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Invariance of Vector Form Operations under Mappings[1])

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Introduction. In a previous paper [4] the authors have presented a theory of derivations on the ring of C^∞ differential forms over a manifold, in which a vital role was played by vector forms[2]). The commutators of certain kinds of derivations were found to be intimately related to a previously published [7] differential concomitant $[L, M]$ of vector forms L and M; and several identities involving this concomitant were derived in a very simple fashion.

If X and Y are C^∞ manifolds, and $F : X \to Y$ is a C^∞ mapping, there are associated mappings F_* on tangent vectors, and F^* on differential forms; the first being a covariant functor; the second a contravariant one.

For mixed tensors on X or Y there are, in general, no induced mappings in either direction. However, they may nevertheless be related in some fashion through F. At present we restrict ourselves to vector forms, and define the concept of *F-relatedness.* This relation is a mapping only in special cases (cf. § 3, 4), but is always transitive with respect to composites of mappings[3]).

We study the behavior of differential concomitants of vector and scalar forms with respect to F-relatedness. The main result (Theorem 1, § 2) is that the notion of F-relatedness is invariant under the formation of the differential concomitants in question[4]). In sections 3 and 4 conditions for F-relatedness will be deduced for special classes of mappings, namely—roughly speaking—imbeddings (§ 3) and projections (§ 4). Section 5 also deals with fiber bundles and develops machinery for LIE differentiation of certain tensor fields which are only defined along the fibers of the bundle.

Two applications of the theory have presented themselves. The first, discussed in § 6, deals with the existence of almost-complex and complex structures which are invariant under the action of a transitive LIE group of transformations. The respective Theorems 8 and 9 of § 6 were proved in [3] (cf. also

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[2]) *Vector forms* are differential forms whose values at any point are not numbers, but *tangent* vectors. Ordinary differential forms are called *scalar forms.*

[3]) Only from a *formal* functorial point of view (cf. [2]) does F-relatedness satisfy the conditions for "mappings" in a category.

[4]) This result is not true for all differential concomitants: if $F : X \to Y$ imbeds X as a submanifold in a RIEMANNian manifold Y, then the induced metric on X is F-related to the metric on Y, while the corresponding curvature tensors are not F-related. In fact, the generalization of GAUSS' *Theorema Egregium* states that the deviation from F-relatedness can be expressed in terms of the second fundamental forms.

[1, 6, 9]), but the proofs have been greatly simplified. A second application, to deformations of complex structures on compact manifolds, requires extensive use of § 5, and is forthcoming as a separate paper [5].

§ 1. Operations on scalar and vector forms. This section is a brief introduction into the concepts used later in this paper. Since most facts can be found elsewhere in the literature, only occasional sketches of proofs will be given.

Let X denote a C^∞ manifold, and Φ_0 the ring of C^∞ functions over X. The field of real numbers, denoted by R, can be considered as subring of Φ_0 (the constant functions). If $f \in \Phi_0$, f_x denotes the value (restriction) of f at $x \in X$. A tangent vector u_x at $x \in X$ is a mapping[5]) $u_x : \Phi_0 \to R$, linear over R, which satisfies the LEIBNITZ rule for products $u_x(fg) = g_x \cdot u_x f + f_x \cdot u_x g$. T_x denotes the tangent space at x; $T = \bigcup_{x \in X} T_x$ is the tangent bundle; and $\tau : T \to X$ the projection mapping in this bundle.

A C^∞ vector field u over X is a map $u : X \to T$ with $x \to u_x \in T_x$, such that for any $f \in \Phi_0$ the map $X \to R$, by $x \to u_x f$, is a C^∞ function, denoted by uf. Thus, u represents a *derivation* on Φ_0.

Tangent vectors are local operators, that is $u_x f = u_x g$ if $f_y = g_y$ for all y in some neighborhood of x. As a consequence, u_x can be defined to operate on C^∞ functions defined only in some neighborhood of x.

Let f be a C^∞ function over an open subset U of X; then df shall denote the real-valued function on the tangent bundle $T(U) = \tau^{-1}(U)$ defined by $df(u_x) = u_x f$. The restriction $(df)_x$ of df to one fiber T_x, $x \in U$, is a linear function on T_x.

If $X^1, \ldots, X^n$ are functions defined on an open set U of X, and if they form a coordinate system there, then $X^1 \circ \tau, \ldots, X^n \circ \tau$, $dX^1, \ldots, dX^n$ are $2n$ functions defined on $T(U)$. Let $Y^1 \ldots, Y^n$ denote coordinate functions in an open set V, then, if $U \cap V$ is not empty, in $U \cap V$ the Y^i are functions of the X^i:

$$Y^i(x) = \Phi^i(X^1(x), \ldots, X^n(x)), \qquad i = 1, \ldots, n, \tag{1.1}$$

or[6])

$$Y^i = \Phi^i(X^1, \ldots, X^n), \qquad i = 1, \ldots, n, \tag{1.2}$$

[5]) In index-notation, $u_x = u^i \left.\frac{\partial}{\partial x^i}\right|_x$. The properties stated constitute a complete axiomatic characterization of tangent vectors to C^∞ manifolds. For manifolds of class $C^k (k < \infty)$, a small modification is needed.

[6]) For the composite of two functions, say f and g, the symbol $g \circ f$ is customary, and stands for $x \to (g \circ f)(x) = g(f(x))$. If g is a function of, say, 2 variables, there is no standard notation for the function $x \to g(f_1(x), f_2(x))$. One may, for instance, use $g \circ (f_1 \times f_2)$, or $g(f_1, f_2)$. We have chosen the latter.

where $\Phi^i(\xi^1, \ldots, \xi^n)$ depends in a C^∞ manner on the real numbers $\xi^1,\ldots,\xi^n$. Then, we have

$$Y^i \circ \tau = \Phi^i(X^1 \circ \tau, \ldots, X^n \circ \tau)\,,$$
$$dY^i = \sum_{j=1}^{n}\left[\frac{\partial \Phi^i}{\partial \xi^j}(X^1\circ\tau, \ldots, X^n\circ\tau)\right]dX^j\,, \tag{1.3}$$

where $(\xi^1, \ldots, \xi^n, \zeta^1, \ldots, \zeta^n) \to \Phi^i(\xi^1, \ldots, \xi^n)$ and

$$(\xi^1, \ldots, \xi^n, \zeta^1, \ldots, \zeta^n) \to \sum_{j=1}^{n}\left[\frac{\partial \Phi^i}{\partial \xi^j}(\xi^1, \ldots, \xi^n)\right]\cdot\zeta^i$$

are $2n$ C^∞ functions in the $2n$ real variables $\xi^1, \ldots, \xi^n, \zeta^1, \ldots \zeta^n$. Thus, the functions $X^1 \circ \tau, \ldots, X^n \circ \tau, dX^1, \ldots, dX^n$ on $\tau^{-1}(U)$, derived from the coordinate functions $X^1, \ldots, X^n$ on U, determine a C^∞ structure on T, and, clearly, τ is a C^∞ projection map.—If f is a C^∞ function on X, then $f \circ \tau$ and df are C^∞ functions on T. A vector field u is of class C^∞ if and only if $u : X \to T$ is a C^∞ section.

A scalar q-form ω_x at $x \in X$ is an R-multilinear, skew-symmetric, real-valued function of q vectors in T_x: $\omega_x(u_1, \ldots, u_q) \in R$ for $u_i \in T_x$, $i = 1, \ldots, q$. Denoting by $\oplus^q T_x$ the set of q-tuples of elements of T_x, ω_x is an R-multilinear, skew-symmetric map $\oplus^q T_x \to R$. Since the bundle $\oplus^q T = \bigcup_{x\in X}(\oplus^q T_x)$ is, in a natural fashion, a C^∞ manifold, a C^∞ scalar q-form ω over X is a C^∞ mapping $\omega : \oplus^q T \to R$ whose restriction $\omega_x = \omega \mid \oplus^q T_x$ to $\oplus^q T_x$, for all $x \in X$, is a scalar q-form at x.—If $u_1, \ldots, u_q$ are C^∞ vector fields over X, then $\omega(u_1, \ldots, u_q)$ is a function over X which, as a composite of C^∞ functions in the sense of footnote 6), is a C^∞ function.

The method of constructing C^∞ scalar forms over X as functions on some bundle over X can be used to give instrinsic definitions of C^∞ tensor fields over X of any degree. For our further investigations, however, we are interested only in certain particular tensors and tensor fields, which we define as follows.

