A New Method in Fixed Point Theory.

A New Method in Fixed Point Theory[1])

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0. Introduction

BOREL [1] has given a proof of SMITH's homology sphere theorem using the spectral sequence of CARTAN and LERAY. The nature of this spectral sequence makes it necessary to consider the orbit space of the complement of the fixed point set. This causes considerable complication in the proof.

By using the Tate cohomology theory [2 Ch. XII] in place of the usual cohomology theory of groups, it is possible to modify the CARTAN-LERAY sequence in such a way that the cohomology of the orbit space no longer appears in the E_∞ term. The resulting spectral sequence gives an immediate proof of SMITH's theorem, and, in fact, shows that for compact homology spheres, no condition of finite dimensionality is required. The sequence also yields generalizations of some inequalities due to FLOYD [4] and HELLER [5].

It is possible to introduce cup products into the spectral sequence and so obtain results involving the cohomology rings of the space and the fixed point set. My results in this direction are rather fragmentary, however.

It is also possible to define the spectral sequence using cohomology with coefficients in a sheaf. This is done by a trivial generalization of the method used here. Since I have no applications for this generalized sequence, I will give no details.

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1. The algebraic construction

Let M be a cochain complex over the group ring $Z(\pi)$ of a finite group π such that $M^n = 0$ for $n < -1$. Choose a Tate complex W (or complete resolution in the terminology of [2 Ch. XII § 3]) for π, and consider the double complex $Hom_\pi(W, M)$ (cf. [2 Ch. IV § 5]). The first filtration of this complex is regular [2 Ch. XV § 6] and so yields a convergent spectral sequence. In general, it will *not* be true that for fixed p, q and large k, $E_k^{p,q}$ will be equal

[1] This work was done while I was a National Science Foundation fellow. The results presented here are contained in my Princeton Doctoral Thesis [8]. The principal result of this thesis, a determination of the homology of cyclic products, will be published elsewhere.

to $E_\infty^{p,q}$. However, for $k > q + 2$, there is a direct system

$$E_k^{p,q} \to E_{k+1}^{p,q} \to \dots$$

with $E_\infty^{p,q}$ as direct limit [2 Ch. XV § 4]. Obviously, $E_2^{p,q} = \hat{H}^p(\pi, H^q(M))$ where $\hat{H}^p$ denotes the *p-th* Tate cohomology group [2 Ch. XII § 2].

It is easy to see that this spectral sequence is functorial in M, and that equivariantly homotopic cochain maps of M induce the same map of the spectral sequence [2 Ch. XV Prop. 6.1]. This also applies to maps of W and shows that, up to natural isomorphism, the spectral sequence is independent of the choice of W [8 Ch. II]. This fact, however, will not be used in the applications.

In order to compute the E_∞ term in certain cases, I will make use of the following lemma.

Lemma 1.1. *Suppose that* $M^n = 0$ *for large* n *as well as for* $n < -1$. *Suppose also that* $\hat{H}^i(\pi, M^j) = 0$ *for all* i *and* j.

Then, $H(Hom_\pi(W, M)) = 0$.

Proof. The conditions imply that the second filtration is regular [2 Ch XV § 6]. Therefore, it is sufficient to show that the E_1 term of the second spectral sequence is zero. This term, however, is just $\hat{H}^i(\pi, M^j)$.

Corollary 1.1. *Let* $0 \to M' \xrightarrow{i} M \xrightarrow{j} M'' \to 0$ *be an exact sequence of equivariant cochain maps. If* M' *satisfies the conditions of Lemma* 1.1, *then* $j_*: H(Hom_\pi(W, M)) \to H(Hom_\pi(W, M''))$ *is an isomorphism.*

Proof. Consider the exact cohomology sequence of

$$0 \to Hom_\pi(W, M') \to Hom_\pi(W, M) \to Hom_\pi(W, M'') \to 0$$

and apply Lemma 1.1.

Remark 1.1. Note that if π acts trivially on M, the spectral sequence is trivial, that is $E_2 = E_\infty$. This follows from the fact that $Hom_\pi(W, M) = \mathrm{Hom}(W/\pi, M)$ in this case. But W/π is Z-free. Therefore it splits into a direct sum of complexes each of which is zero except in two consecutive dimensions. Thus the spectral sequence is the direct sum of spectral sequences each of which is obviously trivial.

Remark 1.2. Note that multiplication by the order of π annihilates the whole spectral sequence, including $H(\mathrm{Hom}_\pi(W, M))$. To prove this, choose a contracting homotopy S (over Z) for W. Define $D = \Sigma g S g^{-1}$, the sum being taken over all $g \in \pi$. Then D is an equivariant homotopy

between the zero map $W \to W$ and the operation of multiplying by the order of π.

2. Simplicial complexes

All results will be stated for the relative cochains $C^*(K, L; G)$ of a simplicial pair (K, L) with coefficients in a group G. All cochains will be alternating unless specified explicitly. For convenience, I will define a symbol «$(K, -1)$» by letting $C^*(K, -1; G)$ be the augmented cochain complex of K, obtained by adjoining a cell of dimension -1 to K in the usual way. This will make it unnecessary to state each result twice, once for ordinary cochains and once for augmented cochains.

Suppose a finite group π acts simplicially on a complex K from the right. Suppose also that L is a π-invariant subcomplex of K. Then each cochain group $C^q(K, L; G)$ is a left π-module in the obvious way. The construction of § 1, applied with $M = C^*(K, L; G)$ yields a spectral sequence whose terms will be denoted by $E_k^{p,q}(K, L; G)$. The term $H^n(\mathrm{Hom}_\pi(W, C^*(K, L; G)))$ will be abbreviated to $J^n(K, L; G)$. It follows from § 1 that

$$E_2^{p,q} = \hat{H}^p(\pi, H^q(K, L; G))$$

The term $J^n(K, L; G)$ is much more complicated and it is necessary to impose rather drastic conditions in order to be able to compute it explicitly.

