end{Bmatrix} T_a{}^h ,$$

where ∂_j represents the partial differentiation with respect to ξ^j. We denote the curvature tensor by

$$K_{kji}{}^h = \partial_k \begin{Bmatrix} h \\ j\,i \end{Bmatrix} - \partial_j \begin{Bmatrix} h \\ k\,i \end{Bmatrix} + \begin{Bmatrix} h \\ k\,a \end{Bmatrix}\begin{Bmatrix} a \\ j\,i \end{Bmatrix} - \begin{Bmatrix} h \\ j\,a \end{Bmatrix}\begin{Bmatrix} a \\ k\,i \end{Bmatrix}$$

and the Ricci tensor and the curvature scalar by

$$K_{ji} = K_{aji}{}^a \qquad \text{and} \qquad K = g^{ji} K_{ji}$$

respectively.

A Killing vector v^h, that is, a vector defining an infinitesimal motion of the space satisfies

$$\nabla_j v_i + \nabla_i v_j = 0 \quad \text{and consequently} \quad \nabla_i v^i = 0 \tag{1.1}$$

and a harmonic vector w_i satisfies

$$\nabla_j w_i - \nabla_i w_j = 0 , \qquad \nabla_i w^i = 0 . \tag{1.2}$$

In case of Killing vector, $\nabla_i v^i = 0$ is a consequence of $\nabla_j v_i + \nabla_i v_j = 0$, but in case of harmonic vector, $\nabla_i w^i = 0$ is not a consequence of $\nabla_j w_i - \nabla_i w_j = 0$.

One of the present authors [2][2]) (See also [5]) proved

[1]) Presented by K. Yano at the International Colloquium on Differential Geometry and Topology, Zürich, June 1960.

[2]) See the Bibliography at the end of the paper.

Theorem 1.1. *A necessary and sufficient condition for a vector field* v^h *in a compact orientable* RIEMANN*ian space to be a* KILLING *vector is that*

$$g^{ji}\nabla_j\nabla_i v^h + K_i{}^h v^i = 0\,, \qquad \nabla_i v^i = 0\,. \tag{1.3}$$

On the other hand the following is well known.

Theorem 1.2. *A necessary and sufficient condition for a vector field* w_i *in a compact orientable* RIEMANN*ian space to be harmonic is that*

$$g^{ji}\nabla_j\nabla_i w_h - K_h{}^i w_i = 0\,. \tag{1.4}$$

In case of KILLING vector $\nabla_i v^i = 0$ is not a consequence of $g^{ji}\nabla_j\nabla_i v^h + K_i{}^h v^i = 0$, but in case of harmonic vector $\nabla_i w^i = 0$ is a consequence of $g^{ji}\nabla_j\nabla_i w_h - K_h{}^i w_i = 0$.

§ 2. Contravariant analytic vectors and covariant analytic vectors

We consider a $2n$-dimensional HERMITian space covered by a system of real coordinate neighborhoods (ξ^h) with a complex structure $F_i{}^h$ and with a positive definite HERMITian metric $ds^2 = g_{ji}(\xi)d\xi^j d\xi^i$, where the indices $h, i, j, \ldots$ run over the range $1, 2, \ldots, n, \bar{1}, \bar{2}, \ldots, \bar{n}$. The tensor $F_i{}^h$ satisfies

$$F_j{}^i F_i{}^h = -A_j{}^h \tag{2.1}$$

and

$$N_{ji}{}^h = 0 \tag{2.2}$$

where

$$N_{ji}{}^h = F_j{}^a(\partial_a F_i{}^h - \partial_i F_a{}^h) - F_i{}^a(\partial_a F_j{}^h - \partial_j F_a{}^h) \tag{2.3}$$

is the so-called NIJENHUIS tensor.

It is well known (A. NEWLANDER and L. NIRENBERG [1]) that the existence of a tensor $F_i{}^h$ satisfying (2.1) and (2.2) characterizes a complex space.

