

Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 34 (1960)

Artikel: A Manifold which does not not admit any Differentiable Structure.
Autor: Kervaire, Michel A.
DOI: <https://doi.org/10.5169/seals-26636>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 12.01.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

A Manifold which does not admit any Differentiable Structure

by MICHEL A. KERVAIRE, New York (USA)

An example of a triangulable closed manifold M_0 of dimension 10 will be constructed. It will be shown that M_0 does not admit any differentiable structure. Actually, M_0 does not have the homotopy type of any differentiable manifold.

Also, a 9-dimensional differentiable manifold Σ^9 is obtained. Σ^9 is homeomorphic but not diffeomorphic to the standard 9-sphere S^9 .

Use is made of a procedure for killing the homotopy groups of differentiable manifolds studied by J. MILNOR in [6]. I am indebted to J. MILNOR for sending me a copy of the manuscript of his paper.

Although much of the constructions (in particular the construction of M_0) generalizes to higher dimensions, I did not succeed disproving the existence of a differentiable structure on the higher dimensional analogues of M_0 . A more general case of some of the constructions below will be published in a subsequent paper, with other applications.¹⁾

§ 1. Construction of an invariant

Let M^{10} be a closed triangulable manifold. Assume that M^{10} is 4-connected. (M^{10} is connected, and $\pi_i(M) = 0$ for $1 \leq i \leq 4$.) It follows from POINCARÉ duality and the universal coefficient theorem that $H^q(M; G) = 0$ for $5 < q < 10$, and $H^5(M)$ is free abelian of even rank $2s$, say. (If no coefficients are mentioned, integer coefficients are understood.)

Let $\Omega = \Omega S^6$ be the loop-space on the 6-sphere. It is well known that $H^5(\Omega) = \mathbb{Z}$, $H^{10}(\Omega) = \mathbb{Z}$, and if $\pi: \Omega \times \Omega \rightarrow \Omega$ is the map given by the product of loops, then

$$\begin{aligned}\pi^*(e_1) &= e_1 \otimes 1 + 1 \otimes e_1, \quad \text{and} \\ \pi^*(e_2) &= e_2 \otimes 1 + 1 \otimes e_2 + e_1 \otimes e_1,\end{aligned}$$

where e_1, e_2 are the generators of $H^5(\Omega)$ and $H^{10}(\Omega)$ respectively, and $H^*(\Omega \times \Omega)$ is identified with $H^*(\Omega) \otimes H^*(\Omega)$ by the KÜNNETH formula. (Compare R. BOTT and H. SAMELSON [1], Theorem 3.1.B.)

Lemma 1.1. *Let $X \in H^5(M)$ be given. There exists a map $f: M \rightarrow \Omega$ such that $f^*(e_1) = X$.*

¹⁾ This paper was presented at the International Colloquium on Differential Geometry and Topology, Zurich, June 1960.

Proof. Let K be a triangulation of M . Define f by stepwise extension on the skeletons $K^{(q)}$ using obstruction theory. $f|K^{(4)}$ is taken to be the constant map into a base point on Ω . Let X_0 be a representative cocycle of X . For every 5-dimensional simplex s_5 of K , define $f|s_5$ to be a representative of $X_0[s_5]$ -times the generator of $\pi_5(\Omega) \cong \pi_6(S^6) \cong \mathbb{Z}$. The obstruction cocycle to extend $f|K^{(5)}$ in dimension 6 is zero. The next obstruction is in dimension 10 with values in $\pi_9(\Omega) \cong \pi_{10}(S^6) = 0$. (See [9], § 41.) Thus the lemma is proven.

Define a function $\varphi_0: H^5(M) \rightarrow \mathbb{Z}_2$ by the following device. For every $X \in H^5(M)$, take a map $f: M \rightarrow \Omega$ such that $f^*(e_1) = X$. Then, $\varphi_0(X) = f^*(u_2)[M]$, where $u_2 \in H^{10}(\Omega; \mathbb{Z}_2)$ is the reduction modulo 2 of $e_2 \in H^{10}(\Omega)$, and $f^*(u_2)[M]$ is the value of the cohomology class $f^*(u_2)$ on the generator of $H_{10}(M^{10}; \mathbb{Z}_2)$.

Lemma 1.2. *The function $\varphi_0: H^5(M) \rightarrow \mathbb{Z}_2$ is well defined, i.e., $\varphi_0(X)$ does not depend on the choice of the map $f: M \rightarrow \Omega$ such that $f^*(e_1) = X$.*

Proof. Let $f, g: M \rightarrow \Omega$ be two maps such that $f^*(e_1) = g^*(e_1)$. We have to show that $f^*(u_2) = g^*(u_2)$. Let K again be a triangulation of M . Since $f^*(e_1) - g^*(e_1) = 0$, it follows that f and g are 5-homotopic. (See S. T. HU [2], Chap. VI.) Since $H^q(M; \pi_q(\Omega)) = 0$ for $5 < q < 10$, it follows that f and g are 9-homotopic. Hence, we may assume that $f|K^{(9)} = g|K^{(9)}$. Let $\omega^{10}(f, g) \in C^{10}(K; \pi_{10}(\Omega))$ be the difference cochain. Then,

$$(f^*(u_2) - g^*(u_2))[s_{10}] = u_2[h\omega^{10}(f, g)[s_{10}]] ,$$

for every 10-simplex s_{10} , where $h: \pi_{10}(\Omega) \rightarrow H_{10}(\Omega)$ is the HUREWICZ homomorphism. According to J. P. SERRE, $u_2[h\alpha]$ is the mod. 2 HOPF invariant of the element in $\pi_{11}(S^6)$ represented by $\alpha \in \pi_{10}(\Omega S^6)$. (Compare [8], Lemme 2.) Since no element of odd HOPF invariant occurs in $\pi_{11}(S^6)$, it follows that $f^*(u_2) = g^*(u_2)$, and the proof is complete.

Lemma 1.3. *Let $X, Y \in H^5(M)$ be two integer cohomology classes of M . Then,*

$$\varphi_0(X + Y) = \varphi_0(X) + \varphi_0(Y) + x \cdot y ,$$

where $x \cdot y$ is the value on the generator of $H_{10}(M^{10}; \mathbb{Z}_2)$ of the cup-product $x \cup y$. (x, y are the mod. 2 reductions of X and Y respectively.)