Lemma 2.1. *Assume that for every simplex e of K, either e is left pointwise fixed by all elements of π, or else that the images of e under π are all distinct. Let (K^π, L^π) be the subcomplex of (K, L) consisting of those points fixed under π. Suppose $K - (K^\pi \cup L)$ is finite dimensional. Then the inclusion map $i : (K^\pi, L^\pi) \to (K, L)$ induces an isomorphism $i^* : J^*(K, L; G) \to J^*(K^\pi, L^\pi; G)$.*

Proof. Consider the exact sequence

$$0 \to C^*(K, K^\pi \cup L) \to C^*(K, L) \to C^*(K^\pi, L^\pi) \to 0$$

The desired result will follow from Corollary 1.1 if we can show that $C^*(K, K^\pi \cup L)$ satisfies the conditions of Lemma 1.1. The first condition of this lemma follows immediately from the finite dimensionality of $K - (K^\pi \cup L)$. Now, it is easy to see that the group π acts freely on the simplexes of $K - (K^\pi \cup L)$. Therefore, the chain complex $C(K, K^\pi \cup L)$ is π-free and hence is weakly projective. Consequently [2 Ch. X Prop. 8.1], $C^*(K, K^\pi \cup L)$ is weakly injective. Thus the second condition of Lemma 1.1 is satisfied [2 Ch. XII Prop. 2.2]. At this point, the use of the Tate cohomology theory is really necessary.

Remark 2.1. The modules $J^n(K, L; G)$ and $J^n(K^\pi, L^\pi; G)$ have natural filtrations. It is very important to remember that i^* need not be an isomorphism of filtered modules even though it is a map of filtered modules and an isomorphism of ordinary modules. In other words, although i^* maps $F^p J^n(K, L; G)$ into $F^p J^n(K^\pi, L^\pi; G)$, it is not necessarily an isomorphism of these submodules.

3. Covering lemmas

The spectral sequence for topological spaces can be defined using only primitive coverings [6] and not special coverings [6]. In this way, the difficult proof of the existence of arbitrarily fine special coverings can be avoided. In this section, I will state the definitions and lemmas which will be required. Many of these can be found in [6] for the case in which π is cyclic of prime order. I will assume throughout that π is a finite group acting on a HAUSDORFF space X from the right, and that A is a π-invariant subspace of X (or else the symbol «-1» defined in § 2). All coverings are to be open. All trivial proofs will be omitted.

Definition 3.1. A π-invariant covering $\mathfrak{U}$ of (X, A) is an indexed covering $\mathfrak{U} = \{U_\alpha \mid \alpha \in (I, I_0)\}$ [3 Ch. X] together with a given action of π on the index set I (from the right) such that I_0 (the index set for A) is π-stable, and such that $U_{\alpha t} = U_\alpha \cdot t$ for all $t \in \pi$.

Obviously, π acts simplicially on the nerve of such a covering.

Definition 3.2. A π-invariant covering $\mathfrak{U} = \{U_\alpha \mid \alpha \in (I, I_0)\}$ is primitive if $U_\alpha \cap U_{\alpha t} = \emptyset$ whenever $t \in \pi$ and $\alpha \in I$ are such that $\alpha \neq \alpha t$.

Lemma 3.1. *Every covering of* (X, A) *has a primitive refinement* (since X is HAUSDORFF).

Lemma 3.2. *Let* $\mathfrak{U}$ *be a primitive covering of* (X, A). *Let* $\mathfrak{V}$ *be a* π*-invariant covering which refines* $\mathfrak{U}$. *Then there is a* π*-equivariant projection from* $\mathfrak{V}$ *to* $\mathfrak{U}$.

Lemma 3.3. *Let* $\mathfrak{U}$ *and* $\mathfrak{V}$ *be primitive coverings of* (X, A) *such that* $\mathfrak{V}$ *refines* $\mathfrak{U}$. *Then any two* π*-equivariant projections from* $\mathfrak{V}$ *to* $\mathfrak{U}$ *are* π*-equivariantly chain homotopic.*

Proof. The index set of $\mathfrak{V}$ falls into disjoint orbits under the action of π. Choose a simple ordering for the set of orbits. This induces a quasi-ordering of the indices themselves. Since $\mathfrak{V}$ is primitive, the vertices of any simplex of $\mathfrak{V}$ will be simply ordered by this quasi-ordering. Let f and g be the two projections. Let $(v_0, v_1, \ldots, v_n)$ be a simplex of $\mathfrak{V}$ with the ordering of the

vertices chosen so that $v_0 < v_1 < \ldots < v_n$. Define

$$D(v_0, \ldots, v_n) = \Sigma_{i=0}^{n} (-1)^i (fv_0, \ldots, fv_i, gv_i, \ldots, gv_n)$$

It is easy to check that D is the required π-equivariant chain homotopy.