A *vector l-form* L_x at $x \in X$ is an R-multilinear, skew-symmetric mapping $L_x : \oplus^l T_x \to T_x$.—A C^∞ vector l-form L over X is a C^∞ map $L : \oplus^l T \to T$ whose restriction $L_x = L \mid \oplus^l T_x$ is a vector l-form at x for all $x \in X$. $L(u_1, \ldots, u_l)$ is a composite of C^∞ maps if $u_1, \ldots, u_l$ are C^∞ vector fields over X, and, therefore, is a C^∞ vector field.

Multiplication by elements of Φ_0 (functions) for vector fields, scalar and vector forms, etc., is defined pointwise; for instance, $(f\omega)_x = f_x\omega_x$. The set Φ_q of scalar q-forms and the set Ψ_l of vector l-forms over X are Φ_0-modules, and every scalar (vector) form is a skew-symmetric Φ_0-multilinear map of Ψ_0 into Φ_0 (Ψ_0). Conversely, we have (cf. [4], Prop. (3.4, 5)):

Lemma 1.1. Every skew-symmetric Φ_0-multilinear map of $\oplus^q \Psi_0$ into $\Phi_0(\Psi_0)$ is a C^∞ scalar (vector) q-form over X.

Remark. The bracket $[u, v]$ of vector fields, defined by $[u, v]f = = u(vf) - v(uf)$ is a skew-symmetric R-bilinear map $\Psi_0 \oplus \Psi_0 \to \Psi_0$, but, as the following equation shows, it is not Φ_0-bilinear.

$$[u, fv] = f[u, v] + (uf)v \, . \tag{1.4}$$

Thus, $(u, v) \to [u, v]$ is *not* representable by a vector 2-form.

Another result, proved in a manner similar to Lemma 1.1, is

Lemma 1.2. The Φ_0-linear mappings of Φ_1 into Φ_l are in 1-1-correspondence with the C^∞ vector l-forms L over X, where L acts on $\varphi \in \Phi_1$ as $\varphi \to \varphi \,\bar{\wedge}\, L$, the latter being defined by

$$(\varphi \,\bar{\wedge}\, L)\,(u_1, \ldots, u_l) = \varphi(L(u_1, \ldots, u_l)) \, . \tag{1.5}$$

Let Ω denote either a scalar or a vector q-form; π a scalar p-form. Then the scalar (vector) $(q+p)$-form $\Omega \wedge \pi$ is defined by

$$(\Omega \wedge \pi)\,(u_1, \ldots, u_{q+p}) = \frac{1}{p!\,q!} \sum_\alpha |\alpha|\, \Omega(u_{\alpha_1}, \ldots u_{\alpha_q}) \cdot \pi(u_{\alpha_{q+1}}, \ldots, u_{\alpha_{q+p}}), \tag{1.6}$$

where $\alpha = \begin{pmatrix} 1, \ldots, p+q \\ \alpha_1, \ldots, \alpha_{p+q} \end{pmatrix}$ runs over all $(p+q)!$ permutations, and $|\alpha|$ denotes the signature of α.—If Ω is as before, and L a vector l-form, $\Omega \,\bar{\wedge}\, L$ denotes the scalar (vector) $(q+l-1)$-form defined by

$$\begin{aligned} &(\Omega \,\bar{\wedge}\, L)\,(u_1, \ldots, u_{q+l-1}) = \\ &\frac{1}{l!\,(q-1)!} \sum_\alpha |\alpha|\, \Omega(L(u_{\alpha_1}, \ldots, u_{\alpha_l}), u_{\alpha_{l+1}}, \ldots, u_{\alpha_{l+q-1}}) \end{aligned} \tag{1.7}$$

if $q > 0$; and by $\Omega \,\bar{\wedge}\, L = 0$ if $q = 0$.

The exterior derivative $d\omega$ of a scalar q-form ω is defined by its action on vector fields as follows:

$$\begin{aligned} d\omega(u_1, \ldots, u_{q+1}) &= \frac{1}{q!} \sum_\alpha |\alpha|\, u_{\alpha_1}(\omega(u_{\alpha_2}, \ldots, u_{\alpha_{q+1}})) \\ &- \frac{1}{2 \cdot (q-1)!} \sum_\alpha |\alpha|\, \omega([u_{\alpha_1}, u_{\alpha_2}], u_{\alpha_3}, \ldots, u_{\alpha_{q+1}}) \, . \end{aligned} \tag{1.8}$$

While the definitions (1.6, 7) are meaningful also if u_i denotes a vector at just one point, in (1.8) it is essential to have *vector fields*. However, a simple computation involving (1.4) shows that the right side of (1.8) is Φ_0-linear in u_i. Lemma 1.1 asserts that, therefore, $d\omega$ is a scalar $(q+1)$-form.

If ω and L are as before, then $[L, \omega]$ is the scalar $(q + l)$-form:

$$[L, \omega] = (d\omega) \bar{\wedge} L + (-1)^l d(\omega \bar{\wedge} L) . \tag{1.9}$$

As the following equation shows, $[L, \omega]$ is not Φ_0-linear in ω:

$$[L, f\omega] = [L, f] \wedge \omega + f[L, \omega] . \tag{1.10}$$

The bracket $[L, M]$ of a vector l-form L and a vector m-form M over X is a vector $(l + m)$-form. If φ denotes any scalar 1-form, then $[L, M]$ is characterized by

$$\varphi \bar{\wedge} [L, M] = (-1)^m [M, \varphi] \bar{\wedge} L - (-1)^{lm} [M, \varphi \bar{\wedge} L] - (-1)^m [M \bar{\wedge} L, \varphi] . \tag{1.11}$$

In fact, the right side (not term for term – cf. (1.10)) is Φ_0 – linear in φ; hence represents a Φ_0-linear map $\Phi_1 \to \Phi_{l+m}$, which, by Lemma 1.2, is given by the $\bar{\wedge}$-action of a well-determined vector $(l + m)$-form.

The operations mentioned here play entirely natural roles in the theory of derivations on the graded ring of scalar forms (cf. [4]). Numerous identities hold, but only a few are needed in what follows. If Ω is a scalar or vector form, L a vector form, then

$$[u, \Omega \bar{\wedge} L] = [u, \Omega] \bar{\wedge} L + \Omega \bar{\wedge} [u, L] . \tag{1.12}$$

This formula is a special case of what is known as LEIBNITZ' rule on the LIE derivative of a product (with or without contractions) of tensor fields. Furthermore, if h is a vector 1-form, then $[h, h]$ is determined by

$$\tfrac{1}{2}[h, h](u, v) = [hu, hv] - h[hu, v] - h[u, hv] + hh[u, v] . \tag{1.13}$$

From (1.12), replacing L by v, one obtains $[u, \Omega \bar{\wedge} v] = [u, \Omega] \bar{\wedge} v + \Omega \bar{\wedge} [u, v]$. Applying this rule repeatedly to $\Omega(v_1, \ldots, v_q) = ((\Omega \bar{\wedge} v_1) \bar{\wedge} \ldots) \bar{\wedge} v_q$ one obtains, after rearrangement of terms:

$$[u, \Omega](v_1, \ldots, v_q) = [u, \Omega(v_1, \ldots, v_q)] - \sum_{i=1}^{q} \Omega(v_1, \ldots, [u, v_i], \ldots, v_q) , \tag{1.14}$$

where $[u, f] = uf$, in agreement with (1.9).

§ 2. Mappings; *F*-relatedness. In this section, $F : X \to Y$ denotes a mapping of C^∞ manifolds X and Y. $T(X)$ denotes the tangent bundle of X; similarly, $\Phi_q(Y)$ the set of scalar q-forms on Y; etc.

The mapping $F : X \to Y$ is of class C^∞ if and only if for every C^∞ function f on Y the composite $f \circ F$, usually denoted as $F^*(f)$, is a C^∞ function on X. Another equivalent condition is that $F^*(Y^i)$ is of class C^∞ on X for all coordinate functions Y^i of a certain covering of Y by coordinate neighborhoods. – F will be assumed of class C^∞; then F^* is a mapping of $\Phi_0(Y)$

into $\Phi_0(X)$; it is a ring homomorphism whose restriction to the constants is an isomorphism.

If $u \in T_x(X)$, the mapping $\Phi_0(Y) \to R$ denoted by $F_*(u_x)$ and defined by $f \to u_x F^*(f)$ is R-linear and satisfies the LEIBNITZ rule as postulated for tangent vectors at $F(x) \in Y$. Hence, $F_*(u_x) \in T_{F(x)}(Y)$, and we have a map $F_*: T(X) \to T(Y)$ which satisfies $\tau_Y \circ F_* = F \circ \tau_X$; and whose restriction $F_* \mid T_x(X)$ is R-linear.

