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# A Manifold which does not admit any Differentiable Structure

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An example of a triangulable closed manifold $M_0$ of dimension 10 will be constructed. It will be shown that $M_0$ does not admit any differentiable structure. Actually, $M_0$ does not have the homotopy type of any differentiable manifold.

Also, a 9-dimensional differentiable manifold $\Sigma^9$ is obtained. $\Sigma^9$ is homeomorphic but not diffeomorphic to the standard 9-sphere $S^9$.

Use is made of a procedure for killing the homotopy groups of differentiable manifolds studied by J. Milnor in [6]. I am indebted to J. Milnor for sending me a copy of the manuscript of his paper.

Although much of the constructions (in particular the construction of $M_0$) generalizes to higher dimensions, I did not succeed disproving the existence of a differentiable structure on the higher dimensional analogues of $M_0$. A more general case of some of the constructions below will be published in a subsequent paper, with other applications.[1])

## § 1. Construction of an invariant

Let $M^{10}$ be a closed triangulable manifold. Assume that $M^{10}$ is 4-connected. ($M^{10}$ is connected, and $\pi_i(M) = 0$ for $1 \leq i \leq 4$.) It follows from Poincaré duality and the universal coefficient theorem that $H^q(M; G) = 0$ for $5 < q < 10$, and $H^5(M)$ is free abelian of even rank $2s$, say. (If no coefficients are mentionned, integer coefficients are understood.)

Let $\Omega = \Omega S^6$ be the loop-space on the 6-sphere. It is well known that $H^5(\Omega) = Z$, $H^{10}(\Omega) = Z$, and if $\pi : \Omega \times \Omega \to \Omega$ is the map given by the product of loops, then

$$\pi^*(e_1) = e_1 \otimes 1 + 1 \otimes e_1, \quad \text{and}$$
$$\pi^*(e_2) = e_2 \otimes 1 + 1 \otimes e_2 + e_1 \otimes e_1,$$

where $e_1, e_2$ are the generators of $H^5(\Omega)$ and $H^{10}(\Omega)$ respectively, and $H^*(\Omega \times \Omega)$ is identified with $H^*(\Omega) \otimes H^*(\Omega)$ by the Künneth formula. (Compare R. Bott and H. Samelson [1], Theorem 3.1.B.)

**Lemma 1.1.** *Let $X \in H^5(M)$ be given. There exists a map $f : M \to \Omega$ such that $f^*(e_1) = X$.*

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[1]) This paper was presented at the International Colloquium on Differential Geometry and Topology, Zurich, June 1960.

*Proof.* Let $K$ be a triangulation of $M$. Define $f$ by stepwise extension on the skeletons $K^{(q)}$ using obstruction theory. $f \mid K^{(4)}$ is taken to be the constant map into a base point on $\Omega$. Let $X_0$ be a representative cocycle of $X$. For every 5-dimensional simplex $s_5$ of $K$, define $f \mid s_5$ to be a representative of $X_0[s_5]$-times the generator of $\pi_5(\Omega) \cong \pi_6(S^6) \cong Z$. The obstruction cocycle to extend $f \mid K^{(5)}$ in dimension 6 is zero. The next obstruction is in dimension 10 with values in $\pi_9(\Omega) \cong \pi_{10}(S^6) = 0$. (See [9], § 41.) Thus the lemma is proven.

Define a function $\varphi_0 \colon H^5(M) \to Z_2$ by the following device. For every $X \in H^5(M)$, take a map $f \colon M \to \Omega$ such that $f^*(e_1) = X$. Then, $\varphi_0(X) = f^*(u_2)[M]$, where $u_2 \in H^{10}(\Omega; Z_2)$ is the reduction modulo 2 of $e_2 \in H^{10}(\Omega)$, and $f^*(u_2)[M]$ is the value of the cohomology class $f^*(u_2)$ on the generator of $H_{10}(M^{10}; Z_2)$.

**Lemma 1.2.** *The function* $\varphi_0 \colon H^5(M) \to Z_2$ *is well defined,* i.e., $\varphi_0(X)$ *does not depend on the choice of the map* $f \colon M \to \Omega$ *such that* $f^*(e_1) = X$.

*Proof.* Let $f, g \colon M \to \Omega$ be two maps such that $f^*(e_1) = g^*(e_1)$. We have to show that $f^*(u_2) = g^*(u_2)$. Let $K$ again be a triangulation of $M$. Since $f^*(e_1) - g^*(e_1) = 0$, it follows that $f$ and $g$ are 5-homotopic. (See S. T. HU [2], Chap. VI.) Since $H^q(M; \pi_q(\Omega)) = 0$ for $5 < q < 10$, it follows that $f$ and $g$ are 9-homotopic. Hence, we may assume that $f \mid K^{(9)} = g \mid K^{(9)}$. Let $\omega^{10}(f, g) \in C^{10}(K; \pi_{10}(\Omega))$ be the difference cochain. Then,

$$(f^*(u_2) - g^*(u_2))[s_{10}] = u_2[h\,\omega^{10}(f, g)[s_{10}]] ,$$

for every 10-simplex $s_{10}$, where $h \colon \pi_{10}(\Omega) \to H_{10}(\Omega)$ is the HUREWICZ homomorphism. According to J. P. SERRE, $u_2[h\alpha]$ is the mod. 2 HOPF invariant of the element in $\pi_{11}(S^6)$ represented by $\alpha \in \pi_{10}(\Omega S^6)$. (Compare [8], Lemme 2.) Since no element of odd HOPF invariant occurs in $\pi_{11}(S^6)$, it follows that $f^*(u_2) = g^*(u_2)$, and the proof is complete.

**Lemma 1.3.** *Let* $X, Y \in H^5(M)$ *be two integer cohomology classes of* $M$. *Then,*

$$\varphi_0(X + Y) = \varphi_0(X) + \varphi_0(Y) + x \cdot y ,$$

*where* $x \cdot y$ *is the value on the generator of* $H_{10}(M^{10}; Z_2)$ *of the cup-product* $x \smile y$. ($x, y$ are the mod. 2 reductions of $X$ and $Y$ respectively.)