Proof. Let $f, g: M \rightarrow \Omega$ be maps such that $f^*(e_1) = X$ and $g^*(e_1) = Y$. By definition, $\varphi_0(X) = f^*(u_2)[M]$, and $\varphi_0(Y) = g^*(u_2)[M]$.

Let $f \times g: M \times M \rightarrow \Omega \times \Omega$ be the product of f and g . (I.e., $f \times g(u, v) = (f(u), g(v))$.) Let $D: M \rightarrow M \times M$ be the diagonal map. Define $F: M \rightarrow \Omega$

by $F = \pi \circ (f \times g) \circ D$, where $\pi: \Omega \times \Omega \rightarrow \Omega$ is given by the multiplication of loops. Since D^* maps the tensor product of cohomology classes into their cup-product, we have $F^*(e_1) = D^*(X \otimes 1 + 1 \otimes Y) = X + Y$. Therefore,

$$\varphi_0(X + Y) = F^*(u_2)[M].$$

On the other hand,

$$\begin{aligned} F^*(u_2) &= D^*(f^*(u_2) \otimes 1 + 1 \otimes g^*(u_2) + f^*(u_1) \otimes g^*(u_1)) \\ &= f^*(u_2) + g^*(u_2) + f^*(u_1) \cup g^*(u_1) \\ &= f^*(u_2) + g^*(u_2) + x \cup y. \end{aligned}$$

(u_1 is the reduction modulo 2 of e_1 .) This proves Lemma 1.3.

The function $\varphi_0: H^5(M) \rightarrow \mathbb{Z}_2$ induces a function $\varphi: H^5(M; \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$ satisfying $\varphi(x + y) = \varphi(x) + \varphi(y) + x \cdot y$. Indeed, if X is an integer class whose reduction modulo 2 yields $x \in H^5(M; \mathbb{Z}_2)$, we define $\varphi(x) = \varphi_0(X)$. It follows from

$$\varphi_0(2Y) = \varphi_0(Y) + \varphi_0(Y) + y \cdot y = y \cdot y = 0,$$

that $\varphi(x) \in \mathbb{Z}_2$ depends only on $x \in H^5(M; \mathbb{Z}_2)$.

The function $\varphi: H^5(M; \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$ is then used to construct the number $\Phi(M)$ as follows. A basis $x_1, \dots, x_s, y_1, \dots, y_s$ of $H^5(M; \mathbb{Z}_2)$ as a vector space over \mathbb{Z}_2 will be called *symplectic* if $x_i \cdot x_j = y_i \cdot y_j = 0$, and $x_i \cdot y_j = \delta_{ij}$ for all $i, j = 1, \dots, s$. Clearly, symplectic bases always exist. Moreover, it is well known that since the function $\varphi: H^5(M; \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$ satisfies the equation

$$\varphi(x + y) = \varphi(x) + \varphi(y) + x \cdot y,$$

the remainder modulo 2

$$\Phi(M) = \sum_i \varphi(x_i) \cdot \varphi(y_i)$$

is independent of the symplectic basis $x_1, \dots, x_s, y_1, \dots, y_s$.

The rest of the paper is devoted to investigating the properties of the invariant Φ .

Clearly, Φ is an invariant of the homotopy type of 4-connected closed manifolds of dimension 10.

Our objective is the proof of the following theorems.

Theorem 1. *If M^{10} has the homotopy type of a C^1 -differentiable 4-connected closed manifold, then $\Phi(M) = 0$.*

(It can be shown that the converse of this theorem would follow from the conjecture that the cohomology ring $H^*(M)$ and $\Phi(M)$ are a complete set of invariants of the homotopy type of the triangulable 4-connected closed manifold M of dimension 10.)

Theorem 2. *There exists a closed 4-connected combinatorial manifold M_0 of dimension 10 for which $\Phi(M_0) = 1$.*

(In fact a specific example will be constructed.)

In § 2, the proof of Theorem 1 will be carried out taking Lemmas 4.2 and 5.1 for granted. (Lemma 4.2 is used in the proof of Lemma 2.2, and Lemma 5.1 is used to deduce Theorem 1 from Lemma 2.4.) The Lemmas 4.2 and 5.1 are proved at the end of the paper, in § 4 and § 5. Theorem 2 will be proved in § 3.

§ 2. Proof of Theorem 1

Let M^{10} be a closed C^1 -differentiable manifold which is 4-connected.

Lemma 2.1. *M^{10} is a π -manifold.*

Proof. Let $M^{10} \subset R^{n+10}$ be an imbedding with n large. We have to show that the normal bundle ν is trivial. This is done by constructing a field of normal n -frames f_n . Let K be a triangulation of M^{10} . Since $\pi_4(SO_n) = 0$, and M^{10} is 4-connected, it follows that $H^{q+1}(M; \pi_q(SO_n)) = 0$ for $0 \leq q < 9$. Thus, there is only one possibly non-vanishing obstruction $\delta(\nu, f_n) \in H^{10}(M; \pi_9(SO_n)) \cong \pi_9(SO_n)$ to the construction of the field f_n of normal n -frames. By Lemma 1 of [7], $\delta(\nu, f_n)$ is in the kernel of the HOPF-WHITEHEAD homomorphism $J_9: \pi_9(SO_n) \rightarrow \pi_{n+9}(S^n)$. But J_9 is a monomorphism. (Compare proof of Lemma 1.2 of [4].) Hence, $\delta(\nu, f_n) = 0$, and the lemma is proved. (Recall that the proof of the assertion: J_9 is a monomorphism, was based on Corollary 2.6 of J. F. ADAMS paper *On the structure and applications of the STEENROD algebra*, Comm. Math. Helv. 32 (1958), 180–214. This statement also follows from the portion of the POSTNIKOV decomposition mod. 2 of S^n given below in § 5.)

The THOM construction associates with every framed manifold $(M; f_n)$, where $M \subset R^{n+\dim M}$, an element $\alpha(M; f_n) \in \pi_{n+\dim M}(S^n)$. We say that $(M^{10}; f_n)$ is *homotopic to zero* if the corresponding element $\alpha(M; f_n)$ is the neutral element of $\pi_{n+10}(S^n)$.