The HERMITian metric satisfies

$$F_j{}^c F_i{}^b g_{cb} = g_{ji}\,. \tag{2.4}$$

Now a KAEHLERian space is characterized as a HERMITian space which has the property

$$\partial_j F_{ih} + \partial_i F_{hj} + \partial_h F_{ji} = 0\,, \tag{2.5}$$

which is equivalent to

$$\nabla_j F_{ih} = 0\,, \tag{2.6}$$

where

$$F_{ji} = F_j{}^a g_{ai} \tag{2.7}$$

is a skew-symmetric tensor.

Suppose that our HERMITian space is covered by a system of complex coordinate neighborhoods (z^κ), where the indices $\kappa, \lambda, \mu, \ldots$ run over the range $1, 2, \ldots, n$. Then a real vector or a self-conjugate vector v^h has the components of the form

$$v^h = (v^\kappa, v^{\bar{\kappa}}),$$

where

$$v^{\bar{\kappa}} = \overline{v^\kappa}.$$

When the components v^κ are functions of z^λ only and $v^{\bar{\kappa}}$ functions of $\bar{z}^\lambda$ only the vector v^h is said to be contravariant analytic. The condition for v^h to be contravariant analytic is written as

$$v^a \partial_a F_i{}^h - F_i{}^a \partial_a v^h + F_a{}^h \partial_i v^a = 0 \tag{2.8}$$

in a real coordinate system. This is equivalent to

$$F_i{}^a \nabla_a v^h + F_a{}^h \nabla_i v^a = 0 \tag{2.9}$$

in a KAEHLERian space.

A real vector or a self-conjugate vector w_i has the components of the form

$$w_i = (w_\lambda, w_{\bar{\lambda}}),$$

where

$$w_{\bar{\lambda}} = \overline{w_\lambda}.$$

When the components w_λ are functions of z^κ only and $w_{\bar{\lambda}}$ are functions of $\bar{z}^\kappa$ only the vector w_i is said to be covariant analytic. The condition for w_i to be covariant analytic is written as

$$(\partial_j F_i{}^h - \partial_i F_j{}^h) w_h = F_j{}^a \partial_i w_a - F_i{}^a \partial_a w_j \tag{2.10}$$

in a real coordinate system. This is equivalent to

$$F_j{}^a \nabla_i w_a - F_i{}^a \nabla_a w_j = 0 \tag{2.11}$$

in a KAEHLERian space.

One of the present authors [3], [4] proved

Theorem 2.1. *A necessary and sufficient condition for a vector field v^h in a compact KAEHLERian space to be contravariant analytic is that*

$$g^{ji} \nabla_j \nabla_i v^h + K_i{}^h v^i = 0. \tag{2.12}$$

On the other hand the following is well known.

Theorem 2.2. *A necessary and sufficient condition for a vector field w_i in a compact KAEHLERian space to be covariant analytic is that*

$$g^{ji} \nabla_j \nabla_i w_h - K_h{}^i w_i = 0. \tag{2.13}$$

In case of KILLING and harmonic vectors the duality between them was not complete, but in case of contravariant and covariant analytic vectors the duality is complete as is seen from (2.12) and (2.13).

We defined the contravariant and covariant vector fields using the complex structure of the space. But equations (2.12) and (2.13) do not depend on the complex structure of the space. They depend only on the RIEMANNian structure of the space.

So it might be interesting to study the properties of a vector in a RIEMANNian space which satisfies (2.12) or (2.13). But following Theorem 1.2 a vector in a compact orientable RIEMANNian space satisfying (2.13) is a harmonic vector.

The purpose of the present paper is to study some of properties of a vector satisfying (2.12) in a compact orientable RIEMANNian space.

