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A Note on the Samelson Product in the Classical Groups

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In a topological group G the correspondance $(x, y) \to y \cdot x \cdot y^{-1} \cdot x^{-1}$ defines a map

$$c: G \# G \to G$$

where as usual $G \# G$ stands for the identification space $G \times G / G \times e \cup e \times G$, with e the identity of G. This map c induces a pairing of $\pi_n(G)$ with $\pi_m(G)$ to $\pi_{n+m}(G)$ in the plausible manner: if $\alpha : S_n \to G$, $\beta : S_m \to G$ are maps based at the identity, the composition $c \circ (\alpha \# \beta) : S_n \# S_m \to G$ determines an element of $\pi_{n+m}(G)$ in view of the homeomorphism of $S_n \# S_m$ with S_{n+m}. That the induced function is actually a bilinear one was shown by Samelson [4]. (See also G. Whitehead [6].)

The commutator c thus induces a ring structure on $\pi_*(G)$. If G is homotopy abelian this product, which we will refer to as the Samelson product and denote by $\langle \alpha, \beta \rangle$, is clearly trivial. Therefore $\langle \alpha, \beta \rangle$ can be thought of as an obstruction to homotopy commutativity.

In [5], Samelson used this criterion to show that the unitary group in two variables, U_2, was not homotopy abelian. He showed that if $\alpha \in \pi_3(U_2)$ was a generator, then $\langle \alpha, \alpha \rangle \neq 0$.

Recently James and Thomas [3], considerably extended this result; they showed, for instance, that among the classical compact groups only the truly commutative ones were homotopy abelian. Their method is again to find elements $\alpha \in \pi_n(G)$ with $\langle \alpha, \alpha \rangle \neq 0$.

Both authors essentially conduct their search for nontrivial squares $\langle \alpha, \alpha \rangle$, not in G, but in the classifying space, B_G of G. This is possible in view of another of Samelson's results [4], according to which the natural isomorphism $T : \pi_{n+1}(B_G) \to \pi_n(G)$ transforms the Whitehead product on B_G, into the "commutator" product on G:

$$T[\alpha, \beta] = \pm \langle T\alpha, T\beta \rangle .$$

In this note we study the ring $\pi_*(U_t)$, where U_t is the unitary group in t-variables, directly from its definition, and show that with the presently known information about $\pi_i(U_t)$, a quite elementary degree argument evaluates the first potentially interesting Samelson products. Recall that [1],

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$$\begin{aligned} \pi_{2i+1}(U_t) &= Z & 0 \le i < t \\ \pi_{2i}(U_t) &= 0 & 0 \le i < t \\ \pi_{2t}(U_t) &= Z/t!Z\,. \end{aligned}$$

Hence the first interesting instance occurs when $\alpha \in \pi_{2r+1}(U_t)$, $\beta \in \pi_{2s+1}(U_t)$; $(t = r + s + 1)$, the product $\langle\alpha, \beta\rangle$ being an element of $Z/t!Z$. Our result can be stated as follows:

Theorem 1. *The kernel of the homomorphism*

$$\pi_{2r+1}(U_t) \otimes \pi_{2s+1}(U_t) \to \pi_{2t}(U_t) \qquad (t = r + s + 1)$$

which takes $\alpha \otimes \beta$ *into* $\langle\alpha, \beta\rangle$ *is divisible by precisely* $t!/r!s!$.

Corollary. If $\alpha \in \pi_{2r+1}(U_t)$, $\beta \in \pi_{2s+1}(U_t)$, $\gamma \in \pi_{2t}(U_t)$, *are suitable generators, then* $\langle\alpha, \beta\rangle = r!s!\gamma$. *This element does not vanish unless* $\gamma = 0$, *that is, unless* $r = s = 1$.

We also give an analogous formula for the symplectic group SP_n. (Theorem 2 of § 2.)

In the orthogonal groups these methods can also be used to show the non-triviality of certain SAMELSON products. However here this product vanishes for stable homotopy classes. We hope to return to this case in the future, and have, for that reason, described the initial constructions for the whole family of classical groups.