Lemma 2.2. *If $(M^{10}; f_n)$ is homotopic to zero, where M^{10} is 4-connected, then $\Phi(M) = 0$.*

Proof. The assumption that $(M; f_n)$ is homotopic to zero implies the existence of a framed manifold $(V^{11}; F_n)$ with boundary M^{10} . (Compare R. THOM [10].) We may assume that V is connected, and hence has a trivial tangent bundle. We can therefore apply to $V - M$ the procedure for killing the homotopy groups of a differentiable manifold studied by J. MILNOR. Specifically, using Theorem 3 of [6], we obtain a new 11-dimensional differen-

tiable manifold with boundary M^{10} which is also 4-connected. This new 4-connected manifold will again be denoted by V^{11} . We can now forget about the fields of normal frames.

We proceed to compute $\Phi(M)$. Consider the cohomology exact sequence of the pair (V, M) with coefficients in \mathbb{Z}_2 ,

$$\dots \rightarrow H^5(V) \xrightarrow{i^*} H^5(M) \xrightarrow{\delta} H^6(V, M) \rightarrow \dots .$$

Using relative Poincaré-Lefschetz duality (over \mathbb{Z}_2), and the formula

$$u \cup \delta x[V, M] = i^*(u) \cup x[M],$$

where $u \in H^5(V)$, $x \in H^5(M)$ and $[V, M]$, $[M]$ are the generators of $H_{11}(V, M; \mathbb{Z}_2)$ and $H_{10}(M; \mathbb{Z}_2)$ respectively, it follows that $H^5(M; \mathbb{Z}_2)$ has a symplectic basis $x_1, \dots, x_s, y_1, \dots, y_s$ say, such that x_1, \dots, x_s is a vector basis of $\text{Ker } \delta$. Consequently, in order to prove $\Phi(M) = 0$, it is sufficient to show that $\varphi(x) = 0$ for every $x \in \text{Ker } \delta$.

Let $X \in H^5(M)$ be an integer class whose reduction modulo 2 is x , and let $f: M^{10} \rightarrow \Omega = \Omega S^6$ be a map such that $f^*(e_1) = X$. We have to show that $f^*(u_2) = 0$, where u_2 generates $H^{10}(\Omega; \mathbb{Z}_2)$. Let Ω^* be the space obtained from Ω by attaching a cell of dimension 6 by a map $S^5 \rightarrow \Omega$ of degree 2. By Lemma 4.2 in § 4, below, for every map $g: S^{10} \rightarrow \Omega^*$, one has $g^*(u_2) = 0$, where we denote by $u_2 \in H^{10}(\Omega^*; \mathbb{Z}_2)$ again the class corresponding to the old $u_2 \in H^{10}(\Omega; \mathbb{Z}_2)$ under the canonical isomorphism $H^{10}(\Omega; \mathbb{Z}_2) \cong H^{10}(\Omega^*; \mathbb{Z}_2)$.

We attempt to extend $f: M \rightarrow \Omega^*$ to a map of V into Ω^* . Let (K, L) be a triangulation of (V, M) . The stepwise extension of f on the skeletons $K^{(q)} \cup L$ leads to obstructions in the groups $H^{q+1}(K, L; \pi_q(\Omega^*))$. For $q < 5$, $\pi_q(\Omega^*) = 0$. We meet a first obstruction for $q = 5$ in $H^6(K, L; \mathbb{Z}_2)$. By the HOPF theorem, this obstruction is δx . (See S. T. Hu [2].) Since $\delta x = 0$, it is possible to extend f on $K^{(6)} \cup L$. Using $H^{q+1}(K, L; \mathbb{G}) = 0$ for $5 < q < 10$ (since V is 4-connected), it follows that there exists a map $F: K - \tau \rightarrow \Omega^*$, where τ is some 11-dimensional simplex in $K - L$, such that $F|L = f$. Let S^{10} denote the boundary of τ , and let $g: S^{10} \rightarrow \Omega^*$ be the restriction of F on S^{10} . Since $\partial(K - \tau) = L - S^{10}$, and $g^*(u_2) = 0$, it follows that $f^*(u_2) = 0$. The proof of Lemma 2.2 is complete.

Corollary 2.3. *If two 4-connected framed manifolds $(M; f_n)$ and $(M'; f'_n)$ of dimension 10 define the same element $\alpha = \alpha(M; f_n) = \alpha(M'; f'_n)$ by the Thom construction, then $\Phi(M) = \Phi(M')$.*

This is obtained by observing that Φ is additive with respect to the connected sum of manifolds.

It follows that Φ provides a homomorphism from a subgroup of $\pi_{n+10}(S^n)$

into Z_2 . We denote this homomorphism by Φ again. Actually, Φ is defined on every element of $\pi_{n+10}(S^n)$. Indeed, using spherical modifications [6], it is easy to see that every element $\alpha \in \pi_{n+10}(S^n)$ is obtainable from a 4-connected framed manifold by the THOM construction. This remark will not be used in the present paper.

It follows from Corollary 2.3 that Theorem 1 is equivalent to the statement that $\Phi(\alpha) = 0$ for every $\alpha \in \pi_{n+10}(S^n)$, provided $\Phi(\alpha)$ is defined.

Since $\Phi(\alpha)$ is obviously zero for every element α of odd order, and by J. P. SERRE's results $\pi_{n+10}(S^n)$ contains no element of infinite order, it is sufficient to show that Φ annihilates the 2-component of the group $\pi_{n+10}(S^n)$. By Lemma 5.1 in § 5 below, every element α in the 2-component of $\pi_{n+10}(S^n)$ is representable in the form

$$\alpha = \beta \circ \eta ,$$

where $\eta \in \pi_{n+10}(S^{n+9})$ is the generator of the stable 1-stem, and $\beta \in \pi_{n+9}(S^n)$. Hence, Theorem 1 will follow from the

Lemma 2.4. *Every element $\alpha \in \pi_{n+10}(S^n)$ of the form $\alpha = \beta \circ \eta$, with $\eta \in \pi_{n+10}(S^{n+9})$, and $\beta \in \pi_{n+9}(S^n)$ is obtainable by the THOM construction from a framed manifold $(\Sigma^{10}; f_n)$, where Σ^{10} has the homotopy type of the 10-sphere S^{10} .*

Proof. We first show that $\beta \in \pi_{n+9}(S^n)$ is obtainable by the THOM construction from a framed manifold $(\Sigma^9; f_n)$, where Σ^9 has the homotopy type of the 9-sphere.