§ 3. Geodesic vector fields

It is well known that a necessary and sufficient condition for an infinitesimal point transformation

$$'\xi^h = \xi^h + v^h(\xi)dt$$

to carry a geodesic $\xi^h(s)$ into a geodesic and to preserve the affine character of the arc length s is that

$$(\nabla_j \nabla_i v^h + K_{kji}{}^h v^k)\frac{d\xi^j}{ds}\frac{d\xi^i}{ds} = 0 . \tag{3.1}$$

Thus for a unit vector λ^h at a point (ξ^h), we call

$$g^h = (\nabla_j \nabla_i v^h + K_{kji}{}^h v^k)\lambda^j\lambda^i , \tag{3.2}$$

the geodesic deviation vector of λ^h with respect to v^h.

Now, take n mutually orthogonal unit vectors $\lambda^h_{(a)}$ $(a = 1, 2, \ldots, n)$ at a point (ξ^h) in an n-dimensional RIEMANNian space and take the mean of geodesic deviation vectors $g^h_{(a)}$ of $\lambda^h_{(a)}$ with respect to v^h then we get

$$\frac{1}{n}\Sigma^n_{a=1} g^h_{(a)} = \frac{1}{n}\Sigma^n_{a=1}(\nabla_j \nabla_i v^h + K_{kji}{}^h v^k)\lambda^j_{(a)}\lambda^i_{(a)}$$

or

$$\frac{1}{n}\Sigma^n_{a=1} g^h_{(a)} = \frac{1}{n}(g^{ji}\nabla_j \nabla_i v^h + K_i{}^h v^i) \tag{3.3}$$

by virtue of

$$g^{ji} = \Sigma^n_{a=1}\lambda^j_{(a)}\lambda^i_{(a)} .$$

Thus the mean of these geodesic deviation vectors does not depend on the choice of n mutually orthogonal unit vectors $\lambda^h_{(a)}$. Thus we call (3.3) the *mean geodesic deviation vector* at (ξ^h) with respect to the vector field v^h.

A vector v^h with respect to which the mean geodesic deviation vector vanishes

$$g^{ji}\nabla_j\nabla_i v^h + K_i^h v^i = 0 \tag{3.4}$$

is called a *geodesic vector field.*

From this definition the following two theorems are evident.

Theorem 3.1. *A KILLING vector is a geodesic vector.*

Theorem 3.2. *A contravariant analytic vector in a KAEHLERian space is a geodesic vector.*

§ 4. Geodesic vector fields in a compact orientable RIEMANNian space

Let v^h be a geodesic vector field in a compact orientable RIEMANNian space, then v^h satisfies

$$g^{ji}\nabla_j\nabla_i v^h + K_i{}^h v^i = 0\,. \tag{4.1}$$

The following integral formula is well known (K. YANO [3], K. YANO and S. BOCHNER [5])

$$\int[(g^{ji}\nabla_j\nabla_i v^h - K_i{}^h v^i)v_h + \tfrac{1}{2}(\nabla^j v^i - \nabla^i v^j)(\nabla_j v_i - \nabla_i v_j) + (\nabla_i v^i)^2]\,d\sigma = 0\,, \tag{4.2}$$

where $d\sigma$ denotes the volume element of the space.

Substituting

$$g^{ji}\nabla_j\nabla_i v^h = -K_i{}^h v^i$$

into (4.2) we find

$$\int K_{ji}v^j v^i d\sigma = \int[\tfrac{1}{4}(\nabla^j v^i - \nabla^i v^j)(\nabla_j v_i - \nabla_i v_j) + \tfrac{1}{2}(\nabla_i v^i)^2]\,d\sigma\,, \tag{4.3}$$

from which

$$\int K_{ji}v^j v^i d\sigma \geqq 0\,.$$

If the equality sign occurs in the above inequality, then we have

$$\nabla_j v_i - \nabla_i v_j = 0 \qquad \nabla_i v^i = 0\,,$$

that is, the vector v_i is harmonic. Combining (4.1) and $\nabla_i v^i = 0$, we see following Theorem 1.1 that v^h is a KILLING vector that is

$$\nabla_j v_i + \nabla_i v_j = 0\,.$$

Thus

$$\nabla_j v_i = 0\,,$$

hence we have

Theorem 4.1. *For a geodesic vector field v^h in a compact orientable RIEMANNian space we have*

$$\int K_{ji}v^j v^i d\sigma \geqq 0\,. \tag{4.4}$$

If the equality sign occurs in (4.4), *then the geodesic vector field* v^h *has vanishing covariant derivative.*