*Proof.* Let $f, g \colon M \to \Omega$ be maps such that $f^*(e_1) = X$ and $g^*(e_1) = Y$. By definition, $\varphi_0(X) = f^*(u_2)[M]$, and $\varphi_0(Y) = g^*(u_2)[M]$.

Let $f \times g \colon M \times M \to \Omega \times \Omega$ be the product of $f$ and $g$. (I.e., $f \times g(u, v) = (f(u), g(v))$.) Let $D \colon M \to M \times M$ be the diagonal map. Define $\mathrm{F} \colon M \to \Omega$

by $F = \pi \circ (f \times g) \circ D$, where $\pi : \Omega \times \Omega \to \Omega$ is given by the multiplication of loops. Since $D^*$ maps the tensor product of cohomology classes into their cup-product, we have $F^*(e_1) = D^*(X \otimes 1 + 1 \otimes Y) = X + Y$. Therefore,

$$\varphi_0(X + Y) = F^*(u_2)[M] .$$

On the other hand,

$$\begin{aligned} F^*(u_2) &= D^*(f^*(u_2) \otimes 1 + 1 \otimes g^*(u_2) + f^*(u_1) \otimes g^*(u_1)) \\ &= f^*(u_2) + g^*(u_2) + f^*(u_1) \smile g^*(u_1) \\ &= f^*(u_2) + g^*(u_2) + x \smile y . \end{aligned}$$

($u_1$ is the reduction modulo 2 of $e_1$.) This proves Lemma 1.3.

The function $\varphi_0 : H^5(M) \to Z_2$ induces a function $\varphi : H^5(M; Z_2) \to Z_2$ satisfying $\varphi(x + y) = \varphi(x) + \varphi(y) + x \cdot y$. Indeed, if $X$ is an integer class whose reduction modulo 2 yields $x \in H^5(M; Z_2)$, we define $\varphi(x) = \varphi_0(X)$. It follows from

$$\varphi_0(2Y) = \varphi_0(Y) + \varphi_0(Y) + y \cdot y = y \cdot y = 0 ,$$

that $\varphi(x) \in Z_2$ depends only on $x \in H^5(M; Z_2)$.

The function $\varphi : H^5(M; Z_2) \to Z_2$ is then used to construct the number $\Phi(M)$ as follows. A basis $x_1, \ldots, x_s, y_1, \ldots, y_s$ of $H^5(M; Z_2)$ as a vector space over $Z_2$ will be called *symplectic* if $x_i \cdot x_j = y_i \cdot y_j = 0$, and $x_i \cdot y_j = \delta_{ij}$ for all $i, j = 1, \ldots, s$. Clearly, symplectic bases always exist. Moreover, it is well known that since the function $\varphi : H^5(M; Z_2) \to Z_2$ satisfies the equation

$$\varphi(x + y) = \varphi(x) + \varphi(y) + x \cdot y ,$$

the remainder modulo 2

$$\Phi(M) = \Sigma_1^s \varphi(x_i) \cdot \varphi(y_i)$$

is independent of the symplectic basis $x_1, \ldots, x_s, y_1, \ldots, y_s$.

The rest of the paper is devoted to investigating the properties of the invariant $\Phi$.

Clearly, $\Phi$ is an invariant of the homotopy type of 4-connected closed manifolds of dimension 10.

Our objective is the proof of the following theorems.

**Theorem 1.** *If $M^{10}$ has the homotopy type of a $C^1$-differentiable 4-connected closed manifold, then* $\Phi(M) = 0$.

(It can be shown that the converse of this theorem would follow from the conjecture that the cohomology ring $H^*(M)$ and $\Phi(M)$ are a complete set of invariants of the homotopy type of the triangulable 4-connected closed manifold $M$ of dimension 10.)

**Theorem 2.** *There exists a closed 4-connected combinatorial manifold $M_0$ of dimension 10 for which $\Phi(M_0) = 1$.*

(In fact a specific example will be constructed.)

In § 2, the proof of Theorem 1 will be carried out taking Lemmas 4.2 and 5.1 for granted. (Lemma 4.2 is used in the proof of Lemma 2.2, and Lemma 5.1 is used to deduce Theorem 1 from Lemma 2.4.) The Lemmas 4.2 and 5.1 are proved at the end of the paper, in § 4 and § 5. Theorem 2 will be proved in § 3.

## § 2. Proof of Theorem 1

Let $M^{10}$ be a closed $C^1$-differentiable manifold which is 4-connected.

**Lemma 2.1.** *$M^{10}$ is a $\pi$-manifold.*

*Proof.* Let $M^{10} \subset R^{n+10}$ be an imbedding with $n$ large. We have to show that the normal bundle $\nu$ is trivial. This is done by constructing a field of normal $n$-frames $f_n$. Let $K$ be a triangulation of $M^{10}$. Since $\pi_4(SO_n) = 0$, and $M^{10}$ is 4-connected, it follows that $H^{q+1}(M; \pi_q(SO_n)) = 0$ for $0 \leqq q < 9$. Thus, there is only one possibly non-vanishing obstruction $\mathfrak{o}(\nu, f_n) \in H^{10}(M; \pi_9(SO_n)) \cong \pi_9(SO_n)$ to the construction of the field $f_n$ of normal $n$-frames. By Lemma 1 of [7], $\mathfrak{o}(\nu, f_n)$ is in the kernel of the HOPF-WHITEHEAD homomorphism $J_9 \colon \pi_9(SO_n) \to \pi_{n+9}(S^n)$. But $J_9$ is a monomorphism. (Compare proof of Lemma 1.2 of [4].) Hence, $\mathfrak{o}(\nu, f_n) = 0$, and the lemma is proved. (Recall that the proof of the assertion: $J_9$ is a monomorphism, was based on Corollary 2.6 of J. F. ADAMS paper *On the structure and applications of the STEENROD algebra*, Comm. Math. Helv. 32 (1958), 180–214. This statement also follows from the portion of the POSTNIKOV decomposition mod. 2 of $S^n$ given below in § 5.)