It is well known that β is obtainable by the THOM construction from some framed manifold $(M^9; f_n)$. We have to show that $(M^9; f_n)$ is homotopic to a framed manifold $(\Sigma^9; f_n)$, where Σ^9 is a homotopy sphere. This is done by simplifying M^9 by a series of spherical modifications. (See J. MILNOR [6].)

Assuming by induction that M^9 is $(p-1)$ -connected ($0 \leq p \leq 4$), we have to prove that $(M^9; f_n)$ is homotopic to a p -connected framed manifold $(M'; f'_n)$. Recall that a spherical modification of type $(p+1, q+1)$ applied to a class $\lambda \in \pi_p(M^9)$ consists of the following construction. Represent λ by an imbedding

$$f: S^p \times D^{q+1} \rightarrow M^9 ,$$

with $p+q+1 = 9$. (This is possible for $p \leq 4$ since M^9 is a π -manifold and the normal bundle of any imbedding $S^p \rightarrow M^9$ is stable in this range of dimensions.) The manifold M is then replaced by

$$M' = (M^9 - f(S^p \times D^{q+1})) \cup (D^{p+1} \times S^q) ,$$

under identification of $f(S^p \times S^q)$ regarded as the boundary of $f(S^p \times D^{q+1})$ with $S^p \times S^q$ regarded as the boundary of $D^{p+1} \times S^q$. By Theorem 2 of

[6], the manifolds M and M' bound a 10-dimensional differentiable manifold $\omega = \omega(M, f)$, and $f: S^p \times D^{q+1} \rightarrow M^9$ can be chosen such that the field f_n (over M) is extendable over ω as a field of normal n -frames. (We can think of ω as imbedded in R^{n+10} with $M \subset R^{n+9} \times (0)$ and $M' \subset R^{n+9} \times (1)$ since n can be taken as large as we please.) Hence spherical modifications of type $(p+1, q+1)$ with $0 \leq p \leq 4$ can be performed so as to carry $(M; f_n)$ into a homotopic framed manifold. It is known (Theorem 3 of [6]) that for $p < 4$, spherical modifications simplify the manifold. More precisely $\pi_p(M')$ is isomorphic to the quotient of $\pi_p(M)$ by the subgroup generated by λ , and $\pi_i(M) \cong \pi_i(M') = 0$ for $i < p$. Hence, it is easy, using [6], to obtain a 3-connected framed manifold homotopic to $(M^9; f_n)$. The case $p = 4$ requires special care. If $\lambda \in \pi_4(M^9)$ is the class we want to kill, there exists an imbedding $f: S^4 \times D^5 \rightarrow M^9$ such that $f|S^4 \times (0)$ represents λ . Let $M' = \chi(M, f)$ be the 9-dimensional manifold obtained from M and f by spherical modification. (f is supposed to be chosen so that $(M'; f'_n)$ with some f'_n is homotopic to $(M; f_n)$.) In general, however, $f|x_0 \times (\text{bdry } D^5)$ represents a non-zero element of $\pi_4(M')$. Thus, it is not clear a priori that a series of spherical modifications of type $(5, 5)$ will carry M into a 4-connected manifold, and hence a homotopy sphere.

If λ is a generator of the free part of $\pi_4(M) \cong H_4(M)$, there exists by POINCARÉ duality a class $\mu \in H_6(M)$ whose intersection coefficient with λ (or $h\lambda$ rather, where h is the HUREWICZ homomorphism) is 1. It follows that in this case the cycle given by $f|x_0 \times (\text{bdry } D^5)$ is homologous to zero in $M - f(S^4 \times D^5)$, and hence in M' . Thus $H_4(M') \cong \pi_4(M')$ has strictly smaller rank than $H_4(M) \cong \pi_4(M)$, and the torsion subgroup is unchanged.

I claim that if $\lambda \in \pi_4(M)$ is a torsion element, the homology class of the cycle $f|x_0 \times (\text{bdry } D^5)$ is of infinite order for any f representing λ . Hence, one more spherical modification will lead to a manifold with 4-dimensional homology group of not bigger rank than $H_4(M)$ and with a strictly smaller torsion subgroup. (I.e., a series of spherical modifications will lead to a 4-connected framed manifold homotopic to $(M^9; f_n)$. By POINCARÉ duality, a closed 4-connected manifold of dimension 9 has the homotopy type of S^9 .)

Since the BETTI numbers p_4, p'_4 of M and M' (in dimension 4) differ at most by 1, and differ indeed by 1 if and only if λ' (represented by $f|x_0 \times (\text{bdry } D^5)$) in M' is of infinite order, it is sufficient to show that $p'_4 + p_4 \equiv 1 \pmod{2}$. Since $p'_i = p_i$ for $0 \leq i \leq 3$, this is equivalent to showing that the semi-characteristics $E^*(M)$ and $E^*(M')$ of M and M' (over the rationals, say) satisfy $E^*(M') + E^*(M) \equiv 1 \pmod{2}$. We use the formula

$$E^*(M') + E^*(M) \equiv E(\omega) + r \pmod{2},$$

where $E(\omega)$ is the EULER characteristic of the manifold ω with boundary $\dot{\omega} = M' - M$, and r is the rank of the bilinear form on $H^5(\omega, \dot{\omega}; \mathbb{Q})$ defined by the cup-product. (Compare M. A. KERVAIRE [3], § 8, formula (8.9).) It is easily seen that $E(\omega) = 1$, up to sign, and since $u \cdot u = 0$ for every $u \in H^5(\omega, \dot{\omega}; \mathbb{Q})$, the rank r must be even: $r \equiv 0 \pmod{2}$. Hence, $E^*(M') + E^*(M) \equiv 1 \pmod{2}$.

Summarizing, we have proved so far that every $\beta \in \pi_{n+9}(S^n)$ is obtainable by the THOM construction from a framed manifold $(\Sigma^9; f_n)$, where the manifold Σ^9 has the homotopy type of S^9 .

Taking a representative $f: S^{n+10} \rightarrow S^{n+9}$ of η such that $f^{-1}(S^{n+9} - x_0)$ is diffeomorphic to $S^1 \times (S^{n+9} - x_0)$, we obtain that $\alpha = \beta \circ \eta$ is obtainable by the THOM construction from $(S^1 \times \Sigma^9; f_n)$.