If f is a C^∞ function on Y, then $f \circ \tau_Y$ and df are C^∞ functions on $T(Y)$. In particular, if Y^i are coordinate functions on Y, then $Y^i \circ \tau_Y$ and dY^i are coordinate functions on $T(Y)$. The map $F_*: T(X) \to T(Y)$ is proved to be of class C^∞ by showing that the induced functions $(F_*)^*(Y^i \circ \tau_Y)$ and $(F_*)^*(dY^i)$ are C^∞ functions on $T(X)$. Let f denote any one of the Y^i for any coordinate system on Y. The C^∞ nature of $(F_*)^*(f \circ \tau_Y)$ on $T(X)$ follows from

$$(F_*)^*(f \circ \tau_Y) = F^*(f) \circ \tau_X , \tag{2.1}$$

since the right side, as composite of C^∞ mappings, is of class C^∞. (2.1) is just another way of writing $(f \circ \tau_Y) \circ F_* = (f \circ F) \circ \tau_X$ which follows from $\tau_Y \circ F_* = F \circ \tau_X$ by associativity. The C^∞ nature of $(F_*)^*(df)$ on $T(X)$ follows from

$$(F_*)^* df = dF^*(f) \tag{2.2}$$

since the right side is a C^∞ function on $T(X)$; (2.2) is proved by evaluation on an arbitrary element $u_x \in T(X)$:

$$\begin{aligned}((F_*)^*(df))(u_x) &= (df \circ F_*)(u_x) = df(F_*(u_x)) = \\ &= F_*(u_x)f = u_x F^*(f) = (dF^*(f))(u_x) .\end{aligned} \tag{2.3}$$

The map $F_*: T(X) \to T(Y)$ induces a map $\oplus^q T(X) \to \oplus^q T(Y)$ which is also denoted by F_*, and is also of class C^∞.

Every scalar q-form ω over Y represents a C^∞ function on $\oplus^q T(Y)$; hence $(F_*)^*(\omega)$ is a C^∞ function over $\oplus^q T(X)$. Since $(F_*)^*(\omega) \mid \oplus^q T_x(X)$ is a scalar q-form at $x \in X$, $(F_*)^*(\omega)$ is a scalar q-form over X. – The more common notation for $(F_*)^*$ is F^*, which will be used henceforth.

The well-known result $d(F^*\omega) = F^*(d\omega)$ for $\omega \in \Phi_q(Y)$ is simplest to prove when F is a map of constant rank, because then every set of vectors $u_{1x}, \ldots, u_{qx} \in T_x(X)$ can be extended to vector fields $u_1, \ldots, u_q$ which, at least locally, can be mapped by F_* into local vector fields on Y. The result is generally true, since every map $F: X \to Y$ can be represented as the composite $F_2 \circ F_1$ of two maps of constant rank, $F_1: X \to X \times Y$ and $F_2: X \times Y \to Y$ being defined by $F_1(x) = (x, F(x))$; $F_2(x, y) = y$.

Since, in general, mappings $F: X \to Y$ do not induce mappings of vector fields and vector forms – the latter not even at one point – we introduce the concepts of (F, x)-relatedness and F-relatedness. The concepts are easily seen to be transitive with respect to composition of mappings; and may be formulated for any types of tensors. For vectors and differential forms at one point, the concept of (F, x)-relatedness is the same as the relation established by the mappings F_* and F^* respectively.

Definition. A vector l-form L_x at $x_x \in X$ and a vector l-form L'_y at $y = F(x) \in Y$ are (F, x)-related if for all $u_1, \ldots, u_l \in T_x(X)$ one has

$$F_*(L_x(u_1, \ldots, u_l)) = L'_y(F_*(u_1), \ldots, F_*(u_l)) \,. \tag{2.4}$$

Definition. Scalar or vector forms Ω over X and Ω' over Y are F-related, if for all $x \in X$ and $y = F(x)$, the forms Ω_x and Ω'_y are (F, x)-related.

Remark. (F, x)-relatedness and F-relatedness can be expressed by commutativity of diagrams like

$$\begin{array}{ccc} \oplus^l T(X) & \xrightarrow{F_*} & \oplus^l T(Y) \\ {\scriptstyle L}\big\downarrow & & {\scriptstyle L'}\big\downarrow \\ T(X) & \xrightarrow{F_*} & T(Y) \end{array} \tag{2.5}$$

Theorem 1. Let X and Y be C^∞ manifolds, and $F: X \to Y$ a C^∞ mapping. The following operations are invariant under F-relatedness[7]):

$$\begin{array}{llll} L, \pi & \to L \wedge \pi & u\,, v & \to [u\,,\, v] \\ \omega, M & \to \omega \bar{\wedge} M & M, \omega & \to [M, \omega] \\ L, M & \to L \bar{\wedge} M & M, L & \to [M, L] \,. \end{array}$$

The following lemma is useful in the proof.

Lemma 2.1. Let L_x be a vector form at $x \in X$; L'_y a vector form at $y = F(x)$. Then L_x and L'_y are (F, x)-related if and only if for every scalar 1-form φ_y at y one has

$$F^*(\varphi_y) \bar{\wedge} L_x = F^*(\varphi_y \bar{\wedge} L'_y) \,. \tag{2.6}$$

The corresponding statement holds for F-related fields L and L'.

Proof of the lemma, for (F, x)-relatedness – the other part being a trivial consequence. Let $u_1, \ldots, u_l \in T_x(X)$, then we have

$$\begin{aligned} (F^*(\varphi_y) \bar{\wedge} L_x)(u_1, \ldots, u_l) &= F^*(\varphi_y)(L_x(u_1, \ldots, u_l)) = \\ &= \varphi_y(F_*(L_x(u_1, \ldots, u_l))) \,; \end{aligned} \tag{2.7}$$

[7]) This means, for instance, that if vector fields u and v over X are F-related to vector fields u', v' respectively over Y, then $[u, v]$ is F-related to $[u', v']$.

$$
\begin{aligned}
F^*(\varphi_y \bar{\wedge} L'_y)(u_1, \ldots, u_l) &= (\varphi_y \bar{\wedge} L'_y)(F_* u_1, \ldots, F_* u_l) = \\
&= \varphi_y(L'_y(F_* u_1, \ldots, F_* u_l)) .
\end{aligned} \tag{2.8}
$$

If L_x and L'_y are (F, x)-related, the expressions in (2.7, 8) are equal for all $u_1, \ldots, u_l$, which proves (2.6). Conversely, if (2.6) holds for all φ_y and $u_1, \ldots, u_l$, then the expressions of (2.7, 8) are equal for all φ_y; hence (2.4) follows and L_x and L'_y are (F, x)-related.

Proof of Theorem 1. The statement concerning $\omega \bar{\wedge} M$ is proved in the same manner as Lemma 2.1, using the general definition (1.7) for the operation $\bar{\wedge}$. For $L \wedge \pi$ and $L \bar{\wedge} M$ the statement follows by putting $\omega'_y = \varphi_y \bar{\wedge} L'_y$, where φ_y is a 1-form at y. Thus we have reduced the questions to the cases $\omega \wedge \pi$ and $\omega \bar{\wedge} M$, the first of which is known to be invariant, while the second is the previous result. Lemma (2.1) then gives the desired invariance under (F, x)-relatedness of $L \wedge \pi$ and $L \bar{\wedge} M$. In the second column, the first part is a special case of the last. The next part, concerning $[M, \omega]$, follows from the behavior of both d and $\bar{\wedge}$ under F, using the definition (1.9). Finally, since the right-hand side of (1.11) contains only operations just proved to be invariant under F-relatedness, Lemma (2.1) gives the result for $[L, M]$.

Remark. It is clear that the operations mentioned in Theorem 1 constitute only a sample of all operations known that are invariant under F-relatedness. However, for the applications in view the ones mentioned are all that is needed. Other invariant operations are, for example, the bracket operations for contravariant tensors due to SCHOUTEN (cf. [8, 7]) and the operation which assigns to an affine connection its RIEMANN-CHRISTOFFEL curvature tensor. For the latter, F-relatedness of connections is defined in the obvious manner using coordinates X^a $(a = 1, \ldots, m)$ in X and Y^i $(i = 1, \ldots, n)$ in Y, and component notation: Γ on X and Γ' on Y are F-related if

$$
\Gamma^a_{cb} \frac{\partial Y^i}{\partial X^a} = \frac{\partial Y^j}{\partial X^c} \frac{\partial Y^k}{\partial X^b} \Gamma'^i_{jk} + \frac{\partial^2 Y^i}{\partial X^c \partial X^b} . \tag{2.8}
$$

However, the operation which assigns to a RIEMANN metric its CHRISTOFFEL symbols is *not* invariant under F-relatedness. More trivial operations not invariant under F-relatedness include: the contraction (trace) of a vector 1-form and the inverse (if it exists) of a covariant tensor of degree 2.