The spectral sequence for the pair (X, A) can now be defined as follows. There is a spectral sequence associated with the nerve of each primitive covering. If one such covering refines another, Lemmas 3.2 and 3.3 show that there is a unique map of the corresponding spectral sequences. We define the spectral sequence of (X, A) to be the direct limit of these spectral sequences. It will be denoted by the same symbols used to denote the sequence for complexes. Since taking direct limits preserves exactness, the resulting sequence has all the usual properties of a spectral sequence. Since taking direct limits commutes with taking Tate cohomology, we have (using Lemma 3.1)

$$E_2^{p,q}(X, A; G) = \hat{H}^p(\pi, H^q(X, A; G))$$

Lemma 3.4. *If $f : (X, A) \to (X', A')$ is a π-equivariant map and $\mathfrak{U}$ is a π-invariant covering of (X', A'), then $f^{-1}(\mathfrak{U})$ is also π-invariant. If $\mathfrak{U}$ is primitive, so is $f^{-1}(\mathfrak{U})$.*

It follows from Lemma 3.4 that a π-equivariant map of spaces induces a homomorphism of the corresponding spectral sequences.

Lemma 3.5. *Suppose X has the property that every covering of X has a finite dimensional refinement. Then every covering of (X, A) has a finite dimensional primitive refinement.*

Proof. Let $\mathfrak{U}$ be any covering of X. We can assume that $\mathfrak{U}$ is primitive by Lemma 3.1. Take any finite dimensional covering $\mathfrak{V}$ refining $\mathfrak{U}$. Take all transforms of sets of $\mathfrak{V}$ by elements of π. This gives a π-invariant finite dimensional refinement $\mathfrak{W}$ of $\mathfrak{U}$. Choose an equivariant projection f from $\mathfrak{W}$ to $\mathfrak{U}$ (by Lemma 3.2) and amalgamate sets of $\mathfrak{W}$ whose indices have the same projection under f. This gives the required covering.

Lemma 3.6. *Suppose π acts on X in such a way that any point of X is either left fixed by all elements of π or else is such that its transforms under π are all distinct. Then the nerve of any primitive covering of (X, A) satisfies the first condition of Lemma 2.1.*

Let X^π and A^π be the subsets of X and A consisting of those points left fixed by all elements of π.

Lemma 3.7. *The coverings of (X^π, A^π) by invariant sets of primitive coverings of (X, A) are cofinal in the directed set of coverings of (X^π, A^π) by sets open in X.*

Lemma 3.8. *Let X be a paracompact HAUSDORFF space. Let Y and B be closed subsets of X such that $B \subset Y$. Then there are arbitrarily fine coverings $\mathfrak{U}$ of (Y, B) by sets open in X such that $\mathfrak{U}$ and $\mathfrak{U} \mid Y$ have the same nerve. Here, $\mathfrak{U} \mid Y$ means the covering of (Y, B) formed by the intersections of Y with the sets of $\mathfrak{U}$.*

Proof. The proof is almost exactly the same as that of [3 Ch. X Lemma 3.6]. The only changes required are the replacement of finite coverings by locally finite coverings and the replacement of the finite inductions by transfinite inductions.

Theorem 3.1. *Let X be a paracompact HAUSDORFF space satisfying the condition that every covering of X has a finite dimensional refinement.*

Let π be a finite group acting on X from the right in such a way that any point of X is either left fixed by all elements of π or else is such that all its transforms under π are distinct.

Let A be a closed, π-stable subset of X and let X^π, A^π be the sets of fixed points in X and A respectively.

Then the inclusion $i:(X^\pi, A^\pi) \to (X, A)$ induces an isomorphism

$$i^* : J^*(X, A; G) \to J^*(X^\pi, A^\pi; G)$$

Proof. By Lemmas 3.5 and 3.6, we can apply Lemma 2.1 to a cofinal system of primitive coverings. This shows that $J^*(X, A; G)$ is isomorphic to the direct limit of groups of the form $J^*(K^\pi, L^\pi; G)$ where (K, L) is the nerve of a primitive covering. By Lemmas 3.7 and 3.8, this direct limit is just $J^*(X^\pi, A^\pi; G)$. Lemma 3.8 is needed here because $J^*(X^\pi, A^\pi; G)$ is defined by coverings of (X^π, A^π), while the invariant subsets of primitive coverings of X are subsets of X, not X^π.

Remark 3.1. The condition about finite dimensional refinements is obviously satisfied if X is compact.

4. Applications

In the applications, except in Theorem 4.8, I will always assume that π is a cyclic group of prime order p. For such a group, the condition of Theorem 3.1 concerning the action of π is always fulfilled.

This restriction on π also implies (by Remark 1.2) that all terms of the spectral sequence are vector spaces over Z_p. Therefore, we can speak of their dimensions over Z_p. The following theorem holds even if these dimensions are infinite provided we interpret them as transfinite cardinals.

Theorem 4.1. *Let (X, A) and π satisfy the conditions of Theorem 3.1 with π cyclic of prime order p.*

Then, for any integers k and n, and any coefficient group

$$\sum_{\substack{j \geqslant k \\ i+j=n}} \dim E_2^{i,j}(X^\pi, A^\pi) \leqslant \sum_{\substack{j \geqslant k \\ i+j=n}} \dim E_2^{i,j}(X, A)$$

