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Slowly Growing Integral and Subharmonic Functions

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1. G. Piranian [3] recently proved the following

Theorem A. *There exists a sequence* $\{t_n, r_n\}$ *such that the integral function*

$$f(z) = \prod_{n=1}^{\infty} \left\{1 - \left(\frac{z}{r_n}\right)^n\right\}^{t_n}$$

has the property that each half-line contains infinitely many disjoint segments of length 1, *on which* $|f(z)| < 1$. *Corresponding to each real-valued function* $h(r)$ *satisfying the condition*

$$\frac{h(r)}{(\log r)^2} \to \infty\,, \tag{1.1}$$

the sequence $\{t_n, r_n\}$ *can be so chosen that the inequality*

$$\log|f(re^{i\theta})| < h(r)$$

holds for $r > r_0$ *and all real* θ.

Erdös conjectured that if on the other hand

$$\log|f(re^{i\theta})| < A(\log r)^2$$

as $r \to \infty$, uniformly in θ, then $|f(z)| > K$ outside a set of bounded regions subtending angles at the origin whose sum is finite. It would follow that for almost every fixed θ, $|f(re^{i\theta})| \to \infty$ as $r \to \infty$.

In this paper the above conjecture will be proved and a little more.

We shall call an $\mathcal{E}$-set any countable set of circles not containing the origin, and subtending angles at the origin whose sum s is finite. The number s will be called the (angular) extent of the $\mathcal{E}$-set.

We make the following remarks

(i) *For almost all fixed* θ *and* $r > r_0(\theta)$, $z = re^{i\theta}$ *lies outside the* $\mathcal{E}$*-set.*

In fact this is the case unless the ray $z = re^{i\theta}$, $0 < r < \infty$ meets infinitely many circles of the $\mathcal{E}$-set. We can write $\mathcal{E} = \mathcal{E}' \cup \mathcal{E}''$, where $\mathcal{E}'$ contains only a finite number of circles and $\mathcal{E}''$ has extent less than ε. If the ray $z = re^{i\theta}$ meets infinitely many circles of $\mathcal{E}$, then this ray meets $\mathcal{E}''$ and the set of such θ has measure at most ε, i. e. measure zero.

(ii) *The set* E, *of* r *for which the circle* $|z| = r$ *meets the circles of an* $\mathcal{E}$*-set has finite logarithmic measure and à fortiori, zero density.*

Let a circle C_n of an $\mathcal{E}$-set have radius r_n and centre distant d_n from the origin. Then the logarithmic measure l_n of the set of r corresponding to circles $|z| = r$ which C_n meets is given by

$$l_n = \int_{d_n - r_n}^{d_n + r_n} \frac{dr}{r} = \log \frac{d_n + r_n}{d_n - r_n} < 3 \frac{r_n}{d_n}, \quad \text{if} \quad r_n < \tfrac{1}{2} d_n .$$

The extent c_n of C_n is $2 \sin^{-1} \frac{r_n}{d_n} > \frac{2 r_n}{d_n}$. Thus for all but a finite number of values of n, $l_n < \frac{3}{2} c_n$, and so $\Sigma l_n < + \infty$. If $c(t)$ is the characteristic function of the set E and

$$\int_1^\infty c(t) \frac{dt}{t}$$

converges then

$$\int_{r_0}^{r} c(t) dt \leqslant [\int_{r_0}^{r} c(t) \frac{dt}{t} \int_{r_0}^{r} t dt]^{\frac{1}{2}} < \varepsilon^{\frac{1}{2}} r$$

if $r > r_0(\varepsilon)$, so that E has zero linear density, but the converse is false.