It remains to show that $(S^1 \times \Sigma^9; f_n)$ is homotopic to a framed manifold $(\Sigma^{10}; f'_n)$, where Σ^{10} is a homotopy sphere.

Apply once more the spherical modification theorems (Theorems 2 and 3 of [6]), this time to the class $\lambda \in \pi_1(S^1 \times \Sigma^9)$ represented by $S^1 \times (z_0)$. The resulting framed manifold is homotopic to $(S^1 \times \Sigma^9; f_n)$ and has the homotopy type of the 10-sphere. This completes the proof of Lemma 2.4.

To complete the proof of Theorem 1 it remains to prove the Lemmas 4.2, and 5.1. This is done in § 4 and § 5.

§ 3. Construction of M_0

This section relies on J. MILNOR's paper [5]. Let $f_0: S^4 \rightarrow SO_4$ be a differentiable map whose homotopy class (f_0) satisfies

$$i_*(f_0) = \partial i_5 ,$$

where $\partial: \pi_5(S^5) \rightarrow \pi_4(SO_5)$ is taken from the homotopy exact sequence of SO_6/SO_5 , and $i: SO_4 \rightarrow SO_5$ is the usual inclusion. Define $f_1 = f_2 = i \circ f_0$. Using $f_1, f_2: S^4 \rightarrow SO_5$, a diffeomorphism $f: S^4 \times S^4 \rightarrow S^4 \times S^4$ is given by $f(x, y) = (x', y')$, where $y' = f_1(x) \cdot y$, and $x = f_2(y') \cdot x'$. Let $M(f_1, f_2)$ be the MILNOR manifold obtained from the disjoint union of $D^5 \times S^4$ and $S^4 \times D^5$ by identifying each point (x, y) in the boundary of $D^5 \times S^4$ with $f(x, y)$, considered as a point on the boundary of $S^4 \times D^5$. By Lemma 1 of [5], together with the remark at the bottom of page 963 in the proof of Lemma 1 in [5], it follows that the differentiable manifold $M(f_1, f_2)$ is homeomorphic to the 9-sphere. It will follow from Theorem 1 in this paper, that $M(f_1, f_2)$ is not diffeomorphic to the standard S^9 . Let W^{10} be the differentiable mani-

fold with boundary $M(f_1, f_2)$ obtained using the construction on page 964 of [5]. W can alternately be described as follows. Let U be a tubular neighborhood of the diagonal Δ in $S^5 \times S^5$. It is well known that U is the space of the fibre bundle $p: U \rightarrow S^5$ with fibre D^5 associated with the tangent bundle of S^5 . The differentiable manifold W is obtained by straightening the angles of the quotient space of the disjoint union of two copies U' and U'' of U under an identification of $p'^{-1}(V)$ with $p''^{-1}(V)$ such that the images of Δ' and Δ'' in W have intersection number 1. (V is an imbedded 5-disc on S^5 , and $p'^{-1}(V) \cong D^5 \times D^5$ is identified with $p''^{-1}(V) \cong D^5 \times D^5$ under $(u, v) \leftrightarrow (v, u)$, $u, v \in D^5$.)

Since W is a 10-dimensional manifold whose boundary $M(f_1, f_2)$ is homeomorphic to S^9 , the union of W with the cone over the boundary is a 10-dimensional closed manifold M_0 . Since $M(f_1, f_2)$ is combinatorially equivalent to S^9 , it follows that M_0 possesses a combinatorial structure. (Compare J. MILNOR, *On the relationship between differentiable manifolds and combinatorial manifolds*, mimeographed notes 1956, § 4.)

It is easily seen that M_0 is 4-connected.

We proceed to compute $\Phi(M_0)$. Let $x, y \in H^5(M_0; \mathbb{Z}_2)$ be the cohomology classes dual to the homology classes of the imbedded spheres $j', j'': S^5 \rightarrow M_0$ given by the images in W of the diagonals Δ' and Δ'' in U' and U'' respectively. Clearly, x, y is a symplectic basis of $H^5(M_0; \mathbb{Z}_2)$. (I.e., $x \cdot x = y \cdot y = 0$, and $x \cdot y = 1$.) To show that $\varphi(x) = \varphi(y) = 1$, observe that the normal bundles of j' and j'' (regarded as imbeddings of S^5 in the *differentiable* manifold W) are non-trivial. These bundles are isomorphic to $p: U \rightarrow S^5$. Let K be the THOM complex of this bundle. (I.e., the space obtained by collapsing the boundary of U to a point.) It is well known that K admits a cell decomposition $S^5 \cup e^{10}$, where the attaching map $S^9 \rightarrow S^5$ is a representative of the WHITEHEAD product $[i_5, i_5]$. On the other hand, the THOM construction provides a map $f_0: M_0 \rightarrow K$ such that $f_0^*(e_1) = X$, the dual class of $j': S^5 \rightarrow M_0$, and $f_0^*(u_2)[M_0] = 1$, where e_1 generates $H^5(K; \mathbb{Z})$ and u_2 generates $H^{10}(K; \mathbb{Z}_2)$. A map $f: M_0 \rightarrow \Omega S^6$ is obtained by composition of f_0 with the usual inclusion $S^5 \cup e^{10} \rightarrow \Omega S^6$. (Recall that ΩS^6 has a cell decomposition $\Omega S^6 = S^5 \cup e^{10} \cup e^{15} \cup e^{20} \cup \dots$, where the attaching map of e^{10} represents $[i_5, i_5]$.) Then, $f: M_0 \rightarrow \Omega S^6$ has the properties $f^*(e_1) = X$, $f^*(u_2) = 1$, showing that $\varphi(x) = 1$. The same construction applied to Y , the dual class of $j'': S^5 \rightarrow M_0$ yields $\varphi(y) = 1$. Hence $\Phi(M_0) = \varphi(x) \cdot \varphi(y) = 1$.

If $M(f_1, f_2)$, with the differentiable structure induced by W (of which $M(f_1, f_2)$ is the boundary) were diffeomorphic to S^9 with the standard differentiable structure, the differentiable structure on W could be extended to a differentiable structure over the cone $C M(f_1, f_2)$, providing a differentiable

structure on M_0 . However, $\Phi(M_0) = 1$ and Theorem 1 show that a differentiable structure on M_0 does not exist. Hence, $M(f_1, f_2)$, homeomorphic to S^9 , is not diffeomorphic to S^9 .