§ 3. Subspaces $i: X \to Y$; restrictions and extensions. In this section the general C^∞-mapping F is replaced by a mapping i whose rank is everywhere equal to the dimension of X (hence i is of constant rank and $\dim Y \geq \dim X$).

Consequently, the induced map $i_* \mid T_x(X)$ is injective and identifies $T_x(X)$ with a vector subspace of $T_{i(x)}(Y)$.

Theorem 2. Let $i : X \to Y$ be a mapping whose rank equals $\dim X$ and L' a vector form over Y. There exists a vector form L over X which is i-related to L' if and only if $L'(u_1, \dots, u_l) \in T_x(X)$ whenever $u_1, \dots, u_l \in T_x(X)$. If such a restriction L exists, it is unique.

Proof. a) Necessity of the condition is obvious. b) Let $u_1, \dots, u_l$ be vectorfields over X. Then $L'(i_* u_1, \dots i_* u_l)_{i(x)} \in i_* (T_x(X))$, and there exists a unique vector $v_x \in T_x(X)$ such that $L'(i_* u_1, \dots, i_* u_l)_{i(x)} = i_* v_x$. If L has to be i-related to L', one must have $L(u_1, \dots, u_l)_x = v_x$. This proves the uniqueness of L, and in order to have the existence it remains to prove that the constructed L is C^∞. If U is a sufficiently small neighborhood of $x \in X$, then $(i_* u_j)_{i(x')}, (x' \in U)$, is the restriction of a C^∞ vectorfield v_j over a neighborhood V of $i(x)$ in Y, and hence $i_* v_{x'}$ is the restriction of a C^∞ vector field w over V. Let f be a C^∞ function over U. If U was taken sufficiently small, then $f = i^* g$, where g is a C^∞ function over V. We have $v_{x'} f = v_{x'}(i^* g) = (i_* v_{x'}) g = w_{i(x')} g = (wg)_{i(x')}$ for $x' \in U$. Hence locally $vf = i^*(wg)$, which proves that vf is C^∞.

Theorem 3. Suppose that the map $i : X \to Y$ is a proper imbedding[8]); that Y is paracompact; and that $i(X)$ is a closed subset of Y. Then every scalar or vector form Ω over X can be extended over Y, i.e. there exists a form Ω' over Y such that Ω and Ω' are i-related.

Proof. a) Special case: $Y = X \times Z$ and $i(x) = (x, z_0)$, where z_0 is a fixed point of the manifold Z. Let $p : X \times Z \to X$ and $i_z : X \to X \times Z$ be defined by $p(x, z) = x$ and $i_z(x) = (x, z)$. ω or L being given over X, we can define ω' and L' over $X \times Z$ by $\omega'(u_1, \dots, u_q) = \omega(p_* u_1, \dots, p_* u_q)$ and $L'(u_1, \dots, u_q) = (i_z)_* L(p_* u_1, \dots, p_* u_q)$ for any vectors $u_j \in T_{(x,z)}(X \times Z)$. b) General case. Since i is one-to-one, we identify x with $i(x)$ and consider X as subset of Y. Using that i is a proper imbedding one finds that every point $x \in X$ has a neighborhood U_x in Y in which there are local coordinates $Y^1, \dots, Y^m$ such that U_x is described by the inequalities $|Y^i| < 1$ $(i = 1, \dots, m)$ and $V_x = U_x \cap X$ by the equations $Y^{n+1} = \dots = Y^m = 0$. U_x is then diffeomorphic to $V_x \times E^{m-n}$, where E^{m-n} is the open $(m-n)$-cell $|Y^i| < 1$ $(i = n+1, \dots m)$, and according to a) Ω (or rather its restrictions to V_x) has an extension Ω'_{U_x} over U_x. The sets U_x together with

[8]) A C^∞ mapping $i : X \to Y$ is a proper imbedding if i) the rank of i is equal $\dim X$; ii) the topology of X coincides with that induced by i.

$U_0 = Y - X$ form an open covering $\mathfrak{U}$ of Y. Put $\Omega'_{U_0} = 0$. Let $\{W_i\}_{i \in I}$ be a locally finite refinement of $\mathfrak{U}$, and $\varrho : I \to X \cup \{0\}$ be a refinement map so that $W_i \subset U_{\rho(i)}$. Define Ω'_i to be the restriction of $\Omega'_{U_{\varrho(i)}}$ to W_i. By construction we have $(\Omega'_{W_i})_x = \Omega_x$ whenever $x \in X \cap W_i$. Taking a partition $\{\varphi_i\}_{i \in I}$ of unity with respect to the covering $\{W_i\}_{i \in I}$ and putting $\Omega' = \sum_{i \in I} \varphi_i \Omega'_{W_i}$ we have obtained the desired extension.

§ 4. Fiber bundles $p : X \to Y$; liftings and projections. In this section F will be a mapping p of X onto Y which is a local product structure; i.e. there exists an (open) covering $\mathfrak{U} = \{U_i\}_{i \in I}$ of Y and a manifold Z such that for all $i \in I$ there is a diffeomorphism $\Psi_i : U_i \times Z \to p^{-1}(U_i)$ which preserves fibers: $p\Psi_i(y, z) = y$ for all $y \in U_i$, $z \in Z$. For brevity $p : X \to Y$ is called a fiber bundle here, though the group of the bundle is not specified.

We study whether a scalar or vector form over Y can be lifted to X. In the case of a scalar form ω one simply takes $p^*(\omega)$. In the case of a vector form L over Y one proceeds as follows: Let L_i be the restriction of L to U_i; $\hat{L}_i$ the extension of L_i to $U_i \times Z$ discussed in part a) of the proof of Theorem 3; L_i^* the vector form over $p^{-1}(U_i)$ corresponding to $\hat{L}_i$ by means of Ψ_i. If now Y is paracompact, there is a locally finite refinement $\mathfrak{V} = \{V_j\}_{j \in J}$ of $\mathfrak{U}$. Choose $\varrho : J \to I$ such that $V_j \subset U_{\rho(j)}$, and let L'_j be the restriction of $L^*_{\varrho(j)}$ to $p^{-1} V_j$. Let $\{\varphi_j\}_{j \in J}$ be a partition of unity for the covering $\mathfrak{V}$. Then $L' = \sum_{j \in J} p^*(\varphi_j) L'_j$ is a global vector form and obviously L' and L are p-related. We have thus proved:

Theorem 4. If $p : X \to Y$ is a fiber bundle, and Y is paracompact, then every scalar or vector form over Y can be lifted to X: there is a scalar or vector form over X which is p-related to the given one on Y.

Definition. A scalar or vector form Ω over X is called *projectable* if there is a p-related scalar or vector form Ω' over Y; the latter is called projection of the former.

Definition. A scalar or vector form Ω over X is called *projectable at* x if there is an $\Omega'_{(x)}$, called projection of Ω_x, at $p(x) \in Y$ which is (p, x)-related to Ω_x. Ω is called *pointwise projectable* if it is projectable at x for all $x \in X$.

The fact that $p : X \to Y$ and $p_* : T_x(X) \to T_{p(x)}(Y)$ are onto implies

Proposition 4.1. If a form Ω over X is projectable (resp. projectable at x), then the projection of Ω (resp. Ω_x) is unique.

Notation. The projection of Ω (or Ω_x), if it exists, is denoted by $p_*(\Omega)$ (or $p_*(\Omega_x)$).

Remarks. a) A pointwise projectable form Ω over X is not necessarily projectable. A necessary and sufficient condition for a pointwise projectable Ω to be projectable is obviously that for all $x_1, x_2 \in X$ with $p(x_1) = p(x_2)$, Ω_{x_1} and Ω_{x_2} have the same projection.

b) A projectable *scalar* form ω over X is completely determined by its projection ω', since this means $\omega = p^*(\omega')$. In particular, ω must vanish if its projection exists and vanishes.

Definition. A vector $v_x \in T_x(X)$ is called *vertical* if $p_*(v_x) = 0$ (i.e. if v_x is "tangent to the fiber through x"). A vector field v over X is called vertical, if it is vertical at each point $x \in X$ (i.e. if the projection of v exists and is the zero vector field over Y).