The THOM construction associates with every framed manifold $(M; f_n)$, where $M \subset R^{n+\dim M}$, an element $\alpha(M; f_n) \in \pi_{n+\dim M}(S^n)$. We say that $(M^{10}; f_n)$ is *homotopic to zero* if the corresponding element $\alpha(M; f_n)$ is the neutral element of $\pi_{n+10}(S^n)$.

**Lemma 2.2.** *If $(M^{10}; f_n)$ is homotopic to zero, where $M^{10}$ is 4-connected, then $\Phi(M) = 0$.*

*Proof.* The assumption that $(M; f_n)$ is homotopic to zero implies the existence of a framed manifold $(V^{11}; F_n)$ with boundary $M^{10}$. (Compare R. THOM [10].) We may assume that $V$ is connected, and hence has a trivial tangent bundle. We can therefore apply to $V - M$ the procedure for killing the homotopy groups of a differentiable manifold studied by J. MILNOR. Specifically, using Theorem 3 of [6], we obtain a new 11-dimensional differentiable manifold with boundary $M^{10}$ which is also 4-connected. This new 4-connected manifold will again be denoted by $V^{11}$. We can now forget about the fields of normal frames.

We proceed to compute $\Phi(M)$. Consider the cohomology exact sequence of the pair $(V, M)$ with coefficients in $Z_2$,

$$\cdots \to H^5(V) \xrightarrow{i^*} H^5(M) \xrightarrow{\delta} H^6(V, M) \to \cdots .$$

Using relative Poincaré-Lefschetz duality (over $Z_2$), and the formula

$$u \smile \delta x[V, M] = i^*(u) \smile x[M] ,$$

where $u \in H^5(V)$, $x \in H^5(M)$ and $[V, M]$, $[M]$ are the generators of $H_{11}(V, M; Z_2)$ and $H_{10}(M; Z_2)$ respectively, it follows that $H^5(M; Z_2)$ has a symplectic basis $x_1, \ldots, x_s, y_1, \ldots, y_s$ say, such that $x_1, \ldots, x_s$ is a vector basis of Ker $\delta$. Consequently, in order to prove $\Phi(M) = 0$, it is sufficient to show that $\varphi(x) = 0$ for every $x \in \operatorname{Ker} \delta$.

Let $X \in H^5(M)$ be an integer class whose reduction modulo 2 is $x$, and let $f: M^{10} \to \Omega = \Omega S^6$ be a map such that $f^*(e_1) = X$. We have to show that $f^*(u_2) = 0$, where $u_2$ generates $H^{10}(\Omega; Z_2)$. Let $\Omega^*$ be the space obtained from $\Omega$ by attaching a cell of dimension 6 by a map $S^5 \to \Omega$ of degree 2. By Lemma 4.2 in § 4, below, for every map $g: S^{10} \to \Omega^*$, one has $g^*(u_2) = 0$, where we denote by $u_2 \in H^{10}(\Omega^*; Z_2)$ again the class corresponding to the old $u_2 \in H^{10}(\Omega; Z_2)$ under the canonical isomorphism $H^{10}(\Omega; Z_2) \cong H^{10}(\Omega^*; Z_2)$.

We attempt to extend $f: M \to \Omega^*$ to a map of $V$ into $\Omega^*$. Let $(K, L)$ be a triangulation of $(V, M)$. The stepwise extension of $f$ on the skeletons $K^{(q)} \cup L$ leads to obstructions in the groups $H^{q+1}(K, L; \pi_q(\Omega^*))$. For $q < 5$, $\pi_q(\Omega^*) = 0$. We meet a first obstruction for $q = 5$ in $H^6(K, L; Z_2)$. By the Hopf theorem, this obstruction is $\delta x$. (See S. T. Hu [2].) Since $\delta x = 0$, it is possible to extend $f$ on $K^{(6)} \cup L$. Using $H^{q+1}(K, L; G) = 0$ for $5 < q < 10$ (since $V$ is 4-connected), it follows that there exists a map $F: K - \tau \to \Omega^*$, where $\tau$ is some 11-dimensional simplex in $K - L$, such that $F \mid L = f$. Let $S^{10}$ denote the boundary of $\tau$, and let $g: S^{10} \to \Omega^*$ be the restriction of $F$ on $S^{10}$. Since $\partial(K - \tau) = L - S^{10}$, and $g^*(u_2) = 0$, it follows that $f^*(u_2) = 0$. The proof of Lemma 2.2 is complete.

**Corollary 2.3.** *If two 4-connected framed manifolds $(M; f_n)$ and $(M'; f'_n)$ of dimension 10 define the same element $\alpha = \alpha(M; f_n) = \alpha(M'; f'_n)$ by the Thom construction, then $\Phi(M) = \Phi(M')$.*

This is obtained by observing that $\Phi$ is additive with respect to the connected sum of manifolds.

It follows that $\Phi$ provides a homomorphism from a subgroup of $\pi_{n+10}(S^n)$

into $Z_2$. We denote this homomorphism by $\Phi$ again. Actually, $\Phi$ is defined on every element of $\pi_{n+10}(S^n)$. Indeed, using spherical modifications [6], it is easy to see that every element $\alpha \in \pi_{n+10}(S^n)$ is obtainable from a 4-connected framed manifold by the THOM construction. This remark will not be used in the present paper.

It follows from Corollary 2.3 that Theorem 1 is equivalent to the statement that $\Phi(\alpha) = 0$ for every $\alpha \in \pi_{n+10}(S^n)$, provided $\Phi(\alpha)$ is defined.

Since $\Phi(\alpha)$ is obviously zero for every element $\alpha$ of odd order, and by J. P. SERRE's results $\pi_{n+10}(S^n)$ contains no element of infinite order, it is sufficient to show that $\Phi$ annihilates the 2-component of the group $\pi_{n+10}(S^n)$. By Lemma 5.1 in § 5 below, every element $\alpha$ in the 2-component of $\pi_{n+10}(S^n)$ is representable in the form

$$\alpha = \beta \circ \eta ,$$

where $\eta \in \pi_{n+10}(S^{n+9})$ is the generator of the stable 1-stem, and $\beta \in \pi_{n+9}(S^n)$. Hence, Theorem 1 will follow from the

**Lemma 2.4.** *Every element* $\alpha \in \pi_{n+10}(S^n)$ *of the form* $\alpha = \beta \circ \eta$, *with* $\eta \in \pi_{n+10}(S^{n+9})$, *and* $\beta \in \pi_{n+9}(S^n)$ *is obtainable by the* THOM *construction from a framed manifold* $(\Sigma^{10}; f_n)$, *where* $\Sigma^{10}$ *has the homotopy type of the* 10-*sphere* $S^{10}$.