§ 4. The auxiliary space Ω^*

Let $Y = S^5 \cup_{2i_5} e^6$ be the space obtained by attaching a 6-cell to S^5 by a map $S^5 \rightarrow S^5$ of degree 2.

Lemma 4.1. *Let $\alpha \in \pi_5(Y) \cong \mathbb{Z}_2$ be the generator, then $[\alpha, \alpha] \neq 0 \in \pi_9(Y)$.*

Proof. We identify Y with the STIEFEL manifold $V_{7,2}$. Consider the exact sequence

$$\cdots \rightarrow \pi_{10}(S^6) \rightarrow \pi_9(S^5) \xrightarrow{i^*} \pi_9(V_{7,2}) \rightarrow \cdots.$$

Since $\pi_{10}(S^6) = 0$, and $[i_5, i_5]$ is non-zero in $\pi_9(S^5)$, it follows that $i_*[i_5, i_5] = [i_*(i_5), i_*(i_5)] = [\alpha, \alpha] \neq 0$.

Let $Y^* = Y \cup e^{10}$ be the space obtained from Y by attaching a 10-cell e^{10} using a representative $f: S^9 \rightarrow Y$ of $[\alpha, \alpha]$. Since Y is 4-connected, the characteristic map $\hat{f}: (D^{10}, S^9) \rightarrow (Y^*, Y)$ of e^{10} induces an *isomorphism*

$$\hat{f}_*: \pi_{10}(D^{10}, S^9) \rightarrow \pi_{10}(Y^*, Y).$$

(Compare J. H. C. WHITEHEAD [12], Theorem 1.) Thus the relative HUREWICZ homomorphism $h_R: \pi_{10}(Y^*, Y) \rightarrow H_{10}(Y^*, Y) \cong \mathbb{Z}$ is an isomorphism. Consider the homotopy-homology ladder of (Y^*, Y) :

$$\begin{array}{ccccccc} \cdots & \rightarrow & \pi_{10}(Y) & \xrightarrow{j_0} & \pi_{10}(Y^*, Y) & \xrightarrow{\partial} & \pi_9(Y) \rightarrow \cdots \\ & & \downarrow & & \downarrow h & & \downarrow \\ \cdots & \rightarrow & 0 & \rightarrow & H_{10}(Y^*) & \xrightarrow{j^*} & H_{10}(Y^*, Y) \rightarrow 0 \rightarrow \cdots. \end{array}$$

Since ∂ sends the generator of $\pi_{10}(Y^*, Y)$ into $[\alpha, \alpha] \neq 0$, and $2[\alpha, \alpha] = 0$, it follows that every element in $\text{Im}\{h: \pi_{10}(Y^*) \rightarrow H_{10}(Y^*)\}$ can be halved.

It follows that for every map $g_0: S^{10} \rightarrow Y^*$, the induced homomorphism $g_0^*: H^{10}(Y^*; \mathbb{Z}_2) \rightarrow H^{10}(S^{10}; \mathbb{Z}_2)$ is zero.

Let Ω be the space of loops over S^6 . Up to homotopy type $\Omega = S^5 \cup e^{10} \cup e^{15} \cup \dots$, with e^{10} attached by a map of class $[i_5, i_5]$. Let $\Omega^* = \Omega \cup e^6$, where e^6 is attached by a map of degree 2 on $S^5 \subset \Omega$. There is a natural inclusion $Y^* \rightarrow \Omega^*$ which induces an isomorphism on cohomology groups in dimension 10. Hence, we have the

Lemma 4.2. *Let $g: S^{10} \rightarrow \Omega^*$ be a map, and let u_2 be the generator of $H^{10}(\Omega^*; \mathbb{Z}_2) \cong \mathbb{Z}_2$. Then, $g^*(u_2) = 0$.*

§ 5. A lemma on homotopy groups of spheres

Lemma 5.1. *The map $\pi_{n+9}(S^n) \rightarrow \pi_{n+10}(S^n)$, for $n \geq 12$, defined by composition with the generator η of $\pi_{n+10}(S^{n+9})$ is surjective on the 2-component.*

This lemma was communicated to me without proof by H. TODA who has also proved that the 2-component of $\pi_{n+10}(S^n)$ is \mathbb{Z}_2 . (See H. TODA [11], Corollary to Proposition 4.10.)

We give a sketch of proof by computation of the POSTNIKOV decomposition modulo 2 of S^n for large n , up to dimension $n+10$.

We begin with a remark which will yield Lemma 5.1 whenever a long enough portion of the POSTNIKOV decomposition of S^n is obtained. Let $X = K(\mathbb{Z}_2, n+9) \times_k K(\mathbb{Z}_2, n+10)$ be the space of the fibration over $K(\mathbb{Z}_2, n+9)$ associated with the k -invariant $k \in H^{n+11}(\mathbb{Z}_2, n+9; \mathbb{Z}_2)$. Let $f: S^{n+9} \rightarrow X$ be a map representing the generator of $\pi_{n+9}(X) \cong \mathbb{Z}_2$. Then, the composition

$$f \circ \eta: S^{n+10} \rightarrow X, \text{ where } \eta: S^{n+10} \rightarrow S^{n+9}$$

represents the generator of $\pi_{n+10}(S^{n+9})$, is essential if and only if $k = Sq^2(\varepsilon)$, where ε is the fundamental class of $H^{n+9}(\mathbb{Z}_2, n+9; \mathbb{Z}_2)$.

Since $Sq^2(\varepsilon)$ generates $H^{n+11}(\mathbb{Z}_2, n+9; \mathbb{Z}_2)$, it follows that $k \neq Sq^2(\varepsilon)$ implies $k = 0$. Hence, $f \circ \eta$ is inessential if $k \neq Sq^2(\varepsilon)$.