If v is a projectable vector field and f a function over X, $f\cdot v$ is in general not projectable; it is, if one (at least) of the two following conditions are satisfied: a) f is constant along the fibers; then $f = p^*g$ for some $g \in \Phi_0(Y)$; b) v is vertical, in which case $f\cdot v$ is also vertical. We conclude that the set $\mathfrak{W}$ of vertical vector fields is a $\Phi_0(X)$-module; the set $\mathfrak{P}$ of projectable vector fields a $\Phi_0(Y)$-module.

Definition. A vector l-form L over X is *vertical-valued* if, for any $u_1, \ldots, u_l$, $L(u_1, \ldots, u_l)$ is vertical; or, equivalently, if the projection of L exists and is the zero vector l-form on Y.

Theorem 5. Let $p: X \to Y$ be a fiber bundle; ω a scalar and L a vector form over X. Then ω is pointwise projectable if and only if $\omega \,\bar{\wedge}\, v = 0$ for all vertical v; L is pointwise projectable if and only if $L \,\bar{\wedge}\, v$ is vertical-valued for all vertical v.

Proof. Suppose ω_x is projectable, i.e. there is $\omega'_{(x)}$ such that

$$\omega_x(u_1, \ldots, u_q) = \omega'_{(x)}(p_*u_1, \ldots, p_*u_q), \qquad u_1, \ldots, u_q \in T_x(X). \qquad (4.1)$$

This shows, that if one of the vectors u_i is vertical, then $\omega_x(u_1, \ldots, u_q) = 0$ and hence $\omega_x \,\bar{\wedge}\, v_x = 0$ if $v_x \in T_x(X)$ is vertical. Suppose now, conversely, that $\omega_x \,\bar{\wedge}\, v_x = 0$ for all vertical v_x. This implies that $\omega_x(u_1, \ldots, u_q) = 0$ if one of the vectors u_i is vertical. Writing

$$\begin{aligned}
&\omega_x(u_1, \ldots, u_q) - \omega_x(u'_1, \ldots, u'_q) = \\
&\quad = \sum_{i=1}^{q} \left(\omega_x(u'_1, \ldots, u'_{i-1}, u_i, \ldots, u_q) - \omega_x(u'_1, \ldots, u'_i, u_{i+1}, \ldots, u_q)\right) \\
&\quad = \sum_{i=1}^{q} \omega_x(u'_1, \ldots, u'_{i-1}, u_i - u'_i, u_{i+1}, \ldots, u_q)
\end{aligned}$$

one finds that therefore $\omega_x(u_1, \ldots, u_q) = \omega_x(u'_1, \ldots, u'_q)$ if $p_*(u_i) = p_*(u'_i)$, $(i = 1, \ldots, q)$; i.e. $\omega_x(u_1, \ldots, u_q)$ depends only on the projections of $u_1, \ldots, u_q$. This shows the existence of a form $\omega'_{(x)}$ at $p(x)$ which is (p, x)-related with ω_x. The result for vector forms can be proved analogously or reduced to the previous one by proving that the following statements are equivalent:

a) L_x is projectable;
b) $(p^*\varphi_y) \bar{\wedge} L_x$ is projectable for all scalar 1-forms φ_y at $y = p(x)$;
c) $(p^*\varphi_y \bar{\wedge} L_x) \bar{\wedge} v = 0$ for all scalar 1-forms φ_y at $y = p(x)$ and all vertical $v \in T_x(X)$;
d) $L_x \bar{\wedge} v$ is vertical-valued for all vertical $v \in T_x(X)$.

c) $\Longleftrightarrow$ d) Associativity gives $(p^*\varphi_y \bar{\wedge} L_x) \bar{\wedge} v = p^*\varphi_y \bar{\wedge} (L_x \bar{\wedge} v)$; the vanishing of the right side for all φ_y is equivalent to d). b) $\Longleftrightarrow$ c) follows from Theorem 5 for scalar forms which was just proved. a) $\Longrightarrow$ b) is part of Theorem 1. Remains to show b) $\Longrightarrow$ a). b) means that for every φ_y there is a unique scalar l-form ω_y at y such that $p^*\varphi_y \bar{\wedge} L_x = p^*\omega_y$. Since L_x is fixed and ω_x depends linearly on φ_y, there is by the pointwise analogue of Lemma 1.2 a vector l-form L'_y at y such that $\omega_y = \varphi_y \bar{\wedge} L'_y$. Hence $p^*\varphi_y \bar{\wedge} L_x = p^*(\varphi_y \bar{\wedge} L'_y)$ for all φ_y, and by Lemma 2.1 L_x and L'_y are (p, x)-related, hence a).

Proposition 4.2. If a vector form is pointwise projectable, it has a restriction to any fiber $p^{-1}(y)$. For vector 1-forms pointwise projectability and restrictability to the fibers are equivalent.

Proof. Follows from Theorems 2 (§ 3) and 5.

Theorem 6. Let $p: X \to Y$ be a fiber bundle whose fiber Z is connected. The following conditions are, respectively, necessary and sufficient that a scalar form ω or a vector form L over X be projectable:

$$\omega \bar{\wedge} v = 0 \text{ and } [v, \omega] = 0 \text{ for all vertical } v;$$
$$L \bar{\wedge} v \text{ and } [v, L] \text{ vertical-valued for all vertical } v.$$

Proof. The necessity of the conditions follows from Theorem 1(§ 2) which gives $p_*(\omega \bar{\wedge} v) = p_*\omega \bar{\wedge} p_*v = 0$ and $p_*[v, \omega] = [p_*v, p_*\omega] = 0$ because $p_*v = 0$; similarly for vector forms. The proof of the converse is based on the fact that if $f \in \Phi_0(X)$, and $vf = 0$ for all vertical v, then f is constant on each fiber, and is the lifting of a C^∞ function f' on Y. Let U be an element of the covering mentioned in the definition of a fiber bundle, and let $u_1, \ldots, u_q$ be vector fields over U; they can be lifted to vector fields $\tilde{u}_i$ over $p^{-1}(U) \subset X$. Then for all vertical v (cf. (1.14))

$$v(\omega(\tilde{u}_1, \ldots, \tilde{u}_q)) = [v, \omega](\tilde{u}_1, \ldots, \tilde{u}_q) + \sum_{i=1}^{q} \omega(\tilde{u}_1, \ldots, [v, \tilde{u}_i], \ldots, \tilde{u}_q). \quad (4.2)$$

The right side vanishes; the last term since $p_*[v, \tilde{u}_i] = [p_* v, p_* \tilde{u}_i] = 0$ (Theorem 1) shows that $[v, \tilde{u}_i]$ is vertical. It follows that the function $\omega(\tilde{u}_1, \ldots, \tilde{u}_q)$ is constant on each fiber. Given $u_1, \ldots, u_q$, by Theorem 5, the value $\omega(\tilde{u}_1, \ldots, \tilde{u}_q)$ is independent of the specific liftings of $u_1, \ldots, u_q$. Let $\sigma\colon U \to X$ be a section of the bundle over U and $\omega'_y = \sigma^*(\omega_{\sigma(y)})$. Since ω is pointwise projectable, $\omega_{\sigma(y)}$ projects onto ω'_y, and therefore $\omega'_y(u_1, \ldots, u_q) = \omega(\tilde{u}_{1,x}, \ldots, \tilde{u}_{q,x})$ for $x = \sigma(y)$. Since however $\omega(\tilde{u}_{1,x}, \ldots, \tilde{u}_{q,x})$ is constant on each fiber, ω' is the projection of ω. Hence ω is projectable, which proves the sufficiency of the condition for scalar forms. For vector forms we introduce again scalar 1-forms φ on Y and remark that by (1.12)

$$[v, p^*(\varphi) \bar{\wedge} L] = [v, p^*(\varphi)] \bar{\wedge} L + p^*(\varphi) \bar{\wedge} [v, L] . \tag{4.3}$$

The middle term vanishes since $p^*(\varphi)$ certainly is projectable, and the third term vanishes because $[v, L]$ is vertical-valued. Furthermore, since $L \bar{\wedge} v$ is vertical-valued, we have

$$(p^*(\varphi) \bar{\wedge} L) \bar{\wedge} v = p^*(\varphi) \bar{\wedge} (L \bar{\wedge} v) = 0 . \tag{4.4}$$

(4.3) and (4.4) imply, by the Theorem for scalar forms which was just proved, that $p^*(\varphi) \bar{\wedge} L$ is projectable, and together with Lemmas 1.2 and 2.1 the projectability of L follows easily.