As a corollary to this theorem, we have

Theorem 4.2. *If, in a compact orientable* RIEMANN*ian space, the* RICCI *curvature* $K_{ji}v^j v^i$ *is negative definite, there exists no geodesic vector field other than the zero vector and if the* RICCI *curvature is negative semi-definite, then the geodesic vector field has vanishing covariant derivative.*

§ 5. Geodesic vectors in an EINSTEIN space

Let us consider a compact orientable EINSTEIN space with positive curvature scalar, that is, a RIEMANNian space satisfying

$$K_{ji} = cg_{ji}, \tag{5.1}$$

where

$$c = \frac{K}{n} > 0 . \tag{5.2}$$

Then a geodesic vector v^h satisfies

$$g^{ji}\nabla_j\nabla_i v^h + cv^h = 0 \tag{5.3}$$

or

$$g^{ji}\nabla_j\nabla_i v_h + cv_h = 0 . \tag{5.4}$$

We now introduce the following notations. We denote by d the operator which operates to a p-form

$$v : \frac{1}{p!} v_{i_p i_{p-1}\dots i_1} d\xi^{i_p} \wedge d\xi^{i_{p-1}} \wedge \dots \wedge d\xi^{i_1} \tag{5.5}$$

and gives

$$\begin{aligned} dv : \frac{1}{(p+1)!} (\nabla_j v_{i_p i_{p-1}\dots i_1} - \nabla_{i_p} v_{j i_{p-1}\dots i_1} - \dots \\ \dots - \nabla_{i_1} v_{i_p i_{p-1}\dots i_2 j}) d\xi^j \wedge d\xi^{i_p} \wedge \dots \wedge d\xi^{i_1} \end{aligned} \tag{5.6}$$

and by δ the operator which operates to a p-form (5.5) and gives

$$\delta v : \frac{1}{(p-1)!} g^{ji}\nabla_j v_{i i_{p-1}\dots i_1} d\xi^{i_{p-1}} \wedge d\xi^{i_{p-2}} \wedge \dots \wedge d\xi^{i_1} \tag{5.7}$$

and put

$$\Delta = \delta d + d\delta . \tag{5.8}$$

Then we have

$$\Delta dv = d\Delta v , \qquad \Delta\delta v = \delta\Delta v . \tag{5.9}$$

If we define the global inner product of two p-forms

$$a : \frac{1}{p!} a_{i_p i_{p-1}\dots i_1} d\xi^{i_p} \wedge d\xi^{i_{p-1}} \wedge \dots \wedge d\xi^{i_1}$$

$$b : \frac{1}{p!} b_{i_p i_{p-1} \dots i_1} d\xi^{i_p} \wedge d\xi^{i_{p-1}} \wedge \dots \wedge d\xi^{i_1}$$

by

$$(a, b) = \frac{1}{p!} \int a_{i_p i_{p-1} \dots i_1} b^{i_p i_{p-1} \dots i_1} d\sigma , \tag{5.10}$$

then we have

$$(du, v) + (u, \delta v) = 0 , \qquad (\delta u, v) + (u, dv) = 0 \tag{5.11}$$

and

$$(\Delta u, u) + (du, du) + (\delta u, \delta u) = 0 . \tag{5.12}$$

If we put

$$v = v_h d\xi^h ,$$

then we have

$$\Delta v : (g^{ji} \nabla_j \nabla_i v_h - K_h{}^i v_i) d\xi^h . \tag{5.13}$$

Thus (4.2) is equivalent to (5.12).