*Proof.* We first show that $\beta \in \pi_{n+9}(S^n)$ is obtainable by the THOM construction from a framed manifold $(\Sigma^9; f_n)$, where $\Sigma^9$ has the homotopy type of the 9-sphere.

It is well known that $\beta$ is obtainable by the THOM construction from some framed manifold $(M^9; f_n)$. We have to show that $(M^9; f_n)$ is homotopic to a framed manifold $(\Sigma^9; f_n)$, where $\Sigma^9$ is a homotopy sphere. This is done by simplifying $M^9$ by a series of spherical modifications. (See J. MILNOR [6].)

Assuming by induction that $M^9$ is $(p-1)$-connected $(0 \leqq p \leqq 4)$, we have to prove that $(M; f_n)$ is homotopic to a $p$-connected framed manifold $(M'; f'_n)$. Recall that a spherical modification of type $(p+1, q+1)$ applied to a class $\lambda \in \pi_p(M^9)$ consists of the following construction. Represent $\lambda$ by an imbedding

$$f: S^p \times D^{q+1} \rightarrow M^9 ,$$

with $p + q + 1 = 9$. (This is possible for $p \leqq 4$ since $M^9$ is a $\pi$-manifold and the normal bundle of any imbedding $S^p \rightarrow M^9$ is stable in this range of dimensions.) The manifold $M$ is then replaced by

$$M' = (M - f(S^p \times D^{q+1})) \cup (D^{p+1} \times S^q) ,$$

under identification of $f(S^p \times S^q)$ regarded as the boundary of $f(S^p \times D^{q+1})$ with $S^p \times S^q$ regarded as the boundary of $D^{p+1} \times S^q$. By Theorem 2 of

[6], the manifolds $M$ and $M'$ bound a 10-dimensional differentiable manifold $\omega = \omega(M, f)$, and $f: S^p \times D^{q+1} \to M^9$ can be chosen such that the field $f_n$ (over $M$) is extendable over $\omega$ as a field of normal $n$-frames. (We can think of $\omega$ as imbedded in $R^{n+10}$ with $M \subset R^{n+9} \times (0)$ and $M' \subset R^{n+9} \times (1)$ since $n$ can be taken as large as we please.) Hence spherical modifications of type $(p+1, q+1)$ with $0 \leqq p \leqq 4$ can be performed so as to carry $(M; f_n)$ into a homotopic framed manifold. It is known (Theorem 3 of [6]) that for $p < 4$, spherical modifications simplify the manifold. More precisely $\pi_p(M')$ is isomorphic to the quotient of $\pi_p(M)$ by the subgroup generated by $\lambda$, and $\pi_i(M) \cong \pi_i(M') = 0$ for $i < p$. Hence, it is easy, using [6], to obtain a 3-connected framed manifold homotopic to $(M^9; f_n)$. The case $p = 4$ requires special care. If $\lambda \in \pi_4(M^9)$ is the class we want to kill, there exists an imbedding $f: S^4 \times D^5 \to M^9$ such that $f \mid S^4 \times (0)$ represents $\lambda$. Let $M' = \chi(M, f)$ be the 9-dimensional manifold obtained from $M$ and $f$ by spherical modification. ($f$ is supposed to be chosen so that $(M'; f'_n)$ with some $f'_n$ is homotopic to $(M; f_n)$.) In general, however, $f \mid x_0 \times (bdry\, D^5)$ represents a *non-zero* element of $\pi_4(M')$. Thus, it is not clear a priori that a series of spherical modifications of type (5, 5) will carry $M$ into a 4-connected manifold, and hence a homotopy sphere.

If $\lambda$ is a generator of the free part of $\pi_4(M) \cong H_4(M)$, there exists by Poincaré duality a class $\mu \in H_6(M)$ whose intersection coefficient with $\lambda$ (or $h\lambda$ rather, where $h$ is the Hurewicz homomorphism) is 1. It follows that in this case the cycle given by $f \mid x_0 \times (bdry\, D^5)$ is homologous to zero in $M - f(S^4 \times D^5)$, and hence in $M'$. Thus $H_4(M') \cong \pi_4(M')$ has strictly smaller rank than $H_4(M) \cong \pi_4(M)$, and the torsion subgroup is unchanged.

I claim that if $\lambda \in \pi_4(M)$ is a torsion element, the homology class of the cycle $f \mid x_0 \times (bdry\, D^5)$ *is of infinite order for any $f$ representing* $\lambda$. Hence, one more spherical modification will lead to a manifold with 4-dimensional homology group of not bigger rank than $H_4(M)$ and with a strictly smaller torsion subgroup. (I.e., a series of spherical modifications will lead to a 4-connected framed manifold homotopic to $(M^9; f_n)$. By Poincaré duality, a closed 4-connected manifold of dimension 9 has the homotopy type of $S^9$.)

Since the Betti numbers $p_4$, $p'_4$ of $M$ and $M'$ (in dimension 4) differ at most by 1, and differ indeed by 1 if and only if $\lambda'$ (represented by $f \mid x_0 \times (bdry\, D^5)$) in $M'$ is of infinite order, it is sufficient to show that $p'_4 + p_4 \equiv 1 \bmod 2$. Since $p'_i = p_i$ for $0 \leqq i \leqq 3$, this is equivalent to showing that the semi-characteristics $E^*(M)$ and $E^*(M')$ of $M$ and $M'$ (over the rationals, say) satisfy $E^*(M') + E^*(M) \equiv 1 \bmod 2$. We use the formula

$$E^*(M') + E^*(M) \equiv E(\omega) + r \qquad \bmod 2,$$

where $E(\omega)$ is the EULER characteristic of the manifold $\omega$ with boundary $\dot{\omega} = M' - M$, and $r$ is the rank of the bilinear form on $H^5(\omega, \dot{\omega}; Q)$ defined by the cup-product. (Compare M. A. KERVAIRE [3], § 8, formula (8.9).) It is easily seen that $E(\omega) = 1$, up to sign, and since $u \cdot u = 0$ for every $u \in H^5(\omega, \dot{\omega}; Q)$, the rank $r$ must be even: $r \equiv 0$ (mod. 2). Hence, $E^*(M') + E^*(M) \equiv 1$ mod. 2.