If $k = Sq^2(\varepsilon)$, let $\hat{f}: S^{n+9} \cup_{\eta} e^{n+11} \rightarrow X \cup_{f \circ \eta} e^{n+11}$ be the map induced by f . Let $s \in H^{n+9}(S^{n+9} \cup_{\eta} e^{n+11}; \mathbb{Z}_2)$ be the generator. We identify $H^{n+9}(X \cup e^{n+11}; \mathbb{Z}_2)$ and $H^{n+9}(X; \mathbb{Z}_2)$ with $H^{n+9}(\mathbb{Z}_2, n+9; \mathbb{Z}_2)$. Since $f^*(\varepsilon) = s$, and $Sq^2(s) \neq 0$, it follows that $Sq^2(\varepsilon) \neq 0$ in $H^{n+11}(X \cup e^{n+11}; \mathbb{Z}_2)$. To show that $f \circ \eta$ is essential, it is therefore sufficient to show that $Sq^2(\varepsilon) = 0$ in $H^{n+11}(X; \mathbb{Z}_2)$. This follows from the commutativity of the diagram

$$\begin{array}{ccccccc} 0 & \leftarrow & H^{n+9}(X; \mathbb{Z}_2) & \leftarrow & H^{n+9}(\mathbb{Z}_2, n+9; \mathbb{Z}_2) & \leftarrow & 0 \\ & \downarrow Sq^2 & \approx & \downarrow Sq^2 & & & \\ H^{n+11}(X; \mathbb{Z}_2) & \leftarrow & H^{n+11}(\mathbb{Z}_2, n+9; \mathbb{Z}_2) & \xleftarrow{\tau} & H^{n+10}(\mathbb{Z}_2, n+10; \mathbb{Z}_2), & & \end{array}$$

where the rows are taken from the exact sequence of the fibration defining X (in the stable range), and τ is the transgression.

Let $Y_{10} \rightarrow Y_9 \rightarrow \dots \rightarrow Y_i \rightarrow Y_{i-1} \rightarrow \dots \rightarrow Y_0 = K(\mathbb{Z}, n)$ be the modulo 2 POSTNIKOV decomposition of S^n . (I.e., $p_i: Y_i \rightarrow Y_{i-1}$ is a fibration with fibre $F_i = K(\pi_i, n+i)$, where π_i is the 2-component of the stable group $\pi_{n+i}(S^n)$, and $H^*(Y_i; \mathbb{Z}_2)$ contains \mathbb{Z}_2 in dimension 0 and n , $H^q(Y_i; \mathbb{Z}_2) = 0$ for $0 < q < n$, and $H^{n+k}(Y_i; \mathbb{Z}_2) = 0$ for $0 < k < i+2$.) By the \mathfrak{C} -theory with \mathfrak{C} = the class of finite groups whose order is prime to

2, a map $S^n \rightarrow Y_i$ inducing an isomorphism $H^n(Y_i; \mathbb{Z}_2) \cong H^n(S^n; \mathbb{Z}_2)$ induces an isomorphism of the 2-component of $\pi_{n+k}(S^n)$ with $\pi_{n+k}(Y_i)$ for $k \leq i$. (Compare J. P. SERRE [8].) We have $\pi_9 \cong \mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2$ and $\pi_{10} \cong \mathbb{Z}_2$ as will be seen below, thus

$$F_9 = K(\mathbb{Z}_2, n+9) \times K(\mathbb{Z}_2, n+9) \times K(\mathbb{Z}_2, n+9),$$

and Lemma 5.1 follows by showing that the restriction of the fibration $Y_{10} \rightarrow Y_9$ over one of the factors of F_9 is $K(\mathbb{Z}_2, n+9) \times_k K(\mathbb{Z}_2, n+10)$ with $k = Sq^2$. This is equivalent to showing that $H^{n+11}(Y_9; \mathbb{Z}_2) \cong \mathbb{Z}_2$ is generated by a class u_9 such that $i_9^*(u_9) = Sq^2(\varepsilon_9)$, where ε_9 is one of the fundamental classes of $H^9(F_9; \mathbb{Z}_2)$, and $i_9: F_9 \rightarrow Y_9$ is the inclusion.

In a similar way, it can be read off from the tables below that composition with η provides injective maps $\pi_{n+7}(S^n) \otimes \mathbb{Z}_2 \rightarrow \pi_{n+8}(S^n)$ and $\pi_{n+8}(S^n) \rightarrow \pi_{n+9}(S^n)$ in the stable range. Using $\pi_7(SO_n) \cong \mathbb{Z}$, $\pi_8(SO_n) \cong \mathbb{Z}_2$, and $\pi_9(SO_n) \cong \mathbb{Z}_2$, this implies that $J_9: \pi_9(SO_n) \rightarrow \pi_{n+9}(S^n)$ is a monomorphism.

We proceed to a partial description of the modulo 2 cohomology of the spaces Y_7 .

$H^*(Y_0)$ is given by J. P. SERRE in [9]. This result of J. P. SERRE and the ADEM relations between the STEENROD squares are the essential tools in computing $H^*(Y_k; \mathbb{Z}_2)$ for $k > 0$. Since we stay in the stable range, the spectral sequences of $p_k: Y_k \rightarrow Y_{k-1}$ reduce to exact sequences

$$\dots \leftarrow H^{n+q+1}(Y_{k-1}) \xleftarrow{\tau} H^{n+q}(F_k) \xleftarrow{i_k^*} H^{n+q}(Y_k) \xleftarrow{p_k^*} H^{n+q}(Y_{k-1}) \leftarrow \dots.$$

It is therefore sufficient to determine at each step the kernel and the image of the transgression τ . Since the cohomology of Y_k is independent of k up to dimension n , we omit to mention the non-vanishing cohomology groups in dimension $\leq n$. The direct sum of the subgroups of $H^*(Y_k; \mathbb{Z}_2)$ in dimensions $> n$ is denoted $H^+(Y_k)$.

The symbol q_k stands for the composition $p_1 \circ p_2 \circ \dots \circ p_k$, and ε_k denotes the fundamental class of $H^{n+k}(G, n+k; G)$.

I omit Y_1 and Y_2 whose cohomology is straightforward, but has to be computed up to dimension $n+17$ and $n+16$ respectively. $H^{n+4}(Y_2; \mathbb{Z}_2)$ is generated by $q_2^*(Sq^4 \varepsilon_0)$, and $H^{n+5}(Y_2; \mathbb{Z}_2)$ by a class u_2 such that $i_2^*(u_2) = Sq^3(\varepsilon_2)$.

$F_3 = K(\mathbb{Z}_8, n+3)$, with $\tau(\varepsilon'_3) = q_2^*(Sq^4 \varepsilon_0)$ and $\tau(\beta \varepsilon_3) = u_2$, where β is the BOCKSTEIN operator associated with the sequence of coefficients $0 \rightarrow \mathbb{Z}_2 \rightarrow \mathbb{Z}_{16} \rightarrow \mathbb{Z}_8 \rightarrow 0$, and ε'_3 is the mod. 2 reduction of ε_3 .