Let $u(z)$ be subharmonic and not constant in the plane and write

$$B(r) = B(r, u) = \sup_{|z|=r} u(z) .$$

Then $B(r)$ is a convex increasing function of $\log r$ and so tends to infinity with r. In the applications we may think of $u(z) = \log |f(z)|$ where $f(z)$ is an integral function, but the more general case has some interest. We then have the following

Theorem 1. *With the above hypotheses suppose that*

$$B(r, u) = O(\log r)^2 \quad \text{as} \quad r \to \infty ; \tag{1.2}$$

then

$$u(re^{i\theta}) \sim B(r) \tag{1.3}$$

uniformly as $re^{i\theta} \to \infty$ *outside an* $\mathcal{E}$*-set.*

Corollary. *The relation* (1.3) *holds as* $r \to \infty$ *for almost every fixed* θ. *It holds uniformly in* θ *as* $r \to \infty$ *outside a set of finite logarithmic measure.*

The special case $u(z) = \log |f(z)|$ where $f(z)$ is regular yields ERDÖS' conjecture and rather more, since ERDÖS only conjectured that $u(z) > 0$ outside an $\mathcal{E}$-set. In this case VALIRON [4, p. 134] showed that (1.3) holds outside a set of linear density 0. As we have just noted an $\mathcal{E}$-set has linear density 0, but the converse is false, so that our result is stronger than that of VALIRON.

We prove a further result generalizing the case $u(z) = \log |f(z)|$, when $f(z)$ is a polynomial.

Theorem 2. *Suppose that* $u(z)$ *is subharmonic and not constant in the plane and that*

$$B(r, u) = O(\log r), \quad \text{as} \quad r \to \infty .$$

Then $u(re^{i\theta}) = B(r, u) + o(1)$, *uniformly as* $re^{i\theta} \to \infty$ *outside an* $\mathcal{E}$*-set.*

Finally we note that if $e^{u(z)}$ is continuous it is not difficult to prove by means of the HEINE-BOREL theorem that we may select a subsystem $\mathcal{E}'$ from our $\mathcal{E}$-set such that only a finite number of the circles of $\mathcal{E}'$ meet any bounded set. In the general case this is not possible since $u(z) = -\infty$ may take place for a set of z which is dense in the plane.

2. Let $u(z)$ be a subharmonic function satisfying $u(0) = 0$. If this condition is not satisfied we replace $u(z)$ inside $|z| < 1$ by the POISSON integral of its values on $|z| = 1$ and leave $u(z)$ unchanged for $|z| \geqslant 1$. The modified function is still subharmonic and is harmonic near $z = 0$, so that $u(0)$ is finite. By subtracting a constant we may suppose that $u(0) = 0$.

It now follows (HEINS [2]) that if the order

$$\varrho = \varlimsup_{r \to \infty} \frac{\log B(r, u)}{\log r} < 1$$

then u can be represented as

$$u(z) = \int \log \left| 1 - \frac{z}{\zeta} \right| d\mu e_{\zeta} \tag{2.1}$$

where $d\mu$ is a positive measure in the plane for which compact sets have finite measure, and the integral extends over the ζ plane. In our applications $\varrho = 0$, so that the above conditions are satisfied. The formula (2.1) reduces to the WEIERSTRASS product expansion

$$\log |f(z)| = \sum_1^{\infty} \log \left| 1 - \frac{z}{\zeta_n} \right| \tag{2.1'}$$

when $u(z) = \log |f(z)|$ and $f(z)$ is an integral function of order less than 1. Further let $n(t) = \mu[|z| < t]$,

$$N(r) = \int_0^r \frac{n(t)\,dt}{t} .$$

Then JENSEN's formula gives ([1], Lemma 1, p. 473 and (1.7) p. 474).

$$\frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta})\,d\theta = N(r)$$

so that in particular

$$N(r) \leqslant B(r) . \tag{2.2}$$

It follows from (2.1) that

$$u(z) \leqslant \int \log\left(1 + \left|\frac{z}{\zeta}\right|\right) d\mu e_\zeta = \int_0^\infty \log\left(1 + \frac{|z|}{t}\right) dn(t) . \tag{2.3}$$