Proof. Since π acts trivially on X^π, we have $E_2^{i,j}(X^\pi, A^\pi) = E_\infty^{i,j}(X^\pi, \mathbf{A}^\pi)$. Furthermore, since i^* is an isomorphism by Theorem 3.1, the map

$$\frac{J^n(X, A)}{F^{n-k+1}J^n(X, A)} \to \frac{J^n(X^\pi, A^\pi)}{F^{n-k+1}J^n(X^\pi, A^\pi)}$$

is an epimorphism. Using these facts and the trivial inequality $\dim E_\infty^{i,j} \leqslant \dim E_2^{i,j}$ we get

$$\Sigma \dim E_2^{i,j}(X^\pi, A^\pi) = \Sigma \dim E_\infty^{i,j}(X^\pi, A^\pi)$$

$$= \dim \frac{J^n(X^\pi, A^\pi)}{F^{n-k+1}J^n(X^\pi, A^\pi)} \leqslant \dim \frac{J^n(X, A)}{F^{n-k+1}J^n(X, A)}$$

$$= \Sigma \dim E_\infty^{i,j}(X, A) \leqslant \Sigma \dim E_2^{i,j}(X, A).$$

Corollary 4.1. *Let (X, A) and π be as in Theorem 4.1. Then, for all n and k,*

$$\sum_{j \geqslant k} \dim H^j(X^\pi, A^\pi; Z_p) \leqslant \sum_{\substack{j \geqslant k \\ i+j=n}} \dim \hat{H}^i(\pi, H^j(X, A; Z_p))$$

This follows from the universal coefficient theorem and the fact that $\hat{H}^i(\pi, Z_p) = Z_p$ for all i.

The inequalities of Floyd [4] follow immediately from Corollary 4.1 and the trivial fact that $\dim \hat{H}^i(\pi, M) \leqslant \dim M$ for all Z_p-modules M (provided π is cyclic). I will not restate these inequalities here. They may be obtained from Corollary 4.1 by omitting the symbol «$H^i(\pi)$».

We can now easily prove the theorems of Smith. For example, if $H^j(X, A; Z_p) = 0$ for all j, Corollary 4.1 shows immediately that $H^j(X^\pi, A^\pi; Z_p) = 0$ for all j. The homology sphere theorem requires an additional argument.

Theorem 4.2. *Let (X, A) and π satisfy the conditions of Theorem 3.1 with π cyclic of prime order p. Suppose there is an integer n such that $H^i(X, A; Z_p) = 0$ for $i \neq n$ and $H^n(X, A; Z_p) = Z_p$.*

Then there is an integer r with $-1 \leqslant r \leqslant n$ such that $H^i(X^\pi, A^\pi; Z_p) = 0$ for $i \neq r$ and $H^r(X^\pi, A^\pi; Z_p) = Z_p$.

Furthermore, $r = -1$ can occur only if A is the symbol -1 of § 2.

Proof. The last part is trivial since $H^{-1}(X^\pi, A^\pi) \neq 0$ can only occur if $A^\pi = -1$, i.e. if $A = -1$.

Suppose now that the conclusion is false. By Corollary 4.1, the only other

possibility is that $H^i(X^\pi, A^\pi; Z_p) = 0$ for all i. The spectral sequence for (X^π, A^π) now shows that $J^i(X^\pi, A^\pi; Z_p) = 0$ for all i. Therefore, by Theorem 3.1, $J^i(X, A; Z_p) = 0$ for all i and so, $E_\infty(X, A; Z_p) = 0$.

Now, the action of π on $H^i(X, A; Z_p)$ must be trivial since π cannot act non-trivially on Z_p. Therefore, we can compute $E_2^{i,j}(X, A; Z_p)$ and show that it is 0 for $j \neq n$ and Z_p for $j = n$. This implies $E_2 = E_\infty$, contradicting the previous conclusion that $E_\infty = 0$.

In order to apply Theorem 4.1 using integral coefficients, we need the following lemma.

Lemma 4.1. *Suppose π acts trivially on the integral cohomology groups of (X, A). Then*

$$\sum_{\substack{j \geq k \\ i+j=k}} \dim E_2^{i,j}(X, A; Z) = \sum_{\substack{j \geq k \\ j-k \text{ even}}} \dim H^j(X, A; Z_p)$$

Proof. Let $H^j = H^j(X, A; Z)$. The universal coefficient theorem shows that

$$\dim H^j(X, A; Z_p) = \dim H^j \otimes Z_p + \dim \operatorname{Tor}(H^{j+1}, Z_p).$$

Since π acts trivially on H^j, we can use the universal coefficient theorem and the known values of $\hat{H}^j(\pi, Z)$ to get

$$E_2^{i,j}(X, A; Z) = H^j \otimes Z_p \quad \text{for i even}$$
$$E_2^{i,j}(X, A; Z) = \operatorname{Tor}(H^j, Z_p) \quad \text{for i odd.}$$

Therefore, if j–k is even,

$$\dim H^j(X, A; Z_p) = \dim E_2^{k-j,j} + \dim E_2^{k-j-1,j+1}$$

Summing on j now gives the required result.

Corollary 4.2. *Let (X, A) and π be as in Theorem 4.1. For all integers k,*

$$\sum_{\substack{j \geq k \\ j-k \text{ even}}} \dim H^j(X^\pi, A^\pi; Z_p) \leq \sum_{j \geq k} \dim \hat{H}^{k-j}(\pi, H^j(X, A; Z))$$

This follows immediately from Theorem 4.1 and Lemma 4.1 applied to the pair (X^π, A^π). By setting $k = 0$ and $k = -1$ in this corollary, we get the inequalities of HELLER [5].

Corollary 4.3. *Let (X, A) and π be as in Theorem 4.1. Suppose π acts trivially on the integral cohomology of (X, A). Then, for all k,*

$$\sum_{\substack{j \geq k \\ j-k \text{ even}}} \dim H^j(X^\pi, A^\pi; Z_p) \leq \sum_{\substack{j \geq k \\ j-k \text{ even}}} \dim H^j(X, A; Z_p)$$

HELLER [5] has defined a generalization of the EULER characteristic. Let

$$\chi_\pi^+(X, A) = \Sigma_{i=0}^\infty \dim \hat{H}^{n-i}(\pi, H^i(X, A; Z)) \quad \text{for } n \text{ even}$$

$$\chi_\pi^-(X, A) = \Sigma_{i=0}^\infty \dim \hat{H}^{n-i}(\pi, H^i(X, A; Z)) \quad \text{for } n \text{ odd.}$$

Because of the periodicity of the Tate cohomology of a cyclic group [2 Ch. XII § 7], the values of χ_π^+ and χ_π^- are independent of the choice of n.