When v^h is a KILLING, harmonic or geodesic vector, we call $v_h d\xi^h$ a KILLING, harmonic or geodesic form respectively.

Relation (5.13) becomes

$$\Delta v : (g^{ji} \nabla_j \nabla_i v_h - c v_h) d\xi^h \tag{5.14}$$

in case of EINSTEIN space. Thus (5.4) can be written as

$$\Delta v = - 2cv , \tag{5.15}$$

v being a geodesic form, from which

$$\Delta \delta v = - 2c \delta v . \tag{5.16}$$

Thus

Theorem 5.1. *The divergence $\nabla_i v^i$ of a geodesic vector field in an EINSTEIN space is a solution of the equation*

$$\Delta f = - 2cf . \tag{5.17}$$

If $f = \nabla_i v^i = 0$ for geodesic vector field in a compact orientable EINSTEIN space then a geodesic vector is a KILLING vector. Thus we have

Theorem 5.2. *If the equation $\Delta f = - 2cf$ admits no solution other than the zero function in a compact orientable EINSTEIN space, then a geodesic vector is a KILLING vector.*

From (5.16), we find

$$\Delta d\delta v = - 2c d\delta v , \tag{5.18}$$

which shows that $d\delta v$ is again a geodesic form.

Thus if we put

$$p = v + \frac{1}{2c} d\delta v , \tag{5.19}$$

then $p = p_i d\xi^i$ is a geodesic form. From (5.19) we obtain

$$\begin{aligned}\delta p &= \delta v + \frac{1}{2c}\,\delta d \delta v = \delta v + \frac{1}{2c}\,(\delta d + d\delta)\,\delta v \\ &= \delta v + \frac{1}{2c}\,\Delta \delta v = \delta v + \frac{1}{2c}\,(-\,2c\,\delta v) = 0 \end{aligned} \tag{5.20}$$

by virtue of $\delta\delta v = 0$ and (5.16) and consequently p is a KILLING form.

From (5.19) we have

$$v = p + df\,, \tag{5.21}$$

where p is a KILLING form and f is a solution of

$$\Delta f = -\,2cf\,. \tag{5.22}$$

Conversely if p is a KILLING form and f is a solution of (5.22) then

$$v = p + df$$

is a geodesic form. Because

$$\Delta v = \Delta p + \Delta df = \Delta p + d\Delta f\,.$$

But we have

$$\Delta p = -\,2cp\,, \qquad d\Delta f = -\,2cdf$$

and consequently

$$\Delta v = -\,2cp + (-\,2cdf) = -\,2cv\,.$$

Moreover if a geodesic form v is decomposed as

$$v = p + df\,,$$

where p is a KILLING form and f is a solution of $\Delta f = -\,2cf$, then this decomposition is unique. In fact suppose that we had another decomposition

$$v = p' + df'\,,$$

then we have from these two equations

$$p - p' + d(f - f') = 0\,.$$

This equation shows that the KILLING form $p - p'$ is a differential of a scalar and consequently the coefficients $p_i - p'_i$ of $p - p'$ are components of a parallel vector. But an EINSTEIN space with positive scalar curvature does not admit a parallel vector field other than the zero vector. Thus we have

$$p = p'\,,$$

from which

$$d(f - f') = 0$$

and consequently

$$f - f' = \text{constant}.$$

Applying Δ to this equation, we find

$$-2c(f - f') = 0,$$

from which

$$f = f'.$$

Thus we have

Theorem 5.3. *A geodesic vector* v^h *in a compact orientable* EINSTEIN *space is decomposed into the form*

$$v^h = p^h + \nabla^h f, \tag{5.23}$$

where p^h *is a* KILLING *vector and* f *is a solution of* $\Delta f = -2cf$ *and the* KILLING *vector* p^h *and the scalar* f *are uniquely determined.*

§ 6. Vector space of geodesic vector fields

We denote by L the vector space of geodesic vector fields by L_1 the LIE algebra of KILLING vector fields and by L_2 the vector space of gradients of solutions of $\Delta f = -2cf$ in a compact orientable EINSTEIN space: $K_{ji} = cg_{ji}$ where c is positive. Then we have

$$L = L_1 + L_2$$

the plus sign denoting the direct sum.