2. A suspension formula. We will follow the notation of JAMES (2): if K is one of the three fields over the real numbers, K_m denotes the right K-module of m-tuples of elements of K:

$$x = (x_1, \ldots, x_m)\,.$$

An inner product is defined on K_m by the formula $(x, y) = \Sigma \bar{x}_i y_i$ where the bar denotes the conjugation in K. The group of automorphisms of K_m which preserve this inner product is denoted by O_m. Hence if K is real field, O_m is the orthogonal group in m variables, and when K is the complex field O_m is the unitary group, U_m, in m variables. In the case when K is the field of quaternions, O_m becomes the symplectic group SP_m in m-variables.

Let e_j, $(j = 1, \ldots, m)$ be the m-tuple with j^{th} coordinate 1 and all others 0. If $n \le m$ we distinguish two imbeddings of O_n in O_m.

(2.1) $i: O_n \to O_m$ identifies O_n with the subgroup of O_m leaving the last $m - n$ elements of the basis $e_1, \ldots e_m$ point wise fixed.

(2.2) $i': O_n \to O_m$ identifies O_n with the subgroup of O_m which leaves the first $m - n$ elements of this basis point wise fixed.

Consider the map:

$$O_n \# O_m \xrightarrow{i \# i'} O_t \# O_t \xrightarrow{c} O_t \tag{2.3}$$

where c is the commutator map of the introduction. If $t \geqslant n+m$, then this composition yields the trivial map as then iO_n and $i'O_m$ commute. If $t = n+m-k$ with $k > O$ this is no longer true, but the map (2.3) may still be factored whenever k is less than both n and m. (We will make this assumption throughout the rest of this note.)

Indeed the two image groups of the following compositions:

$$O_{n-k} \xrightarrow{i} O_n \xrightarrow{i} O_t$$
$$O_{m-k} \xrightarrow{i'} O_m \xrightarrow{i'} O_t$$

commute with iO_m and $i'O_n$ respectively. Hence if we write $O_{n,k}$ for O_n/iO_{n-k} and $O'_{m,k}$ for $O_m/i'O_{m,k}$ and denote the natural projections by π and π' respectively, the map (2.3) induces a map: $O_{n,k} \# O'_{m,k} \xrightarrow{\lambda} O_t$ which makes the following diagram commutative:

$$\begin{tikzcd} O_n \# O_m \arrow[r, "c \circ (i \# i')"] \arrow[d, "\pi \# \pi'"'] & O_t \\ O_{n,k} \# O'_{m,k} \arrow[ur, "\lambda"'] & \end{tikzcd}$$

Consider next the fibering. $O_t \xrightarrow{i} O_{t+k} \xrightarrow{\lambda} O_{t+k,k}$, where again $O_{t+k,k}$ is equal to O_{t+k}/iO_t in accordance with the James notation.

We assert that the map λ, which takes values in the fiber of this fibering, is suspendable in the total space, and we will construct an explicit suspension λ^E: $E(O_{n,k} \# O'_{m,k}) \to O_{t+k,k}$ for it. That λ can be suspended is plausible enough: It was already remarked earlier that the map

$$O_n \# O_m \xrightarrow{c \circ (i \# i')} O_{t+k}$$

was trivial because $O_{t+k} = O_{n+m}$. However $c \circ (i \# i')$ and $i \circ \lambda \circ (\pi \# \pi')$ are homotopic as maps into O_{t+k}. A deformation between them followed by the projection on $O_{t+k,k}$ should therefore yield the suspension.

Explicitly we proceed as follows. Let $X = O_{n,k} \# O'_{m,k}$. For convenience we represent the suspension of X, that is EX, as the quotient $X \times [0, \pi/2]/(X \times 0 \cup X \times \pi/2)$. Also, for each $\theta \in [0, \pi/2]$ we determine an element $a_\theta \in O_{t+k}$ according to this prescription: Let $\alpha = n-k$, then

$$\begin{aligned} a_\theta e_{\alpha+i} &= \cos\theta\, e_{\alpha+i} + \sin\theta\, e_{t+i} && 0 \leq i \leq k \\ a_\theta e_{t+i} &= -\sin\theta\, e_{\alpha+i} + \cos\theta\, e_{t+i} && 0 \leq i \leq k\,. \end{aligned}$$