Summarizing, we have proved so far that every $\beta \in \pi_{n+9}(S^n)$ is obtainable by the THOM construction from a framed manifold $(\Sigma^9; f_n)$, where the manifold $\Sigma^9$ has the homotopy type of $S^9$.

Taking a representative $f: S^{n+10} \to S^{n+9}$ of $\eta$ such that $f^{-1}(S^{n+9} - x_0)$ is diffeomorphic to $S^1 \times (S^{n+9} - x_0)$, we obtain that $\alpha = \beta \circ \eta$ is obtainable by the THOM construction from $(S^1 \times \Sigma^9; f_n)$.

It remains to show that $(S^1 \times \Sigma^9; f_n)$ is homotopic to a framed manifold $(\Sigma^{10}; f'_n)$, where $\Sigma^{10}$ is a homotopy sphere.

Apply once more the spherical modification theorems (Theorems 2 and 3 of [6]), this time to the class $\lambda \in \pi_1(S^1 \times \Sigma^9)$ represented by $S^1 \times (z_0)$. The resulting framed manifold is homotopic to $(S^1 \times \Sigma^9; f_n)$ and has the homotopy type of the 10-sphere. This completes the proof of Lemma 2.4.

To complete the proof of Theorem 1 it remains to prove the Lemmas 4.2, and 5.1. This is done in § 4 and § 5.

## § 3. Construction of $M_0$

This section relies on J. MILNOR's paper [5]. Let $f_0: S^4 \to SO_4$ be a differentiable map whose homotopy class $(f_0)$ satisfies

$$i_*(f_0) = \partial i_5 ,$$

where $\partial: \pi_5(S^5) \to \pi_4(SO_5)$ is taken from the homotopy exact sequence of $SO_6/SO_5$, and $i: SO_4 \to SO_5$ is the usual inclusion. Define $f_1 = f_2 = i \circ f_0$. Using $f_1, f_2: S^4 \to SO_5$, a diffeomorphism $f: S^4 \times S^4 \to S^4 \times S^4$ is given by $f(x, y) = (x', y')$, where $y' = f_1(x) \cdot y$, and $x = f_2(y') \cdot x'$. Let $M(f_1, f_2)$ be the MILNOR manifold obtained from the disjoint union of $D^5 \times S^4$ and $S^4 \times D^5$ by identifying each point $(x, y)$ in the boundary of $D^5 \times S^4$ with $f(x, y)$, considered as a point on the boundary of $S^4 \times D^5$. By Lemma 1 of [5], together with the remark at the bottom of page 963 in the proof of Lemma 1 in [5], it follows that the differentiable manifold $M(f_1, f_2)$ is homeomorphic to the 9-sphere. It will follow from Theorem 1 in this paper, that $M(f_1, f_2)$ is not diffeomorphic to the standard $S^9$. Let $W^{10}$ be the differentiable manifold with boundary $M(f_1, f_2)$ obtained using the construction on page 964 of [5]. $W$ can alternately be described as follows. Let $U$ be a tubular neighborhood of the diagonal $\Delta$ in $S^5 \times S^5$. It is well known that $U$ is the space of the fibre bundle $p: U \to S^5$ with fibre $D^5$ associated with the tangent bundle of $S^5$. The differentiable manifold $W$ is obtained by straightening the angles of the quotient space of the disjoint union of two copies $U'$ and $U''$ of $U$ under an identification of $p'^{-1}(V)$ with $p''^{-1}(V)$ such that the images of $\Delta'$ and $\Delta''$ in $W$ have intersection number $1$. ($V$ is an imbedded 5-disc on $S^5$, and $p'^{-1}(V) \cong D^5 \times D^5$ is identified with $p''^{-1}(V) \cong D^5 \times D^5$ under $(u, v) \longleftrightarrow (v, u)$, $u, v \in D^5$.)

Since $W$ is a 10-dimensional manifold whose boundary $M(f_1, f_2)$ is homeomorphic to $S^9$, the union of $W$ with the cone over the boundary is a 10-dimensional closed manifold $M_0$. Since $M(f_1, f_2)$ is combinatorially equivalent to $S^9$, it follows that $M_0$ possesses a combinatorial structure. (Compare J. Milnor, *On the relationship between differentiable manifolds and combinatorial manifolds*, mimeographed notes 1956, § 4.)

It is easily seen that $M_0$ is 4-connected.