The application in § 6 requires the following lemma.

Lemma 4.3. *If $\mathfrak{V}$ is a set of vertical vector fields on X such that for every $x \in X$ the set of values v_x for all $v \in \mathfrak{V}$ spans the space of vertical vectors at x, then the conditions in Theorems 5 and 6 involving all vertical vector fields v may be restricted to only those v that belong to $\mathfrak{V}$.*

Proof. If $x \in X$, every vertical vector field v can, in a neighborhood of x, be written as $v = \Sigma g_i v_i$, where $v_i \in \mathfrak{V}$ and the $g_i's$ are functions. Then

$$\omega \bar{\wedge} v = \Sigma g_i(\omega \bar{\wedge} v_i) , \qquad L \bar{\wedge} v = \Sigma g_i(L \bar{\wedge} v_i) , \tag{4.4}$$

$$[v, \omega] = \Sigma g_i[v_i, \omega] + \Sigma(\omega \bar{\wedge} v_i) \wedge dg_i , \tag{4.5}$$

$$[v, L] = \Sigma g_i[v_i, L] + \Sigma(L \bar{\wedge} v_i) \wedge dg_i - \Sigma v_i \wedge (dg_i \bar{\wedge} L) . \tag{4.6}$$

These formulas show, for instance, that $\omega \bar{\wedge} v = 0$ for $v \in \mathfrak{V}$ implies $\omega \bar{\wedge} v = 0$ for all vertical v; and that $\omega \bar{\wedge} v = 0, [v, \omega] = 0$ for $v \in \mathfrak{V}$ implies the same for all vertical v. The converse is, of course, trivial.

The proof of the identities (4.5) and (4.6) is a computation in which

$$(gv)f = g(vf) \tag{4.7}$$

$$[gv, u] = g[v, u] - (dg \bar{\wedge} u)v \tag{4.8}$$

are applied to the right side of (1.14) with u replaced by gu.

§ 5. Vertical forms over a bundle. Various geometric problems on a fiber bundle $p: X \to Y$ lead to considering objects which are very similar to scalar or vector forms over X, the difference being essentially that the domain of definition of the objects is restricted. Many of the previously considered operations can be defined for the modified objects and have similar properties.

The first concept is that of a vector field v on the bundle space X, as in § 2, which is, however, defined only at the points of *one* fiber $Z_y = p^{-1}(y)$, $y \in Y$; but is *not* required to be tangent to the fiber. Such a *vector field at the fiber* Z_y is thus simply a section in $T(X)$ over the subset $Z_y \subset X$. The concept of being projectable or not applies to vector fields at Z_y. If u is a projectable vector field at Z_y, its projection $p_*(u)$ is a vector of $T_y(Y)$. Conversely, if $u_y \in T_y(Y)$ is given, there is at least one vector field $\tilde{u}$ at the fiber Z_y with $p_* \tilde{u} = u_y$. Any such $\tilde{u}$ is called a *lifting* of u_y. If $\tilde{u}$ is a vector field at Z_y and f a C^∞ function in a neighborhood of Z_y, then $\tilde{u} f$ is a C^∞ function on Z_y.

The next concept is that of an object acting on q-tuples of vectors in exactly the same way as a scalar q-form over X, with the only difference that it acts only on q-tuples of vertical vectors and is therefore called a vertical scalar q-form over X[9]). Denoting by $V_x(X)$ the subspace of $T_x(X)$ formed by the vertical vectors at $x \in X$, and by $V(X) = \bigcup_{x \in X} V_x(X)$ the submanifold of $T(X)$ formed by all vertical tangent vectors to X, we have therefore the analogous definition as for scalar q-forms (cf. § 1).

Definition. A *vertical scalar q-form* ω over X is a real-valued C^∞ function on $\oplus^q V(X)$ whose restriction $\omega_x = \omega \mid \oplus^q V_x(X)$ is, for each x, R-multilinear and skew-symmetric.

Since $V_x(X)$ is identified with $T_x(Z_y)$, $y = p(x)$, by means of the imbedding of Z_y in X, a vertical scalar q-form ω determines for each fiber Z_y an ordinary scalar q-form over Z_y. Intuitively speaking, ω is the collection of these scalar q-forms over the fibers; the given definition insures, however, that that transition from fiber to fiber is sufficiently smooth. If $v_1, \ldots, v_q$ are vertical C^∞ vector fields over X, then $\omega(v_1 \ldots, v_q)$ is a C^∞ function over X, because it is a composite of C^∞ mappings. $\omega(v_1, \ldots, v_q)$ is $\Phi_0(X)$-linear in each of the v_i's. Conversely, one can prove (in analogy to Lemma 1.1) that the $\Phi_0(X)$-multilinear skew-symmetric mappings of the $\Phi_0(X)$-

[9]) The English language has good terms to restrict *ranges* of functions (e.g. "real-valued," "vertical-valued"), but the situation for domains seems to force some kind of abuse of language, since vertical scalar forms are not scalar forms in the proper sense.

module $\mathfrak{W}$ of vertical vector fields into $\Phi_0(X)$ are in 1-1-correspondence with the vertical scalar forms over X.

Another way (for paracompact Y) to look at vertical scalar forms is as follows. Call two scalar forms over X equivalent if for all $y \in Y$, their restrictions to Z_y are the same. The equivalence classes of this relation are in 1-1-correspondence with the vertical scalar forms as defined above. In fact, any such class can be assigned to the common restriction to the fibers; and conversely, every vertical form ω' on X is the restriction to the fibers of some form ω on X. A construction of ω can be given by a method analogous to the proof of Proposition 5.1 below.

Definition. A *vertical vector l-form* L over X is a C^∞ mapping $L: \oplus^l V(X) \to V(X)$ whose restriction $L_x = L \mid \oplus^l V_x(X)$ is, for each x, a R-multilinear and skew-symmetric mapping $L_x: \oplus^l V_x(X) \to V_x(X)$.

A vertical vector field is obviously the same as a vertical-valued vector 0-form. Not all vector forms over X are restrictable to the fibers; but for the ones that are, the remarks made for scalar forms hold here, mutatis mutandis.

Proposition 5.1. Let h' be a vector 1-form on the paracompact base space Y of a fiber bundle $p: X \to Y$ and h'' a vertical vector 1-form over X. Suppose that they satisfy the equations $\chi(h') = \chi(h'') = 0$, where χ is a polynomial whose coefficients are functions on Y. Then there is a vector 1-form h over X such that:

i) h is projectable, its projection being h';
ii) the restriction of h to the fibers is h'';
iii) $\chi(h) = 0$.

Proof. a) In the special case where the fiber bundle is actually a direct product, $X = Y \times Z$, the proposition is obvious. b) In order to reduce the general case to the special one, let $\mathfrak{U} = \{U_i\}_{i \in I}$ and $\Psi_i: U_i \times Z \to p^{-1}(U_i)$ have the meanings stated in the definition of a fiber bundle (§ 4). We can suppose that $\mathfrak{U}$ is a locally finite covering. Each set $p^{-1}(U_i)$ has, through the diffeomorphism Ψ_i, a structure of direct product, and according to the case a) there exists a vector 1-form h_i over $p^{-1}(U_i)$ whose projection is h' (restricted to U_i of course) and whose restriction to the fibers is h'' (restricted to $p^{-1}(U_i)$, of course) and which satisfies $\chi(h_i) = 0$. Let $\{\varphi_i\}_{i \in I}$ be a partition of unity with respect to the covering $\mathfrak{U}$. Then $h = \sum_{i \in I} (p^* \varphi_i) \cdot h_i$ is a vector 1-form over X which projects to h' and restricts to h''. It remains to prove that $\chi(h) = 0$. This is a pointwise property and follows from the equations $\chi(h_i) = 0$ according to the following:

Lemma 5.2. Suppose given

a) $p\colon T \to H$, a homomorphism of a vector space T onto an other H;
b) $h''\colon V \to V$, an endomorphism of V, V being the kernel of p;
c) $h'\colon H \to H$, an endomorphism of H.