If $\chi_\pi^+(X, A)$ and $\chi_\pi^-(X, A)$ are both finite, the HELLER characteristic of (X, A) is defined to be

$$\chi_\pi(X, A) = \chi_\pi^+(X, A) - \chi_\pi^-(X, A).$$

The next theorem was proved by HELLER [5] for finite dimensional complexes. The use of the spectral sequence gives a simple proof for topological spaces.

Theorem 4.3. *Let (X, A) and π satisfy the conditions of Theorem 3.1 with π cyclic of prime order p.*

If $\chi_\pi^+(X, A)$ and $\chi_\pi^-(X, A)$ are both finite, then $\chi_\pi(X, A) = \chi(X^\pi, A^\pi; Z_p)$, the ordinary EULER characteristic of (X^π, A^π).

Proof. The existence of $\chi(X^\pi, A^\pi; Z_p)$ follows from Corollary 4.2 with $k = 0$ and $k = -1$.

We can choose the Tate complex W to be periodic of period 2 [2 Ch. XII § 7]. Therefore, the spectral sequence $E_k^{i,j}$ will have period 2 in i (i.e. it will be unchanged if i is replaced by $i + 2$).

Let

$$E_k^n = \sum_{i+j=n} E_k^{i,j}$$

These complexes have period 2 in n. Therefore, a trivial argument shows that

$$\dim E_k^m - \dim E_k^{m+1} = \dim E_{k+1}^m - \dim E_{k+1}^{m+1}.$$

Since χ_π^+ and χ_π^- are finite, there is some N such that $E_2^{i,j} = 0$ for $j > N$. Because of the periodicity in i, we can assume N is independent of i. Therefore, the spectral sequence terminates in a finite number of steps and induction shows that

$$\begin{aligned}
&\dim E_2^m(X, A; Z) - \dim E_2^{m+1}(X, A; Z) = \\
&\dim E_\infty^m(X, A; Z) - \dim E_\infty^{m+1}(X, A; Z) = \\
&\dim J^m(X, A; Z) - \dim J^{m+1}(X, A; Z)
\end{aligned}$$

By Theorem 3.1, this is equal to the corresponding expression with (X^π, A^π) substituted for (X, A). If m is any even integer, the left hand side is just $\chi_\pi(X, A)$. If (X^π, A^π) is substituted for (X, A), Lemma 4.1 shows that the left hand side becomes $\chi(X^\pi, A^\pi; Z_p)$.

Corollary 4.4. *Let (X, A) and π be as in Theorem 4.3. Suppose π acts trivially on the integral cohomology of (X, A). Then $\chi(X^\pi, A^\pi; Z_p) = \chi(X, A; Z_p)$.*

This follows immediately from the theorem and from Lemma 4.1 which shows that $\chi_\pi(X, A) = \chi(X, A; Z_p)$ in this case.

Corollaries 4.3 and 4.4 give fairly complete information regarding the relations between the betti numbers of the space and of the fixed point set in the case where π acts trivially on $H^*(X, A; Z)$. This statement is made precise by the next theorem.

Theorem 4.4. *Suppose we are given two sequences of non-negative integers $a_0, a_1, \ldots$ and $b_0, b_1, \ldots$ such that $a_n = b_n = 0$ if n is large enough. Suppose also that*

$$\text{(i)} \quad \textit{For all } n, \ \sum_{\substack{j \geqslant n \\ j-n \text{ even}}} b_j \leqslant \sum_{\substack{j \geqslant n \\ j-n \text{ even}}} a_j$$

$$\text{(ii)} \quad \Sigma_{n=0}^{\infty} (-1)^n a_n = \Sigma_{n=0}^{\infty} (-1)^n b_n$$

Then there are spaces X and A satisfying the conditions of Corollaries 4.3 and 4.4 such that

$$a_n = \dim H^n(X, A; Z_p) \ \textit{and} \ b_n = \dim H^n(X^\pi, A^\pi; Z_p).$$

If in addition, $a_0 > 0$ and $b_0 > 0$, we can even let A be empty.

Proof. The required space will be a union, with common base point, of certain elementary spaces. If $a_0 > 0$ and $b_0 > 0$, then A is to be empty. Otherwise, A is to be the base point.

The elementary spaces are as follows:

(1) A sphere S^n with π acting as a rotation group such that the fixed point set is a non-empty subsphere S^r with $n-r$ even. The base point is chosen in S^r.

(2) Let π act on S^{2m+1} without fixed points by rotation. Let π act trivially on S^n. Then π acts on $S^{2m+1} \times S^n$. Let e be a point of S^n and attach a cone over $S^{2m+1} \times e$ to $S^{2m+1} \times S^n$. The result will be the second type of elementary space. The group π acts on the cone in the obvious way, keeping the vertex fixed. The base point is taken to be the vertex of the cone.

The theorem is now easily proved by induction on Σa_n.

Using Theorem 4.3, we can give a simple proof of a recent result of LIAO [7]. This theorem concerns a space which is a cohomology n-sphere over Z_p and which has finitely generated integral cohomology groups. The universal coefficient theorem shows that such a space is an integral cohomology n-sphere modulo non-p-torsion. The action of π is said to preserve orientation if π acts trivially on the n-th co-betti group (i.e. H^n with all torsion factored out). Otherwise π is said to reverse orientation.