All other basis vectors are to be pointwise fixed. We also let $A_\theta \colon O_{t+k} \to O_{t+k}$

be the inner automorphism induced by a_θ: $A_\theta f = a_\theta \cdot f \cdot a_\theta^{-1}$, $f \in O_{t+k}$. Clearly a_0 is the identity whereas $a_{\pi/2}$ maps the plane spanned by the $e_{\alpha+i}$, $0 \leq i \leq k$, onto the plane spanned by the e_{t+i}, $0 \leq i \leq k$. As a result, if i_1 and i_2 are the compositions

$$i_1 : O_n \xrightarrow{i} O_t \xrightarrow{i} O_{tk}$$

$$i_2 : O_m \xrightarrow{i'} O_t \xrightarrow{i'} O_{tk}$$

the two groups $i_1(O_n)$ and $A_{\pi/2} i_2(O_m)$ commute.

Further the elements in the image of:

$$O_{n-k} \xrightarrow{i} O_n \xrightarrow{i_1} O_{t+k} \quad \text{and} \quad O_{m-k} \xrightarrow{i'} O_m \xrightarrow{i_2} O_{t+k}$$

commute with a_θ for all $0 \leq \theta \leq \pi/2$.

It is now easily verified that the function

$$s : O_n \# O_m \times [0, \pi/2] \to O_{t+k} = O_{n+m}$$

defined by:

$$s(f, g, \theta) = [A_\theta \circ i_2 g, i_1 f], \qquad f \in O_n, g \in O_m, \theta \in [o, \pi/2]$$

induces a map of $CX = X \times [0, \pi/2]/X \times [\pi/2]$ into O_{t+k} whose restriction to $X \times [0]$ is precisely λ. This proves that λ is suspendable, and we may take for λ^E the map induced by $\tau \circ s$, on EX.

These constructions have the following consequence:

Proposition 2.1. *Consider the map* $c \circ (i \# i') : O_n \# O_m \to O_t$. *Then the induced homomorphism in homotopy has the following factorization:*

$$\{c \circ (i \# i')\}_* = \Delta \circ \lambda_*^E \circ E \circ (\pi \# \pi')_*$$

where $\pi \# \pi' : O_n \# O_m \to O_{n,k} \# O'_{m,k}$ *is the natural projection and* E *denotes suspension. The homomorphism* λ_*^E *is induced by the map* $\lambda^E : E(O_{n,k} \# O'_{m,k}) \to O_{t+k,k}$, *and* Δ *denotes the boundary operator in the exact sequence of the fibering* $O_t \xrightarrow{i} O_{t+k} \xrightarrow{\tau} O_{t+k,k}$.

Consider now the case $k = 1$. If d is the dimension of K over the real field, $O_{n,1}$ is homeomorphic to a sphere of dimension $dn - 1$. In this case, then, λ^E will be a map from $E\ (S_{dn-1} \# S_{dm-1})$ to $S_{d(n+m)-1}$. Thus λ^E is a map between spheres of equal dimension. In the next section we will prove the following proposition:

Proposition 2.2. *Consider the situation of proposition* 2.1, *with* $k = 1$. *The map* λ^E *is then a map of* $S_{d(n+m)-1}$ *onto* $S_{d(n+m)-1}$ *with degree* 1.

As a corollary to these two propositions we bring the proof of theorem 1.

Let then K be the complex field, whence $d = 2$. From the homotopy sequence of the fibering: $O_{r-1} \xrightarrow{i} O_r \to S_{dr-1}$, one concludes that: [1],

(a) $\pi_{dr-1}(O_r)$ is stable. (Use $r+1$ for r in the above sequence.)

(b) The section of the sequence $\pi_{dr-1}(O_r) \to \pi_{dr-1}(S_{dr-1}) \to \pi_{dr-2}(O_{r-1}) \to 0$ is given by $Z \xrightarrow{(r-1)!} Z \to Z/(r-1)!Z \to 0$.