We proceed to compute $\Phi(M_0)$. Let $x$, $y \in H^5(M_0; Z_2)$ be the cohomology classes dual to the homology classes of the imbedded spheres $j', j'': S^5 \to M_0$ given by the images in $W$ of the diagonals $\Delta'$ and $\Delta''$ in $U'$ and $U''$ respectively. Clearly, $x, y$ is a symplectic basis of $H^5(M_0; Z_2)$. (I.e., $x \cdot x = y \cdot y = 0$, and $x \cdot y = 1$.) To show that $\varphi(x) = \varphi(y) = 1$, observe that the normal bundles of $j'$ and $j''$ (regarded as imbeddings of $S^5$ in the *differentiable* manifold $W$) are non-trivial. These bundles are isomorphic to $p: U \to S^5$. Let $K$ be the Thom complex of this bundle. (I.e., the space obtained by collapsing the boundary of $U$ to a point.) It is well known that $K$ admits a cell decomposition $S^5 \cup e^{10}$, where the attaching map $S^9 \to S^5$ is a representative of the Whitehead product $[i_5, i_5]$. On the other hand, the Thom construction provides a map $f_0: M_0 \to K$ such that $f_0^*(e_1) = X$, the dual class of $j': S^5 \to M_0$, and $f_0^*(u_2)[M_0] = 1$, where $e_1$ generates $H^5(K; Z)$ and $u_2$ generates $H^{10}(K; Z_2)$. A map $f: M_0 \to \Omega S^6$ is obtained by composition of $f_0$ with the usual inclusion $S^5 \cup e^{10} \to \Omega S^6$. (Recall that $\Omega S^6$ has a cell decomposition $\Omega S^6 = S^5 \cup e^{10} \cup e^{15} \cup e^{20} \cup \ldots$, where the attaching map of $e^{10}$ represents $[i_5, i_5]$.) Then, $f: M_0 \to \Omega S^6$ has the properties $f^*(e_1) = X$, $f^*(u_2) = 1$, showing that $\varphi(x) = 1$. The same construction applied to $Y$, the dual class of $j'': S^5 \to M_0$ yields $\varphi(y) = 1$. Hence $\Phi(M_0) = \varphi(x) \cdot \varphi(y) = 1$.

If $M(f_1, f_2)$, with the differentiable structure induced by $W$ (of which $M(f_1, f_2)$ is the boundary) were diffeomorphic to $S^9$ with the standard differentiable structure, the differentiable structure on $W$ could be extended to a differentiable structure over the cone $CM(f_1, f_2)$, providing a differentiable structure on $M_0$. However, $\Phi(M_0) = 1$ and Theorem 1 show that a differentiable structure on $M_0$ does not exist. Hence, $M(f_1, f_2)$, homeomorphic to $S^9$, is not diffeomorphic to $S^9$.

## § 4. The auxiliary space $\Omega^*$

Let $Y = S^5 \cup_{2i_5} e^6$ be the space obtained by attaching a 6-cell to $S^5$ by a map $S^5 \to S^5$ of degree 2.

**Lemma 4.1.** *Let* $\alpha \in \pi_5(Y) \cong Z_2$ *be the generator, then* $[\alpha, \alpha] \neq 0 \in \pi_9(Y)$.

*Proof.* We identify $Y$ with the STIEFEL manifold $V_{7,2}$. Consider the exact sequence

$$\cdots \to \pi_{10}(S^6) \to \pi_9(S^5) \xrightarrow{i_*} \pi_9(V_{7,2}) \to \cdots .$$

Since $\pi_{10}(S^6) = 0$, and $[i_5, i_5]$ is non-zero in $\pi_9(S^5)$, it follows that $i_*[i_5, i_5] = [i_*(i_5), i_*(i_5)] = [\alpha, \alpha] \neq 0$.

Let $Y^* = Y \cup e^{10}$ be the space obtained from $Y$ by attaching a 10-cell $e^{10}$ using a representative $f: S^9 \to Y$ of $[\alpha, \alpha]$. Since $Y$ is 4-connected, the characteristic map $\hat{f}: (D^{10}, S^9) \to (Y^*, Y)$ of $e^{10}$ induces an *isomorphism*

$$\hat{f}_*: \pi_{10}(D^{10}, S^9) \to \pi_{10}(Y^*, Y) .$$

(Compare J. H. C. WHITEHEAD [12], Theorem 1.) Thus the relative HUREWICZ homomorphism $h_R: \pi_{10}(Y^*, Y) \to H_{10}(Y^*, Y) \cong Z$ is an isomorphism. Consider the homotopy-homology ladder of $(Y^*, Y)$:

$$\begin{array}{ccccccccc}
\cdots \to \pi_{10}(Y) & \to & \pi_{10}(Y^*) & \xrightarrow{j_0} & \pi_{10}(Y^*, Y) & \xrightarrow{\partial} & \pi_9(Y) & \to & \cdots \\
\downarrow & & \downarrow h & & \downarrow h_R & & \downarrow & & \\
\cdots \to 0 & \to & H_{10}(Y^*) & \xrightarrow{j_*} & H_{10}(Y^*, Y) & \to & 0 & \to & \cdots .
\end{array}$$

Since $\partial$ sends the generator of $\pi_{10}(Y^*, Y)$ into $[\alpha, \alpha] \neq 0$, and $2[\alpha, \alpha] = 0$, it follows that every element in $\operatorname{Im}\{h: \pi_{10}(Y^*) \to H_{10}(Y^*)\}$ can be halved.

It follows that for every map $g_0: S^{10} \to Y^*$, the induced homomorphism $g_0^*: H^{10}(Y^*; Z_2) \to H^{10}(S^{10}; Z_2)$ is zero.

Let $\Omega$ be the space of loops over $S^6$. Up to homotopy type $\Omega = S^5 \cup e^{10} \cup e^{15} \cup \ldots$, with $e^{10}$ attached by a map of class $[i_5, i_5]$. Let $\Omega^* = \Omega \cup e^6$, where $e^6$ is attached by a map of degree 2 on $S^5 \subset \Omega$. There is a natural inclusion $Y^* \to \Omega^*$ which induces an isomorphism on cohomology groups in dimension 10. Hence, we have the

**Lemma 4.2.** *Let* $g: S^{10} \to \Omega^*$ *be a map, and let* $u_2$ *be the generator of* $H^{10}(\Omega^*; Z_2) \cong Z_2$. *Then,* $g^*(u_2) = 0$.

## § 5. A lemma on homotopy groups of spheres

**Lemma 5.1.** *The map* $\pi_{n+9}(S^n) \to \pi_{n+10}(S^n)$, *for* $n \geqq 12$, *defined by composition with the generator* $\eta$ *of* $\pi_{n+10}(S^{n+9})$ *is surjective on the* 2-*component.*

This lemma was communicated to me without proof by H. TODA who has also proved that the 2-component of $\pi_{n+10}(S^n)$ is $Z_2$. (See H. TODA [11], Corollary to Proposition 4.10.)

We give a sketch of proof by computation of the POSTNIKOV decomposition modulo 2 of $S^n$ for large $n$, up to dimension $n+10$.