Theorem 4.5. (LIAO) *Let X, A, n, r be as in Theorem 4.2. Suppose the integral cohomology groups of (X, A) are finitely generated. If π preserves orientation, $n-r$ is even. If π reverses orientation, $n-r$ is odd.*

Proof. It is sufficient to compute the EULER characteristic of the fixed point set. This can be done using Theorem 4.3 if we can compute $\hat{H}^i(\pi, H^j(X, A; Z))$. Since we can ignore non-p-torsion in computing Tate cohomology over a group of order p, we can replace $H^j(X, A; Z)$ by the corresponding co-betti group. The result now follows from known computations of Tate cohomology [2 Ch. XII § 7].

It is also possible to deduce local properties of X^π from those of X by using the spectral sequence. The theorem requires a rather artificial hypothesis if X is not finite dimensional.

Theorem 4.6. *Let X and π satisfy the conditions of Theorem 3.1 with π cyclic of prime order p and $A = \emptyset$.*

Let $x \in X^\pi$ be such that x has arbitrarily small closed π-invariant neighborhoods U with the property that $H^n(U; Z_p) = 0$ for $n \geqslant N$, where N is a large integer depending on U.

If X is cohomology locally connected over Z_p at x, then so is X^π.

Proof. Let U be any neighborhood of x having the property mentioned in the hypothesis. Choose closed π-invariant neighborhoods $V_0 \subset V_1 \subset \ldots \subset V_{N+1} = U$ such that the inclusions $V_i \subset V_{i+1}$ induce the zero map on the augmented cohomology groups. The induced map of spectral sequences is then zero on E_2 and hence is zero on E_∞. Therefore, it increases the filtration of all elements of $J^*(V_{i+1}; Z_p)$ by at least 1. Since $H^n(U; Z_p) = 0$ for $n > N$, it follows that all elements of $J^n(U; Z_p)$ have filtration $\geqslant n - N$. Therefore $J^*(U; Z_p) \to J^*(V_0; Z_p)$ is zero. This shows that $H^*(U^\pi; Z_p) \to H^*(V_0^\pi; Z_p)$ is zero.

I will conclude this section with two applications which are not directly connected with fixed point theory.

Theorem 4.7. *Let X be a cohomology n-sphere over Z_p. Let π be cyclic of prime order p and act on X.*

Then there is a map $d: H^n(X; Z_p) \to H^0(X; Z_p)$ natural with respect to equivariant maps of X into other cohomology n-spheres on which π acts.

If π acts freely on X and X satisfies the conditions of Theorem 3.1, this map d is an isomorphism.

Proof. The map d is just the operator d^{n+1} of the spectral sequence. Since π cannot act non-trivially on Z_p, we have $E_2^{i,0} \approx H^0(X; Z_p)$ and $E_2^{i,n}$

$\approx H^n(X; Z_p)$ both isomorphisms being natural. If $X^\pi = 0$ and Theorem 3.1 applies, we have $E_\infty = 0$. Therefore d must be an isomorphism in this case.

Corollary 4.5. *Let X and Y be cohomology n-spheres over Z_p. Let π be cyclic of prime order p. Let π act on X and Y and let $f: X \to Y$ be π-equivariant. Suppose that Y satisfies the conditions of Theorem 3.1 and that π acts freely on Y. Then f induces an isomorphism $f^*: H^n(Y; Z_p) \to H^n(X; Z_p)$.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc} H^n(Y) & \to & H^n(X) \\ d\downarrow & & d\downarrow \\ H^0(Y) & \to & H^0(X) \end{array}$$

The result follows immediately from the fact that d is an isomorphism for Y, f^* is an isomorphism in dimension 0, and $H^n(Y; Z_p)$ and $H^n(X; Z_p)$ are both isomorphic to Z_p.

The BORSUK-ULAM theorem is an immediate consequence of Corollary 4.5 with $p = 2$.

We can also give a slight generalization of the results of [2 Ch. XVI § 9 Appl. 4]. In the following theorem we allow π to be non-cyclic.

Theorem 4.8. *Let X satisfy the conditions of Theorem 3.1. Let π act freely on X. If X is an integral cohomology n-sphere, then π has periodic cohomology with period $n+1$ if n is odd. If n is even, $\pi = 1$ or Z_2.*

Proof. Since $X^\pi = \emptyset$, $E_\infty = 0$. Therefore d^{n+1} must be an isomorphism $d^{n+1}: \hat{H}^i(\pi, H^n(X)) \to \hat{H}^{i+n+1}(\pi, Z)$ for all i. The same must be true for all subgroups of π because these also act freely on X.

If n is even, this result applied to the cyclic subgroups of π shows that every element of π except 1 must reverse orientation. Therefore $\pi = 1$ or Z_2.

If n is odd, the result applied to the cyclic subgroups of π shows that all elements of π preserve orientation.

Now, applying the result to π itself gives $\hat{H}^{n+1}(\pi, Z) \approx \hat{H}^0(\pi, Z) \approx Z_h$ where h is the order of π. The conclusion of the theorem then follows from [2 Ch. XII Prop. 11.1].