Suppose now that $\alpha \in \pi_{dn-1}(O_n)$ and $\beta \in \pi_{dm-1}(O_m)$ are generators. Then according to the factorization of proposition (2.1),

$$\langle i\alpha, i'\alpha\rangle = \Delta \circ \lambda^E_* \circ E \circ (\pi_* \alpha \# \pi'_* \beta) .$$

According to (b) π_* multiplies by $(n-1)!$ while π'_* multiplies by $(m-1)!$. The suspension E is a bijection in the pertinent dimension, and according to proposition (2.2) so is λ^E_*. Finally Δ projects Z onto $Z/(n+m-1)!Z$. Hence the order of $\langle i\alpha, i'\beta\rangle$ is $(n+m-1)!/(n-1)!(m-1)!$. Finally because we are in the stable range i and i' are bijections. This proves theorem 1, once n and m are replaced by $n+1$ and $m+1$ respectively.

The quaternionic case can be treated entirely the same way. In this case $\pi_{nd-1}(O_n)$ is again stable and isomorphic to Z, and the sequence in question has the form:

$$\begin{gathered}\pi_{dr-1}(O_r) \to \pi_{dr-1}(S_{dr-1}) \to \pi_{dr-2}(O_{r-2}) \to 0\\ Z \xrightarrow{k_r} Z \to Z/k_r Z \to 0\end{gathered}$$

where $k_r = (2r-1)!$ if r is odd, and $k_r = (2r-1)!2$ if r is even. The analogue of theorem 1 therefore takes the following form:

Theorem 2. *The kernel of the homomorphism*

$$\pi_{4n-1}(SP_n) \otimes \pi_{4m-1}(SP_m) \to \pi_{4(n+m)-2}(SP_{m+n-1})$$

induced by the SAMELSON *product is precisely divisible by $k_{n+m}/k_n \cdot k_m$ where*

$$k_r = \begin{matrix} (2r-1)!2 & r \text{ even} \\ (2r-1)! & r \text{ odd.}\end{matrix}$$

For the real field, this argument fails because $\pi_{dn-1}(O_n)$ is not stable anymore. As a consequence the SAMELSON product of two stable elements in $\pi_*(O_n)$ vanishes.

3. Proof of proposition 2.2. We have to show that our map

$$\lambda^E : E\{O_{n,1} \# O'_{m,1}\} \to O_{n+m,1}$$

has degree one.

For this purpose let $K_{n+m} = A + a + B + b$ be the orthogonal decomposition in which A is spanned by $e_1, \ldots, e_{n-1}$, the plane B is spanned by $e_{n+1}, \ldots, e_{n+m-1}$, while a is the line spanned by $e_n = e_a$ and b is the line spanned by $e_{n+m} = e_b$.

Let O^*_n be the image of $O_n \xrightarrow{i} O_{n+m-1} \xrightarrow{i} O_{n+m}$, and let O^*_{n-1} be the image of $iO_{n-1} \subset O_n$ under this map. Similarly let $O^*_m[O^*_{m-1}]$ be the image of O_m

and $i'(O_{m-1}) \subset O_m$ under the map $O_m \xrightarrow{i'} O_{m+n-1} \xrightarrow{i'} O_{m+n}$. Clearly these subgroups are characterized by the following properties:

$$\begin{array}{lll} O_n^* - \text{leaves} & B + b & \text{pointwise fixed} \\ O_{n-1}^* \text{ leaves} & a + B + b & \text{pointwise fixed} \\ O_m^* \ \text{ leaves} & A + b & \text{pointwise fixed} \\ O_{m-1}^* \text{leaves} & A + b + B & \text{pointwise fixed.} \end{array}$$

Let S_{dn-1} be the unit sphere of $A + a$. Clearly the map $f \to f e_a$, $f \in O_n^*$ identifies $O_{n,1}^*$ with S_{dn-1}. Similarly $g \to g e_a$, $g \in O_m^*$ identifies $O_{m,1}^*$ with the unit sphere S_{dm-1} of the plane $a + B$. Finally the map $f \to f e_b$, $f \in O_{n+m}$ identifies $O_{n+m,1}$ with the unit sphere $S_{d(n+m)-1}$ in K_{n+m}. In this realization our map

$$\lambda^E : E\{S_{dn-1} \# S_{dm-1}\} \to S_{d(n+m)-1}$$

is described in the following manner:

If $x \in S_{dn-1}$, $y \in S_{dm-1}$, $\theta \in [0, \pi/2]$ are given, then

$$\lambda^E(x, y, \theta) = [A_\theta(g), f]e_b$$

where g and f are any elements of O_m^* and O_n^* respectively subject to:

$$f e_a = x\,; \qquad g e_a = y\,,$$

and A_θ is, as in the earlier section, the inner automorphism by $a_\theta \in O_{n+m}$. (Recall that in the present case $a_\theta e_a = \cos\theta e_a + \sin\theta e_b$; $a_\theta e_b = -\sin\theta e_a + {}$ $+ \cos\theta e_b$ while all other basis elements are held fixed.)

Let us write g_θ for $A_\theta g$. Then $\lambda^E(x, y, \theta) = (g_\theta f g_\theta^{-1} f^{-1})e_b = g_\theta f g_\theta^{-1} e_b$, (the last step follows from $f e_b = e_b$).

Consider the orthogonal decompositions:

$$\begin{array}{ll} g_\theta^{-1} e_b = e_a \bar\beta + w & w \in B + b \\ x = f^{-1} e_a = x' + e_a \alpha & x' \in A\,, \end{array}$$

so that $\bar\beta = (e_a, g_\theta^{-1} e_b) = (g_\theta e_a, e_b)$, whence $\beta = (e_b, g_\theta e_a)$ while $\alpha = (e_a, x)$.

We have therefore, in order:

$$\begin{aligned} f g_\theta^{-1} e_b &= (x' + e_a \alpha)\bar\beta + w \\ g_\theta f g_\theta^{-1} e_b &= x' + (g_\theta e_a)\alpha\bar\beta + g_\theta w \\ &= x'\bar\beta + (g_\theta e_a)\alpha\bar\beta + e_b - (g_\theta e_a)\bar\beta \end{aligned}$$

$$\lambda^E(x, y, \theta) = \{x' + (g_\theta e_a)(\alpha - 1)\}\bar\beta + e_b\,. \tag{3.1}$$

It is clear that x', α and β are functions of x, y and θ above. On the other hand, so is $g_\theta e_a$. Indeed a straight-forward computation yields:

$$\begin{aligned} a_\theta^{-1} e_a &= e_a \cos\theta - e_b \sin\theta \\ g a_\theta^{-1} e_a &= y \cos\theta - e_b \sin\theta\,. \end{aligned}$$

Let $y = e_a\gamma + y'$ with $y' \in B$, be an orthogonal decomposition of y, so that:

$$g a_\theta^{-1} e_a = e_a \gamma \cos\theta + y' \cos\theta - e_b \sin\theta$$

whence

$$a_\theta g a_\theta^{-1} e_a = (e_a \cos\theta + e_b \sin\theta)\gamma \cos\theta + y' \cos\theta - (-e_a \sin\theta + e_b \cos\theta)\sin\theta$$

so that

$$g_\theta e_a = e_a(\gamma \cos^2\theta + \sin^2\theta) + y' \cos\theta + e_b(\gamma - 1)\sin\theta\cos\theta. \qquad (3.2)$$

The formulae (3.1) and (3.2) completely describe the map λ^E, and show furthermore, that considered a function on the manifold $S_{dn-1} \times S_{dm-1} \times [0, \pi/2]$, the map λ^E is smooth. To determine its degree it is therefore sufficient to examine the inverse image of a regular point.