We begin with a remark which will yield Lemma 5.1 whenever a long enough portion of the POSTNIKOV decomposition of $S^n$ is obtained. Let $X = K(Z_2, n+9) \times_k K(Z_2, n+10)$ be the space of the fibration over $K(Z_2, n+9)$ associated with the $k$-invariant $k \in H^{n+11}(Z_2, n+9; Z_2)$. Let $f: S^{n+9} \to X$ be a map representing the generator of $\pi_{n+9}(X) \cong Z_2$. Then, the composition

$$f \circ \eta : S^{n+10} \to X, \quad \text{where } \eta : S^{n+10} \to S^{n+9}$$

represents the generator of $\pi_{n+10}(S^{n+9})$, is essential if and only if $k = Sq^2(\varepsilon)$, where $\varepsilon$ is the fundamental class of $H^{n+9}(Z_2, n+9; Z_2)$.

Since $Sq^2(\varepsilon)$ generates $H^{n+11}(Z_2, n+9; Z_2)$, it follows that $k \neq Sq^2(\varepsilon)$ implies $k = 0$. Hence, $f \circ \eta$ is inessential if $k \neq Sq^2(\varepsilon)$.

If $k = Sq^2(\varepsilon)$, let $\hat{f}: S^{n+9} \cup_\eta e^{n+11} \to X \cup_{f \circ \eta} e^{n+11}$ be the map induced by $f$. Let $s \in H^{n+9}(S^{n+9} \cup_\eta e^{n+11}; Z_2)$ be the generator. We identify $H^{n+9}(X \cup e^{n+11}; Z_2)$ and $H^{n+9}(X; Z_2)$ with $H^{n+9}(Z_2, n+9; Z_2)$. Since $f^*(\varepsilon) = s$, and $Sq^2(s) \neq 0$, it follows that $Sq^2(\varepsilon) \neq 0$ in $H^{n+11}(X \cup e^{n+11}; Z_2)$. To show that $f \circ \eta$ is essential, it is therefore sufficient to show that $Sq^2(\varepsilon) = 0$ in $H^{n+11}(X; Z_2)$. This follows from the commutativity of the diagram

$$\begin{array}{ccccccc} 0 \leftarrow & H^{n+9}(X; Z_2) & \xleftarrow{\approx} & H^{n+9}(Z_2, n+9; Z_2) & \leftarrow 0 & & \\ & \downarrow Sq^2 & & \downarrow Sq^2 & & & \\ & H^{n+11}(X; Z_2) & \leftarrow & H^{n+11}(Z_2, n+9; Z_2) & \xleftarrow{\tau} & H^{n+10}(Z_2, n+10; Z_2), & \end{array}$$

where the rows are taken from the exact sequence of the fibration defining $X$ (in the stable range), and $\tau$ is the transgression.

Let $Y_{10} \to Y_9 \to \cdots \to Y_i \to Y_{i-1} \to \cdots \to Y_0 = K(Z, n)$ be the modulo 2 POSTNIKOV decomposition of $S^n$. (I.e., $p_i: Y_i \to Y_{i-1}$ is a fibration with fibre $F_i = K(\pi_i, n+i)$, where $\pi_i$ is the 2-component of the stable group $\pi_{n+i}(S^n)$, and $H^*(Y_i; Z_2)$ contains $Z_2$ in dimension 0 and $n$, $H^q(Y_i; Z_2) = 0$ for $0 < q < n$, and $H^{n+k}(Y_i; Z_2) = 0$ for $0 < k < i+2$.) By the $\mathfrak{C}$-theory with $\mathfrak{C}$ = the class of finite groups whose order is prime to

2, a map $S^n \to Y_i$ inducing an isomorphism $H^n(Y_i; Z_2) \cong H^n(S^n; Z_2)$ induces an isomorphism of the 2-component of $\pi_{n+k}(S^n)$ with $\pi_{n+k}(Y_i)$ for $k \leq i$. (Compare J. P. SERRE [8].) We have $\pi_9 \cong Z_2 + Z_2 + Z_2$ and $\pi_{10} \cong Z_2$ as will be seen below, thus

$$F_9 = K(Z_2, n+9) \times K(Z_2, n+9) \times K(Z_2, n+9),$$

and Lemma 5.1 follows by showing that the restriction of the fibration $Y_{10} \to Y_9$ over one of the factors of $F_9$ is $K(Z_2, n+9) \times {}_k K(Z_2, n+10)$ with $k = Sq^2$. This is equivalent to showing that $H^{n+11}(Y_9; Z_2) \cong Z_2$ is generated by a class $u_9$ such that $i_9^*(u_9) = Sq^2(\varepsilon_9)$, where $\varepsilon_9$ is one of the fundamental classes of $H^9(F_9; Z_2)$, and $i_9 : F_9 \to Y_9$ is the inclusion.

In a similar way, it can be read off from the tables below that composition with $\eta$ provides *injective* maps $\pi_{n+7}(S^n) \otimes Z_2 \to \pi_{n+8}(S^n)$ and $\pi_{n+8}(S^n) \to \pi_{n+9}(S^n)$ in the stable range. Using $\pi_7(SO_n) \cong Z$, $\pi_8(SO_n) \cong Z_2$, and $\pi_9(SO_n) \cong Z_2$, this implies that $J_9 : \pi_9(SO_n) \to \pi_{n+9}(S^n)$ is a monomorphism.

We proceed to a partial description of the modulo 2 cohomology of the spaces $Y_7$.

$H^*(Y_0)$ is given by J. P. SERRE in [9]. This result of J. P. SERRE and the ADEM relations between the STEENROD squares are the essential tools in computing $H^*(Y_k; Z_2)$ for $k > 0$. Since we stay in the stable range, the spectral sequences of $p_k : Y_k \to Y_{k-1}$ reduce to exact sequences

$$\cdots \leftarrow H^{n+q+1}(Y_{k-1}) \xleftarrow{\tau} H^{n+q}(F_k) \xleftarrow{i_k^*} H^{n+q}(Y_k) \xleftarrow{p_k^*} H^{n+q}(Y_{k-1}) \leftarrow \cdots .$$

It is therefore sufficient to determine at each step the kernel and the image of the transgression $\tau$. Since the cohomology of $Y_k$ is independent of $k$ up to dimension $n$, we omit to mention the non-vanishing cohomology groups in dimension $\leq n$. The direct sum of the subgroups of $H^*(Y_k; Z_2)$ in dimensions $> n$ is denoted $H^+(Y_k)$.