5. Products

Let φ be a diagonal map for the Tate complex W [2 Ch. XII § 4]. If M and N are cochain complexes, the map

$$\mathrm{Hom}_\pi(W_i, M^j) \otimes \mathrm{Hom}_\pi(W_k, N^l) \to \mathrm{Hom}_\pi(W_{i+k}, M^j \otimes N^l)$$

defined by $f \cup g = (-1)^{jk}(f \otimes g)\varphi_{i,k}$ gives a map of double complexes. Therefore, it induces a map of spectral sequences

$$E(\mathrm{Hom}_\pi(W, M)) \otimes E(\mathrm{Hom}_\pi(W, N)) \to E(\mathrm{Hom}_\pi(W, M \otimes N))$$

by [2 Ch. XII Ex. 1, 2, 4]. On the E_2 terms, the products

$$\hat{H}^i(\pi, H^j(M)) \otimes \hat{H}^k(\pi, H^l(N)) \to \hat{H}^{i+k}(\pi, H^{j+l}(M \otimes N))$$

is obtained by composing the cup product for Tate cohomology with the coefficient homomorphism

$$\alpha : H^j(M) \otimes H^l(N) \to H^{k+l}(M \otimes N)$$

and prefixing the sign $(-1)^{jk}$.

Suppose now that π acts simplicially on a simplicial pair (K, L). Let $C^*(K, L; G)$ be the group of ordered cochains [3 Ch. VI § 2] with values in a group G. Then the usual ALEXANDER-ČECH-WHITNEY formula gives a π-equivariant map

$$C^*(K, L; G) \otimes C^*(K, L; G) \to C^*(K, L; G \otimes G')$$

This map then yields products in the spectral sequence of (K, L) based on *ordered* cochains. We must now show that this sequence agrees with the one previously considered which was based on *alternating* cochains. This is done using the following lemmas.

Lemma 5.1. *Let K and K' be simplicial complexes on which π acts simplicially. Suppose that whenever a simplex σ of K is mapped onto itself by an element $t \in \pi$, then σ is left pointwise fixed by t. Let C be an acyclic carrier from K to K' which is π-equivariant i.e. $C(t\sigma) = tC(\sigma)$ for all $t \in \pi$. Assume finally that if $t\sigma = \sigma$, then t leaves $C(\sigma)$ pointwise fixed.*

Then the conclusions of Theorem 5.7 of [3 Ch. VI] *hold for C if all maps and homotopies are required to be π-equivariant and the subcomplex L of K is required to be π-stable.*

Proof. This is proved in exactly the same way as Theorem 5.7 of [3 Ch. VI] except that whenever the value $f(\sigma)$ of a map or homotopy has been found, we immediately set $f(t\sigma) = tf(\sigma)$ for all $t \in \pi$. The hypothesis insures that this f is carried by C and if $t\sigma = \sigma$, then $tf(\sigma) = f(\sigma)$.

Lemma 5.2. *Suppose π acts simplicially on a simplicial complex K in such a way that if v is a vertex of K and $t \in \pi$ is such that $tv \neq v$, then tv and v do not belong to a common simplex of K.*

Then the natural map of ordered chains of K into alternating chains of K is a π-equivariant homotopy equivalence.

Proof. The condition imposed on K is easily seen to be equivalent to that imposed on K in Lemma 5.1. Quasi-order the vertices of K as in the proof of Lemma 3.3. Then follow the proof of Theorem 6.10 of [3 Ch. VI] using the above Lemma 5.1 in place of Theorem 5.7 of [3 Ch. VI].

The conditions of Lemma 5.2 are obviously satisfied by the nerves of primitive coverings. Therefore, the spectral sequence of a topological space obtained by using ordered cochains is isomorphic to the one obtained by using alternating cochains. We can now define products in the spectral sequence based on alternating cochains by using this isomorphism.

On the E_2 terms, this product

$$\hat{H}^i(\pi, H^j(X, A; G)) \otimes \hat{H}^k(\pi, H^l(X, A; G')) \to \hat{H}^{i+k}(\pi, H^{j+l}(X, A; G \otimes G'))$$

is obtained by composing the cup product for Tate cohomology with the cup product for (X, A) and prefixing the sign $(-1)^{jk}$. This shows that the products on E_2 (and so on E_k for $k \geqslant 2$) are associative, (skew) commutative, and independent of the choice of the diagonal maps. The corresponding result for $J^*(X, A)$ may be proved by showing that, up to equivariant homotopy, a diagonal map for W is unique, associative, and (skew) commutative [8]. The analogous result for the diagonal map of a simplicial complex follows immediately from Lemma 5.1. If, however, $G = G' = Z_p$, we can avoid this rather long proof by using the following lemma.

Lemma 5.3. *The natural isomorphism*

$$\alpha : \hat{H}^*(\pi, Z_p) \otimes H^*(X^\pi, A^\pi; Z_p) \to J^*(X^\pi, A^\pi; Z_p)$$

preserves products, the products on the left hand side being the usual cup products.

Proof. This follows immediately from the KUNNETH theorem and the fact that if π acts trivially on a cochain complex M, then

$$\mathrm{Hom}_\pi(W, M) \approx \mathrm{Hom}(W/\pi, M) \approx \mathrm{Hom}(W/\pi, Z_p) \otimes M .$$

The last isomorphism follows from the fact that W is free and finitely generated.

6. An application of products

In [5], there is a theorem to the effect that, under certain conditions, if X has the cohomology groups of a product of spheres then so does X^π. Unfortunately, this theorem is false, many counterexamples being given by Theorem 4.4. The word «product» should, of course, be replaced by the word «wedge» to make the theorem true.

The following theorem gives a simple case in which we can conclude that

the fixed point set has the cohomology of a product of spheres. Surprisingly enough, even this theorem becomes false if the condition that X^π be connected is omitted.