Lemma 3.7. *Let P be the point $-e_b \in S_{d(n+m)-1}$. The inverse image of P under λ^E consists of the single point $Q = (-e_a, -e_a, \pi/4)$.*

Proof. The condition $\lambda^E(x, y, \theta) = -e_b$ implies that

$$(e_b, g_\theta e_a)(\alpha - 1)\bar{\beta} = -2$$

as is aparent from (3.1). From our definition of β this is equivalent to $\beta(\alpha - 1)\bar{\beta} = -2$. Now $|\beta|$ and $|\alpha|$ are non-negative numbers less than or equal to 1. Hence this relation holds only if $\alpha = -1$, and $|\beta| = 1$. From (3.2) we see that $\beta = (\gamma - 1)\sin\theta\cos\theta$. Therefore, as $0 \leq |\gamma| \leq 1$; $0 \leq \theta \leq \pi/2$ the condition $|\beta| = 1$, implies $\gamma = -1$, and $\theta = \pi/4$. Now $\alpha = -1$ implies $x = -e_a$, and similarly $\gamma = -1$ implies $y = -e_a$. The lemma is therefore established.

Lemma 3.2. *The differential of λ^E at Q is an isomorphism. Thus P is a regular point of λ^E.*

Proof. We may identify the tangent space to P with the real vector space K_{n+m}/L_b where L_b denotes the real line $e_b \cdot r$, r a real number. We also write I_b for the real vector space spanned by those $e_b q$, $q \in K$ for which $q = -\bar{q}$, and define I_a analogously. Hence K_{n+m}/L_b is spanned by the real vector space $A + a + B + I_b$, mod L_b. Similarly, the tangent space at Q is identified with the real vector space $\mathfrak{A} + \mathfrak{B} + \mathfrak{C}$ where $\mathfrak{A}$ is spanned by $A + I_a$, $\mathfrak{B}$ is spanned by $I_a + B$ and $\mathfrak{C}$ is spanned by $\partial/\partial\theta$. If $z = (\dot{x}, \dot{y}, \dot{\theta})$ is a triple in this space, the differential $d\lambda^E$, is seen to take the form:

$$d\lambda^E(\dot{x}, \dot{y}, \dot{\theta}) = -\dot{x} + 2\dot{g}_\theta e_a + e_b(\dot{\alpha} + 2\dot{\bar{\beta}}) \bmod L_b$$

where the dot denotes differentiation in the direction z. (Recall that at Q, $g_\theta e_a = -e_b$, while $\alpha = \beta = \gamma = -1$.)

Using this formula it is clear enough that the image of $d\lambda^E$ spans all of the tangent space at P, except possibly the real multiples of e_a. (The space I_b can be obtained by setting $\dot{x} = \mu e_a$, $(\mu = -\bar{\mu})$, $\dot{y} = 0$, $\dot{\theta} = 0$). To obtain these multiples of e_a, consider the variation $(0, 0, \partial/\partial\theta)$. One finds $\dot{\alpha} = 0$, $\dot{\beta} = 0$, $\dot{x} = 0$, while $\dot{g}_\theta e_a = 2e_a$. Hence $d\lambda^E$ is onto, and therefore an isomorphism.

The two lemmas clearly imply that λ^E is a map of degree one. It was this that was to be established.

A question. In [2] I. JAMES considers a map of the join of $O_{n,k}$ with $O_{m,k}$ into $O_{n+m,k}$ which is the natural extension of the usual join map of $S_n * S_m$ onto S_{n+m+1}. If we think of a point of $O_{n,k}$ as a k-frame $f = (f_1, \ldots, f_k)$ in K_n and of $O_{m,k}$ as a k-frame $g = (g_1, \ldots, g_k)$ in K_m, then the JAMES map, which we denote by λ^J, attaches the frame

$$\cos\theta f + \sin\theta g \quad \text{in} \quad K_n + K_m$$

to the triple $(f, \theta, g) \in O_{n,k} * O_{m,k}$. (We have again used the interval $[0, \pi/2]$ to parametrise the join.) In view of the fact that $O_{n,k} * O_{m,k}$ and $E\{O_{n,k} \# O_{m,k}\}$ are of the same homotopy type, λ^J has the same domain of definition, and image space, as our map λ^E. For $k = 1$, the JAMES map clearly has degree one. Hence proposition (2.2) can be thought of as showing that the two maps are equivalent in this case, and the problem of comparing the two maps in general immediately arises. In view of the beautiful properties which JAMES discovered for his map, it would be very encouraging if λ^J and λ^E turned out to be homotopic.

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