The symbol $q_k$ stands for the composition $p_1 \circ p_2 \circ \cdots \circ p_k$, and $\varepsilon_k$ denotes the fundamental class of $H^{n+k}(G, n+k; G)$.

I omit $Y_1$ and $Y_2$ whose cohomology is straightforward, but has to be computed up to dimension $n + 17$ and $n + 16$ respectively. $H^{n+4}(Y_2; Z_2)$ is generated by $q_2^*(Sq^4 \varepsilon_0)$, and $H^{n+5}(Y_2; Z_2)$ by a class $u_2$ such that $i_2^*(u_2) = Sq^3(\varepsilon_2)$.

$F_3 = K(Z_8, n+3)$, with $\tau(\varepsilon_3') = q_2^*(Sq^4 \varepsilon_0)$ and $\tau(\beta \varepsilon_3) = u_2$, where $\beta$ is the BOCKSTEIN operator associated with the sequence of coefficients $0 \to Z_2 \to Z_{16} \to Z_8 \to 0$, and $\varepsilon_3'$ is the mod. 2 reduction of $\varepsilon_3$.

*$H^+(Y_3)$ has a basis consisting of*
$u_3$ in dimension $n+7$, such that $i_3^*(u_3) = Sq^4\varepsilon_3'$;
$Sq^1(u_3)$, $q_3^*(Sq^8\varepsilon_0)$; $Sq^2(u_3)$, $v_3$ such that $i_3^*(v_3) = Sq^5\beta\varepsilon_3$; $Sq^3(u_3)$;
$Sq^4(u_3)$; $Sq^5(u_3)$, $Sq^4Sq^1(u_3)$, $q_3^*(Sq^{12}\varepsilon_0)$; $Sq^6(u_3)$, $Sq^4Sq^2(u_3)$, $Sq^4(v_3)$;
$Sq^6Sq^1(u_3)$, $Sq^5Sq^2(u_3)$, $q_3^*(Sq^{14}\varepsilon_0)$;
$Sq^8(u_3)$, $Sq^7Sq^1(u_3)$, $Sq^6Sq^2(u_3)$, $Sq^6(v_3)$, $q_3^*(Sq^{15}\varepsilon_0)$; ...

$Y_4 = Y_5 = Y_3$. $(\pi_4 = \pi_5 = 0.)$

$F_6 = K(Z_2, n+6)$ with $\tau(\varepsilon_6) = p_5^*p_4^*(u_3)$.
*$H^+(Y_6)$ has a basis consisting of*
$q_6^*(Sq^8\varepsilon_0)$; $p_6^*p_5^*p_4^*(v_3)$, $u_6$ such that $i_6^*(u_6) = Sq^2Sq^1\varepsilon_6$;
$Sq^1(u_6)$; nothing in dimension $n+11$; $q_6^*(Sq^{12}\varepsilon_0)$, $Sq^2Sq^1(u_6)$;
$p_6^*p_5^*p_4^*(Sq^4v_3)$, $Sq^4(u_6)$, $v_6$ such that $i_6^*(v_6) = Sq^7\varepsilon_6$;
$q_6^*(Sq^{14}\varepsilon_0)$, $Sq^5(u_6)$; $q_6^*(Sq^{15}\varepsilon_0)$, $p_6^*p_5^*p_4^*(Sq^6v_3)$, ...
(and possibly other classes of dimension $n+15$).

$F_7 = K(Z_{16}, n+7)$ with $\tau(\varepsilon_7') = q_6^*(Sq^8\varepsilon_0)$ and $\tau(\beta'\varepsilon_7) = p_6^*p_5^*p_4(v_3)$, where $\beta'$ is the BOCKSTEIN operator of $0 \to Z_2 \to Z_{32} \to Z_{16} \to 0$, and $\varepsilon_7'$ is the reduction modulo 2 of $\varepsilon_7$.

*$H^+(Y_7)$ has a basis consisting of*
$u_7$ in dimension $n+9$, such that $i_7^*(u_7) = Sq^2(\varepsilon_7')$, $p_7^*(u_6)$;
$Sq^1(u_7)$, $p_7^*(Sq^1u_6)$, $v_7$ such that $i_7^*(v_7) = Sq^2\beta'\varepsilon_7$;
$Sq^1(v_7)$; $Sq^2Sq^1(u_7)$, $p_7^*(Sq^2Sq^1u_6)$, ... $(Sq^2(v_7) = 0.)$

$F_8 = K(Z_2 + Z_2, n+8)$ with $\tau(\varepsilon_8') = u_7$, $\tau(\varepsilon_8'') = p_7^*(u_6)$, where $\varepsilon_8'$ and $\varepsilon_8''$ are the two fundamental classes in $H^{n+8}(F_8; Z_2)$.

*$H^+(Y_8)$ has a basis consisting of*
$p_8^*(v_7)$, $u_8$, $v_8$, where $i_8^*(u_8) = Sq^2(\varepsilon_8')$ and $i_8^*(v_8) = Sq^2(\varepsilon_8'')$;
$Sq^1(u_8)$, $Sq^1(v_8)$, $p_8^*(Sq^1v_7)$;
$Sq^2(u_8)$, $Sq^2(v_8)$, ...

$F_9 = K(Z_2 + Z_2 + Z_2, n+9)$ with fundamental classes $\varepsilon_9$, $\varepsilon_9'$, $\varepsilon_9''$ which are send by transgression on $p_8^*(v_7)$, $u_8$, $v_8$ respectively.

$$H^{n+11}(Y_9; Z_2) \cong Z_2(u_9), \quad \text{where} \quad i_9^*(u_9) = Sq^2(\varepsilon_9).$$

We have seen that this statement implies Lemma 5.1, hence the proof is complete.

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