Theorem 6.1. *Let X and π satisfy the conditions of Theorem 3.1 with $A = \emptyset$ and π cyclic of prime order p. Assume that π acts trivially on $H^*(X; Z)$ and on $H^*(X; Z_p)$. Assume also that $H^*(X; Z_p)$ is isomorphic to $H^*(S^{2m} \times S^{2n}; Z_p)$ as a ring. Here $2m$ and $2n$ are any even positive integers.*

If X^π is connected, then $H^*(X^\pi; Z_p)$ is isomorphic as a group to $H^*(S^{2r} \times S^{2s}; Z_p)$ where $2r$ and $2s$ are even and positive and $r \leqslant m, s \leqslant n$.

Furthermore, if $r \neq s$, then $H^*(X^\pi; Z_p)$ is isomorphic to $H^*(S^{2r} \times S^{2s}; Z_p)$ as a ring.

Proof. Suppose we have proved all except the inequalities $r \leqslant m$, $s \leqslant n$. We can assume (by changing notation) that $m \leqslant n$, $r \leqslant s$. The required inequalities then follow immediately from Corollary 4.1.

By Corollaries 4.3 and 4.4, $H^i(X^\pi; Z_p) = 0$ for i odd and $\chi(X^\pi; Z_p) = \chi(X; Z_p) = 4$. Therefore, $H^*(X^\pi; Z_p)$ has 4 linearily independent elements. Since X^π is connected, $H^0(X^\pi; Z_p) = Z_p$.

Now, let 1, x, y, z be a (homogeneous) base for $H^*(X^\pi; Z_p)$. Suppose some product, say xy, of distinct basis elements (excluding 1) is non-zero. Since $\dim x, \dim y > 0$, we have $\dim xy > \dim x, \dim y$, so we can choose $z = xy$. The theorem is obviously true in this case.

Therefore, we may assume that either all products are zero or that $xy = xz = yz = 0$ but some square, say x^2, is non-zero. In this latter case, we can choose $y = x^2$. We may also assume that in this case, $\dim z \neq \dim x$ otherwise the theorem will again be true. Therefore, we can suppose $z^2 = 0$. Clearly, $y^2 = 0$.

Consider the spectral sequence with Z_p as coefficient group. Since π acts trivially on $H^*(X; Z_p)$, we have

$$E_2(X; Z_p) = \hat{H}^*(\pi, Z_p) \otimes H^*(X; Z_p)$$

In any given total dimension, there are 4 linearly independent elements in E_2. Since the same is true of $J^*(X; Z_p) = J^*(X^\pi; Z_p)$, we must have $E_2 = E_\infty$.

Now, for p odd, $\hat{H}^*(\pi, Z_p)$ is generated as a ring by an element $v \in \hat{H}^1(\pi, Z_p)$ and elements $u \in \hat{H}^2(\pi, Z_p)$ and $u^{-1} \in \hat{H}^{-2}(\pi, Z_p)$ with relations $v^2 = 0$ and $uu^{-1} = 1$. For $p = 2$, $\hat{H}^*(\pi, Z_2)$ is generated by elements $v \in \hat{H}^1(\pi, Z_2)$ and $v^{-1} \in \hat{H}^{-1}(\pi, Z_2)$ with relation $vv^{-1} = 1$. In this case, we set $u = v^2$ and $u^{-1} = v^{-2}$. These results follow from the results of [2 Ch. XII § 7].

The ring $H^*(X; Z_p)$ is generated by elements $a \in H^{2m}$ and $b \in H^{2n}$ with relations $a^2 = b^2 = 0$. Therefore, the ring $E_2(X; Z_p)$ is generated by products of u, v, u^{-1}, v^{-1}, a and b. I will identify u with $u \otimes 1$, a with $1 \otimes a$, etc. In total dimension 0, there are the elements 1, $u^{-m}a$, $u^{-n}b$, and $u^{-m-n}ab$. These are represented by elements in $J^0(X; Z_p) = J^0(X^\pi; Z_p)$. But,

$$J^*(X^\pi; Z_p) \approx \hat{H}^*(\pi, Z_p) \otimes H^*(X^\pi; Z_p)$$

as an algebra (Lemma 5.3). Since $(u^{-m}a)\ (u^{-n}b) = u^{-m-n}ab \neq 0$, the products in $H^*(X^\pi; Z_p)$ cannot all be trivial. Therefore, we have only to eliminate the case $x^2 = y$, $z^2 = 0$, $xy = xz = yz = 0$.

Choose representatives for $u^{-m}a$ and $u^{-n}b$ in $J^0(X; Z_p)$. Let F^k denote the filtration in $J^0(X; Z_p)$ rather than that in $J^0(X^\pi; Z_p)$. Since $1 \in F^0J^0(X; Z_p)$, we can choose a representative α for $u^{-m}a$ of the form $\alpha = Ax + By + Cz \in F^{-2m}J^0$ because we can subtract off any multiple of 1 which occurs in α. Similarly, we can choose a representative $\beta = Dx + Ey + Fz \in F^{-2n}J^0$ for $u^{-n}b$.

Now $\alpha\beta \in F^{-2m-2n}J^0$ represents $(u^{-m}a)\ (u^{-n}b) \neq 0$. Therefore, $\alpha\beta \notin F^{1-2m-2n}J^0$. Suppose $m \leqslant n$. Since α^2 represents 0, $\alpha^2 \in F^{1-4m}J^0$. But, $\alpha\beta = ADy$ and so $A \neq 0$, $D \neq 0$, and $y \notin F^{1-2m-2n}J^0$. But, $A^2y = \alpha^2 \in F^{1-4m}J^0$ and so $y \in F^{1-4m}J^0$. This is a contradiction since $F^{1-4m}J^0 \subset F^{1-2m-2n}J^0$.

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