Now take a KILLING vector p^h and a solution f of $\Delta f = -2cf$ and put

$$f_i = \nabla_i f.$$

Then we have

$$[p, f]_i = p^j \nabla_j f_i - f^j \nabla_j p_i. \tag{6.1}$$

Taking account of $\nabla_j f_i = \nabla_i f_j$ and $\nabla_j p_i = -\nabla_i p_j$, we have from (6.1)

$$[p, f]_i = p^j \nabla_i f_j + f^j \nabla_i p_j = \nabla_i (p^j f_j). \tag{6.2}$$

On the other hand we have

$$\nabla^k \nabla_k (p^j f_j) = (\nabla^k \nabla_k p^j) f_j + 2(\nabla^k p^j)(\nabla_k f_j) + p^j (\nabla^k \nabla_k f_j).$$

Substituting

$$\nabla^k \nabla_k p^j = -cp^j \qquad \text{and} \qquad \nabla^k \nabla_k f_j = -cf_j$$

in this equation and taking account of

$$\nabla^k p^j = -\nabla^j p^k \qquad \text{and} \qquad \nabla_k f_j = \nabla_j f_k$$

we find

$$\nabla^k \nabla_k (p^j f_j) = -2c(p^j f_j) . \tag{6.3}$$

Equations (6.2) and (6.3) show that

$$[L_1, L_2] \subset L_2 .$$

Take next a solution g of $\Delta g = -2cg$ and put

$$g_i = \nabla_i g . \tag{6.4}$$

Then we have

$$\nabla_i [f, g]^i = \nabla_i (f^j \nabla_j g^i - g^j \nabla_j f^i) = f^j \nabla_i \nabla_j g^i - g^j \nabla_i \nabla_j f^i .$$

But

$$\nabla_i \nabla_j g^i - \nabla_j \nabla_i g^i = K_{ji} g^i = c g_j ,$$

from which

$$\nabla_i \nabla_j g^i = -c g_j$$

by virtue of

$$\nabla_i g^i = \nabla_i \nabla^i g = -2cg .$$

Similarly

$$\nabla_i \nabla_j f^i = -c f_j .$$

Thus we have

$$\nabla_i [f, g]^i = 0 .$$

Consequently we have

$$[L_2, L_2] \subset L_3 ,$$

where L_3 is the LIE algebra of the vector fields whose divergences are zero. Thus we have

Theorem 6.1. *In a compact orientable EINSTEIN space with positive curvature scalar we have*

$$L = L_1 + L_2 , \quad [L_1, L_2] \subset L_2 , \quad [L_2, L_2] \subset L_3 ,$$

where L is the vector space of geodesic vector fields, L_1 LIE algebra of KILLING vector fields, L_2 vector space of gradients of the solutions of $\Delta f = -2cf$ and L_3 the LIE algebra of the vector fields whose divergences are zero.

BIBLIOGRAPHY

[1] A. NEWLANDER and L. NIRENBERG, *Complex analytic coordinates in almost complex spaces.* Annals of Math., *65* (1957), 391–404.

[2] K. YANO, *On harmonic and* KILLING *vectors.* Annals of Math., *55* (1952), 38–45.

[3] K. YANO, *The theory of* LIE *derivatives and its applications.* Amsterdam (1957).

[4] K. YANO, *Some integral formulas and their applications.* Michigan J. of Math., *5* (1958), 63–73.

[5] K. YANO and S. BOCHNER, *Curvature and* BETTI *numbers.* Annals of Math. Studies, *32* (1953).

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