$H^+(Y_3)$ has a basis consisting of

u_3 in dimension $n + 7$, such that $i_3^*(u_3) = Sq^4 \varepsilon_3'$;
 $Sq^1(u_3), q_3^*(Sq^8 \varepsilon_0); Sq^2(u_3), v_3$ such that $i_3^*(v_3) = Sq^5 \beta \varepsilon_3; Sq^3(u_3); Sq^4(u_3); Sq^5(u_3), Sq^4 Sq^1(u_3), q_3^*(Sq^{12} \varepsilon_0); Sq^6(u_3), Sq^4 Sq^2(u_3), Sq^4(v_3); Sq^6 Sq^1(u_3), Sq^5 Sq^2(u_3), q_3^*(Sq^{14} \varepsilon_0); Sq^8(u_3), Sq^7 Sq^1(u_3), Sq^6 Sq^2(u_3), Sq^6(v_3), q_3^*(Sq^{15} \varepsilon_0); \dots$

$$Y_4 = Y_5 = Y_3. \quad (\pi_4 = \pi_5 = 0.)$$

$F_6 = K(Z_2, n + 6)$ with $\tau(\varepsilon_6) = p_5^* p_4^*(u_3)$.

$H^+(Y_6)$ has a basis consisting of

$q_6^*(Sq^8 \varepsilon_0); p_6^* p_5^* p_4^*(v_3), u_6$ such that $i_6^*(u_6) = Sq^2 Sq^1 \varepsilon_6; Sq^1(u_6);$ nothing in dimension $n + 11; q_6^*(Sq^{12} \varepsilon_0), Sq^2 Sq^1(u_6); p_6^* p_5^* p_4^*(Sq^4 v_3), Sq^4(u_6), v_6$ such that $i_6^*(v_6) = Sq^7 \varepsilon_6; q_6^*(Sq^{14} \varepsilon_0), Sq^5(u_6); q_6^*(Sq^{15} \varepsilon_0), p_6^* p_5^* p_4^*(Sq^6 v_3), \dots$
 (and possibly other classes of dimension $n + 15$).

$F_7 = K(Z_{16}, n + 7)$ with $\tau(\varepsilon_7') = q_6^*(Sq^8 \varepsilon_0)$ and $\tau(\beta' \varepsilon_7) = p_6^* p_5^* p_4^*(v_3)$, where β' is the BOCKSTEIN operator of $0 \rightarrow Z_2 \rightarrow Z_{32} \rightarrow Z_{16} \rightarrow 0$, and ε_7' is the reduction modulo 2 of ε_7 .

$H^+(Y_7)$ has a basis consisting of

u_7 in dimension $n + 9$, such that $i_7^*(u_7) = Sq^2(\varepsilon_7'), p_7^*(u_6); Sq^1(u_7), p_7^*(Sq^1 u_6), v_7$ such that $i_7^*(v_7) = Sq^2 \beta' \varepsilon_7; Sq^1(v_7); Sq^2 Sq^1(u_7), p_7^*(Sq^2 Sq^1 u_6), \dots \quad (Sq^2(v_7) = 0.)$

$F_8 = K(Z_2 + Z_2, n + 8)$ with $\tau(\varepsilon_8') = u_7$, $\tau(\varepsilon_8'') = p_7^*(u_6)$, where ε_8' and ε_8'' are the two fundamental classes in $H^{n+8}(F_8; Z_2)$.

$H^+(Y_8)$ has a basis consisting of

$p_8^*(v_7), u_8, v_8$, where $i_8^*(u_8) = Sq^2(\varepsilon_8')$ and $i_8^*(v_8) = Sq^2(\varepsilon_8'');$
 $Sq^1(u_8), Sq^1(v_8), p_8^*(Sq^1 v_7); Sq^2(u_8), Sq^2(v_8), \dots$

$F_9 = K(Z_2 + Z_2 + Z_2, n + 9)$ with fundamental classes $\varepsilon_9, \varepsilon_9', \varepsilon_9''$ which are send by transgression on $p_8^*(v_7), u_8, v_8$ respectively.

$$H^{n+11}(Y_9; Z_2) \cong Z_2(u_9), \quad \text{where } i_9^*(u_9) = Sq^2(\varepsilon_9).$$

We have seen that this statement implies Lemma 5.1, hence the proof is complete.

BIBLIOGRAPHY

- [1] R. BOTT and H. SAMELSON, *On the Pontryagin product in spaces of paths*. Comment. Math. Helv. 27 (1953), 320–337.
- [2] S. T. HU, *Homotopy theory*. Academic Press, 1959.
- [3] M. A. KERVAIRE, *Relative characteristic classes*. Amer. J. Math., 79 (1957), 517–558.
- [4] M. A. KERVAIRE, *Some non-stable homotopy groups of LIE groups*. Illinois J. Math., 4 (1960), 161–169.
- [5] J. MILNOR, *Differentiable structures on spheres*. Amer. J. Math., 81 (1959), 962–972.
- [6] J. MILNOR, *A procedure for killing homotopy groups of differentiable manifolds*. Proceedings of the Symposium on Differential Geometry. Tucson, 1960, to appear.
- [7] J. MILNOR and M. A. KERVAIRE, *BERNOULLI numbers, homotopy groups, and a theorem of ROHILIN*. Proceedings of the Int. Congress of Math., Edinburgh, 1958.
- [8] J. P. SERRE, *Groupes d'homotopie et classes de groupes abeliens*. Ann. of Math., 58 (1953), 258–294.
- [9] J. P. SERRE, *Cohomologie modulo 2 des complexes d'EILENBERG-MACLANE*. Comment. Math. Helv., 27 (1953), 198–232.
- [10] R. THOM, *Quelques propriétés globales des variétés différentiables*. Comment. Math. Helv., 28 (1954), 17–86.
- [11] H. TODA, *On exact sequences in STEENROD algebra mod. 2*. Memoirs of the College of Science, University of Kyoto, 31 (1958), 33–64.
- [12] J. H. C. WHITEHEAD, *Note on suspension*. Quart. J. Math. Oxford, Ser. (2), 1 (1950), 9–22.

Received May 7, 1960