Then for any polynomial Ψ (with real coefficients), any endomorphisms $h_1, \ldots, h_n$ of T satisfying $p \circ h_i = h' \circ p$ and $h_i \mid V = h''$ $(i = 1, \ldots, n)$, and any numbers $\lambda_1, \ldots, \lambda_n$ satisfying $\sum_{i=1}^{n} \lambda_i = 1$ one has:

$$\Psi(\lambda_1 h_1 + \cdots + \lambda_n h_n) = \lambda_1 \Psi(h_1) + \cdots + \lambda_n \Psi(h_n) .$$

Proof. For at least one i, say $i = n$, one has $\lambda_i \neq 1$ (unless $n = 1$, in which case there is nothing to prove). Writing then

$$\lambda_1 h_1 + \cdots + \lambda_n h_n = (1 - \lambda_n)(\mu_1 h_1 + \cdots + \mu_{n-1} h_{n-1}) + \lambda_n h_n ,$$

one has $\sum_{i=1}^{n-1} \mu_i = 1$, and therefore the lemma follows by induction on n, provided we prove it for $n = 2$. Given Ψ there is a polynomial Λ in two non-commutative variables such that for any two endomorphisms ξ, ζ of T one has

$$\Psi(\xi + \zeta) = \Psi(\xi) + \Lambda(\xi, \zeta) . \tag{5.1}$$

With $\lambda_1 + \lambda_2 = 1$ and $k = h_2 - h_1$ we have $\lambda_1 h_1 + \lambda_2 h_2 = h_1 + \lambda_2 k$; hence

$$\Psi(\lambda_1 h_1 + \lambda_2 h_2) = \Psi(h_1) + \Lambda(h_1, \lambda_2 k) . \tag{5.2}$$

According to the assumptions on h_1 and h_2, their difference k (and hence also $\lambda_2 k$) has projection zero and restriction zero, i.e. $k(T) \subset V$ and $k(V) = 0$. Together with $h_1(V) \subset V$ this implies that all those terms of $\Lambda(h_1, \lambda_2 k)$ which contain at least two factors k vanish, and since Λ has no terms of degree 0, we conclude that $\Lambda(h_1, \lambda_2 k)$ is actually linear in $\lambda_2 k$, i.e. $\Lambda(h_1, \lambda_2 k) = \lambda_2 \Lambda(h_1, k)$. Using this and (5.1), we obtain from (5.2); $\lambda_1 + \lambda_2 = 1$ and $k = h_2 - h_1$:

$$\begin{aligned} \Psi(\lambda_1 h_1 + \lambda_2 h_2) &= \Psi(h_1) + \Lambda(h_1, \lambda_2 k) = \Psi(h_1) + \lambda_2 \Lambda(h, k) \\ &= \Psi(h_1) + \lambda_2(\Psi(h_1 + k) - \Psi(h_1)) = \lambda_1 \Psi(h_1) + \lambda_2 \Psi(h_2) \end{aligned} \tag{5.4}$$

which gives the desired formula.

The main subject of this section is to introduce and study LIE derivatives of vertical forms with respect to projectable vector fields. Intuitively, these LIE derivatives are meaningful since the mappings $\exp tu$, for projectable u, send fibers into fibers. However, the domains in which these mappings can be defined may create difficulties (unless the fibers are compact, or certain additional conditions are imposed on the vector fields). Besides, it turns out that in the more formal approach which is given here, the assumption on the domain over which u is defined, can be reduced greatly.

Proposition 5.3. Let u be a vector field at one fiber Z_y; v a vertical vector field defined in a neighborhood V of Z_y. Then the following holds:

a) The usual definition of $[u, v]_x$ is meaningful for all $x \in Z_y$;
b) $[u, fv] = f\cdot[u, v] + (uf)\cdot v$ for all $f \in \Phi_0(V)$;
c) $[fu, v] = f\cdot[u, v] - (vf)\cdot u$ for all $f \in \Phi_0(Z_y)$;
d) u projectable implies $[u, v]$ vertical.

Proof. a) The defining expression for $[u, v]_x$ is

$$[u, v]_x f = u_x(vf) - v_x(uf)\,, \qquad f \in \Phi_0(V)\,. \tag{5.4}$$

Here vf is defined in a neighborhood of Z_y, and since $x \in Z_y$, the action of u_x on vf is defined. uf is defined only over Z_y, but v_x being tangent to Z_y, its action on vf is also defined. b) and c) are as usual, except for the domain of f, which is obvious. d) If $\tilde{u}$ is an extension of u over a neighborhood of Z_y, then $[\tilde{u}, v]_x = [u, v]_x$ for $x \in Z_y$; u being projectable, $\tilde{u}$ can be chosen projectable. By Theorem 1, $p_*[\tilde{u}, v] = [p_*\tilde{u}, p_* v] = 0$ since v is vertical, which proves d).

Thus the action of a projectable vector field u at Z_y on functions of $\Phi_0(X)$ and on vertical vector fields are meaningful. This is used in order to extend the action of u to any vertical scalar or vector q-form Ω. The action of u shall have the usual derivation property of a Lie derivative, i.e.

$$[u, \Omega](v_1, \ldots, v_q) = [u, \Omega(v_1, \ldots, v_q)] - \sum_{i=1}^{q} \Omega(v_1, \ldots, [u, v_i], \ldots, v_q) \tag{5.5}$$

Theorem 7. Let u be a vector field at the fiber Z_y; Ω a vertical scalar or vector q-form over a neighborhood of Z_y; and $v_1, \ldots, v_q$ vertical vector fields over a neighborhood of Z_y. Then the right side of (5.5) is meaningful at points $x \in Z_y$; is skew-symmetric and $\Phi_0(X)$-multilinear in the v_i's; and is vertical in case Ω is a vertical vector form. Thus (5.5) defines a vertical scalar resp. vector q-form $[u, \Omega]$ over Z_y.

Proof. The skew-symmetry statement is obvious; all the rest is a consequence of Proposition 5.3.

The brackets $[L, \omega]$ and $[L, M]$ of vertical forms over X are again vertical forms over X and could be defined according to the derivation approach; or by defining their action on (vertical) vector fields analogously as for ordinary forms over X. If then the vertical forms ω, L, M are the restrictions to the fibers of forms $\tilde{\omega}, \tilde{L}, \tilde{M}$ over X, then the brackets $[L, \omega]$, $[L, M]$ so defined are the restrictions to the fibers of the forms $[\tilde{L}, \tilde{\omega}]$, $[\tilde{L}, \tilde{M}]$ (which by Theorem 1 are restrictable to the fibers). This can be used to give a shorter definition of $[\omega, L]$ and $[L, M]$: for given vertical forms ω, L, M, choose extensions $\tilde{\omega}, \tilde{L}, \tilde{M}$[10]); by Theorem 1 the restrictions to the fibers of $[\tilde{L}, \tilde{\omega}]$ and $[\tilde{L}, \tilde{M}]$ exist and do not depend on the choice of the extensions, and therefore can be defined to be $[L, \omega]$ and $[L, M]$. The same procedure works of course for $\omega \bar{\wedge} L$ and $M \bar{\wedge} L$. From these definitions it follows at once, that for vertical forms over X there hold the analogous formulas as for forms over X.

With LIE derivatives of vertical forms with respect to a projectable vector field at one fiber Z_y, as introduced in Theorem 7, one has to be more careful. If e.g. u and v are projectable vector fields at Z_y, then it is of course not possible to take LIE derivatives with respect to u and v in succession – unless u and v are both vertical. However, many properties of ordinary LIE derivatives still hold; in particular the following „LEIBNITZ rules":

Proposition 5.4. Let u be a projectable vector field at the fiber Z_y; Ω a vertical scalar or vector form and L a vertical vector form over a neighborhood of Z_y. Then

$$[u, [L, \Omega]] = [[u, L], \Omega] + [L, [u, \Omega]]\,; \tag{5.6}$$

$$[u, \Omega \bar{\wedge} L] = [u, \Omega] \bar{\wedge} L + \Omega \bar{\wedge} [u, L]\,. \tag{5.7}$$

The proof follows from the definition of the operations for vertical forms and the following.

Lemma 5.5. Let Ω be a vertical form over a neighborhood of Z_y and u a projectable vector field at Z_y. If $\tilde{\Omega}$ is any form over a neighborhood of Z_y whose restriction to the fibers is Ω, and $\tilde{u}$ any projectable vector field over a neighborhood of Z_y which is an extension of u, then $[u, \Omega]$ is the restriction to Z_y of $[\tilde{u}, \tilde{\Omega}]$. Moreover, such extensions always exist.

The proof consists in evaluating $[\tilde{u}, \tilde{\Omega}]$ on vertical vector fields and comparing with the equation (5.5) by which $[u, \Omega]$ was defined. That extensions exist is trivial if one chooses a neighborhood of Z_y which is a direct product.