We suppose in all cases that

$$B(r) < C(\log r)^2, \qquad r > r_0 . \tag{2.4}$$

Using (2.2) we deduce

$$n(r) \log r \leqslant \int_r^{r^2} n(t) \frac{dt}{t} \leqslant N(r^2) < 4C(\log r)^2 , \qquad r > r_0$$

i.e.

$$n(r) < 4C \log r , \qquad r > r_0 . \tag{2.5}$$

Let

$$\lim_{t \to \infty} n(t) = n . \tag{2.6}$$

If $n = 0$, $u(z) \equiv 0$ which is contrary to our hypotheses. If $0 < n < \infty$

$$N(r) \sim n \log r , \quad \text{as} \quad r \to +\infty . \tag{2.7}$$

If $n = +\infty$

$$\frac{N(r)}{\log r} \to +\infty , \quad \text{as} \quad r \to +\infty . \tag{2.8}$$

In the case (2.1′), (2.7) corresponds to the case when $f(z)$ is a polynomial and (2.8) to the case when $f(z)$ is transcendental. In this case VALIRON [4, p. 132] noted that if (2.4) is satisfied then

$$B(r) \sim N(r) \tag{2.9}$$

as $r \to \infty$, and his argument extends at once to subharmonic functions. In fact from (2.3) we obtain

$$B(r) \leqslant \int_0^\infty \log\left(1 + \frac{r}{t}\right) dn(t) = r \int_0^\infty \frac{n(t)\,dt}{t(t + r)} .$$

Suppose now first that n is finite in (2.6). Let η be a fixed small positive number and choose r so large that $n(t) > n - \eta$ for $t \geqslant \eta r$. Then

$$\begin{aligned} B(r) &\leqslant \int_0^{\eta r} \frac{r n(t)\,dt}{t(t + r)} + \int_{\eta r}^\infty \frac{n r\,dt}{t(t + r)} \leqslant N(\eta r) + n \log \frac{r + \eta r}{\eta r} \\ &= N(\eta r) + n \log\left(\frac{r}{\eta r}\right) + n \log(1 + \eta) \\ &\leqslant N(\eta r) + \int_{\eta r}^{r} (n(t) + \eta) \frac{dt}{t} + n \log(1 + \eta) \\ &= N(r) + \eta \log \frac{1}{\eta} + n \log(1 + \eta) . \end{aligned}$$

Since η may be chosen as small as we please, we deduce in this case that

$$B(r) \leqslant N(r) + o(1), \quad \text{as} \quad r \to \infty .$$

In the case (2.8), when (2.4) holds we deduce from (2.5)

$$B(r) \leqslant N(r) + r \int_r^\infty \frac{O(\log t)}{t^2} dt \leqslant N(r) + O(\log r) \sim N(r) .$$

Since (2.2) holds in all cases we deduce (2.9) and in the case (2.7) the stronger result

$$B(r) = N(r) + o(1), \quad \text{as} \quad r \to \infty . \tag{2.10}$$

3. In order to prove our results we note that (2.1) and (2.3) give

$$u(z) - B(r) \geqslant \int \log \frac{|\zeta - z|}{|\zeta| + |z|} d\mu e_\zeta = I_1 + I_2 + I_3 \tag{3.1}$$

say, where I_1 is taken over the range $|\zeta| \leqslant \frac{1}{2}|z|$, I_2 over the range $\frac{1}{2}|z| < |\zeta| < 2|z|$, and I_3 over the range $|\zeta| \geqslant 2|z|$.