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[10]) Extensions over X do not necessarily exist. However, since the operations to be defined are local operations, it is sufficient to choose extensions over one of the sets $p^{-1}(U)$ which has a product structure (cf. the definition of a fiber bundle).

§ 6. Homogeneous almost-complex and complex structures. Let G be a real Lie group, H a closed connected subgroup and G/H the set of left-cosets of H in G. Then $p: G \to G/H$, where $p(a)$ is the left-coset aH containing the element $a \in G$, is a fiber bundle of class C^ω. Since for any $a \in G$ the left-translation $l_a: G \to G$ preserves left-cosets, l_a induces a C^ω mapping $W_a: G/H \to G/H$ characterized by

$$W_a \circ p = p \circ l_a . \tag{6.1}$$

A vector form – or any other tensor field – over G/H is called invariant or homogeneous, if it is invariant under all W_a; or, equivalently, if it is W_a-related to itself for all $a \in G$. A vector 1-form J thus is invariant if and only if

$$J_{W_a(x)} \circ (W_a)_* = (W_a)_* \circ J_x \tag{6.2}$$

for all $x \in G/H$ and all $a \in G$.

Lemma 6.1. Let J be a vector 1-form over G/H. Then the following statements are equivalent:

i) J is invariant

ii) J is the projection of a (projectable) left-invariant vector 1-form K over G whose restriction to the fibers is zero.

Proof. i) $\Rightarrow$ ii). We first construct K. Let e be the identity element of G, and $x_0 = p(e)$. Choose a subspace V of $T_e(G)$ which is complementary to $T_e(H)$. Then p_* maps V isomorphically onto $T_{x_0}(G/H)$. Let $q: T_{x_0}(G/H) \to V$ be the inverse map. Define the value of K at e by

$$K_e = q \circ J_{x_0} \circ p_* \tag{6.3}$$

and define K over G by left-invariance:

$$K_a = (l_a)_* \circ K_e \circ (l_{a^{-1}})_* . \tag{6.4}$$

From (6.4) we obtain, using (6.3) and (6.1):

$$K_a = (l_a)_* \circ q \circ J_{x_0} \circ (W_{a^{-1}})_* \circ p_* \tag{6.5}$$

which shows, that K_a maps vertical vectors into zero (cf. the definition of a vertical vector in § 4). Multiplying (6.5) from the left with p_*, and using (6.1), (6.2) and the fact that $p_* \circ q$ is the identity on $T_{x_0}(G/H)$, one obtains

$$p_* \circ K_a = J_{W_a(x_0)} \circ p_* ; \tag{6.6}$$

and since

$$W_a(x_0) = W_a(p(e)) = p(l_a(e)) = p(a \cdot e) = p(a) ,$$

(6.6) states that J is the projection of K.

ii) $\to$ i) We consider the diagram

$$\begin{tikzcd}
T(G/H) \arrow[rrr, "(W_a)_*"] \arrow[ddd, "J"'] & & & T(G/H) \arrow[ddd, "J"] \\
 & T(G) \arrow[ul, "p_*"'] \arrow[r, "(l_a)_*"] \arrow[d, "K"'] \arrow[dr, phantom, "1"] & T(G) \arrow[ur, "p_*"] \arrow[d, "K"] & \\
 & T(G) \arrow[dl, "p_*"] \arrow[r, "(l_a)_*"] \arrow[dr, phantom, "4"] & T(G) \arrow[dr, "p_*"'] & \\
T(G/H) \arrow[rrr, "(W_a)_*"] & & & T(G/H)
\arrow[from=1-2, to=2-3, phantom, "5"]
\arrow[from=2-1, to=3-2, phantom, "2"]
\arrow[from=2-3, to=3-4, phantom, "3"]
\end{tikzcd}$$

a) Square 1 is commutative, since K is left-invariant.

b) Squares 2 and 3 are commutative, since K and J are p-related (cf. (2.4)).

c) Squares 4 and 5 are commutative according to (6.1).

a), b), c) together with the fact that p^* is onto immediately imply the commutativity of the outside square, i.e. the invariance of J under W_a.

Let g denote the LIE algebra of left-invariant vector fields on G, h the subalgebra of those that are tangent to H. g and h are isomorphic to the LIE algebras of G and H, respectively, and the elements of h form a set $\mathfrak{V}$ in the sense of Lemma 4.3. A left-invariant vector 1-form K on G sends left-invariant vector fields into such, hence induces a linear transformation on g (also denoted by K). Conversely, a linear transformation of g induces a left-invariant vector 1-form over G.

Lemma 6.2. Let K be a left-invariant vector 1-form over G whose restriction to the fibers is zero. Then the following are equivalent:

i) K is projectable

ii) $[v, Ku] - K[v, u] \in h$ for all $u \in g$ and $v \in h$.

Proof. K, having restriction zero, is certainly pointwise-projectable (Proposition 4.2). Condition ii) is equivalent to saying that $[v, K]$ is vertical-valued for all vertical v (cf. (1.12) and Lemma 4.3). This reduces the Lemma to Theorem 6 (§ 4).

A homogeneous vector 1-form J on G/H is a homogeneous almost-complex structure if $J \bar{\wedge} J = -I$. An associated K (cf. Lemma 6.1) then has the property $KKv = -v$ for all $v \in V$; where now V is precisely the set $K(T_e(G))$. Denoting by W the subset of those $v \in g$ for which $v_e \in V$ and combining with Lemmas 6.1 and 6.2 one thus has:

Theorem 7. G/H admits a homogeneous almost-complex J if and only if the LIE algebra g of G admits a linear transformation K satisfying

a) $Ku = 0$ for all $u \in h$;
b) $KKv = -v$ for all v in a subspace W of g complementary to h;
c) $[v, Ku] - K[v, u] \in h$ for all $v \in h$ and $u \in g$.

The relation between J and K is not a one-to-one correspondence, since the choice of V was involved in the construction of K. The action of K on g/h, however, is uniquely determined by J. Denoting by $\langle u \rangle$ the equivalence class $u + h \in g/h$ of the element $u \in g$, h acts on g/h, the action A_v of $v \in h$ being

$$A_v \langle u \rangle = \langle [v, u] \rangle . \tag{6.7}$$

Theorem 8. The homogeneous almost-complex structures over G/H are in one-to-one correspondence with those endomorphisms of g/h with square minus identity which commute with the action (6.7) of h on g/h.

Remark. So far we assumed that H is connected. The motivation is that the transformations A_v, $v \in h$ (cf. (6.7)) should generate the action of H on g/h. Since the group which acts effectively on g/h is a factor group H/H_0 of H, it is enough to require that H/H_0 is connected. Here, H_0 is the normal subgroup of those elements of H which, when acting on g by the adjoint representation, keep every element of g fixed modulo h. Another way of formulating the condition is that H_0 should intersect every component of H.

For the following theorem dealing with complex structures we make use of the following two facts. The complex structures correspond precisely to those almost-complex structures J that satisfy the integrability condition $[J, J] = 0$; in this one-to-one correspondence the homogeneous complex structures correspond to the invariant integrable almost complex structures J.

Theorem 9. G/H admits a homogeneous complex structure if and only if the Lie algebra g of G admits a linear transformation K satisfying:
α) $Ku = 0$ for all $u \in h$;
β) $KKu = -u$ for all u in a subspace W of g complementary to h;
γ) $[Ku, Kv] + KK[u, v] - K[Ku, v] - K[u, Kv] \in h$ for all u, $v \in g$.

Proof. a) Suppose J is a homogeneous complex structure. Then $[J, J] = 0$. According to Theorem 8 we obtain K satisfying conditions α) and β). K is p-related to J; hence by Theorem 1 $[K, K]$ is p-related to $[J, J] = 0$, i.e. $[K, K]$ must be vertical-valued, and (1.13) therefore shows that condition γ) is also verified.

b) Suppose we have K satisfying α), β), γ). We first show that then condition c) of Theorem 8 is satisfied. γ) gives, for $u \in g$ and $v \in h$ (since $Kv = 0$ according to α):

$$K(K[u, v] - [Ku, v]) \in h . \tag{6.8}$$

This gives condition c) of Theorem 8; in fact, for any $w \in g$, $Kw \in h$ implies $Kw = 0$ (because $K(g) = W$) and therefore $w \in h$ (because h is the kernel of K). By Theorem 8 it follows, that K projects onto an invariant almost-complex structure J over G/H. According to γ), $[K, K]$ is vertical-valued, and thus its projection, which by Theorem 1 is $[J, J]$, is zero. Hence J is a homogeneous complex structure over G/H.

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