We note that $\log \dfrac{1+x}{1-x} < 3x$, for $0 < x < \frac{1}{2}$, so that for $|z| = r$

$$- I_1 \leqslant \int_{|\zeta| \leqslant \frac{1}{2}|z|} \log \frac{1 + \left|\frac{\zeta}{z}\right|}{1 - \left|\frac{\zeta}{z}\right|} d\mu e_\zeta < \frac{3}{|z|} \int_{|\zeta| \leqslant \frac{1}{2}|z|} |\zeta| \, d\mu e_\zeta = \frac{3}{r} \int_0^{\frac{1}{2}r} t dn(t) .$$

Similarly

$$- I_3 < 3r \int_{2r}^\infty \frac{1}{t} dn(t) .$$

In case n is finite in (2.6), suppose that $n(t) > n - \varepsilon$, $t > t_0$. Then if $r > 2t_0$, we have

$$\int_0^{\frac{1}{2}r} t dn(t) \leqslant \int_0^{t_0} t dn(t) + \int_{t_0}^{\frac{1}{2}r} t dn(t) \leqslant t_0 n + \tfrac{1}{2} r \varepsilon ,$$

so that

$$I_1 \to 0 , \quad \text{as} \quad r \to \infty .$$

Similarly we have for $r > t_0$

$$I_3 < \frac{3r}{2r} \int_{2r}^\infty dn(t) < \tfrac{3}{2} \varepsilon .$$

Thus in this case

$$I_1 \to 0 , \quad I_3 \to 0 , \quad \text{as} \quad r \to \infty . \tag{3.2}$$

Consider next the case when (2.4) and hence (2.5) holds. In this case we have for $r > r_0$,

$$I_1 \leqslant \frac{3 \cdot \frac{1}{2} r}{r} \int_0^{\frac{1}{2}r} dn(t) \leqslant 6C \log r \,,$$

$$I_3 \leqslant 3r \int_{2r}^{\infty} \frac{1}{t}\, dn(t) = 3r\left[-\frac{n(2r)}{2r} + \int_{2r}^{\infty} \frac{n(t)\,dt}{t^2}\right] \leqslant 12Cr \int_{2r}^{\infty} \frac{\log t\,dt}{t^2}$$

$$= 6C[\log(2r) + 1] \,.$$

Thus in case (2.4) holds we have, uniformly as $z \to \infty$,

$$I_1 = O(\log |z|) \,, \qquad I_3 = O(\log |z|) \,. \tag{3.3}$$

4. It remains to estimate I_2 and this estimation is the crux of the paper. We need a form (Lemma 2) of the BOUTROUX-CARTAN Lemma applicable to subharmonic functions.

In order to prove this we use the following result ([1], Lemma 4, p. 482).

Lemma 1. *Suppose that* $\mu[|z| < h] = n \geqslant 0$, *and that* $0 < d < \frac{1}{2}h$. *Then there exists a set of circles* S *the sum of whose radii is at most* d *and such that for* $|z| < \frac{1}{2}h$, *and* z *outside* S *we have*

$$\int_{|z-\zeta|<\frac{1}{2}h} \log \left| \frac{h}{2(z-\zeta)} \right| d\mu e_\zeta < n \log \frac{16h}{d} \,.$$

We deduce

Lemma 2. *Suppose that* μ *is a positive measure in the plane vanishing outside a compact set*[1]*, and such that the measure* n *of the whole plane satisfies* $0 < n < \infty$. *Then we have*

$$\int \log |z - \zeta|\, d\mu e_\zeta \geqslant n \log \varrho$$

outside a set of circles the sum of whose radii is at most 32ϱ.

Suppose that $\mu[|\zeta| > R] = 0$. In this case we have for $|z| > R + \varrho$

$$\int \log |z - \zeta|\, d\mu e_\zeta \geqslant \int \log (|z| - R) d\mu e_\zeta = n \log (|z| - R) \geqslant n \log \varrho \,.$$

Thus we may confine ourselves to points in the circle $|z| < R + \varrho$. In Lemma 1 choose $h = 4(R + \varrho)$. Then we have for $|z| < \frac{1}{4}h$ and z lying outside the set S of circles, the sum of whose radii is at most d

$$\int_{|z-\zeta|<\frac{1}{2}h} \left\{\log \frac{h}{2} + \log \frac{1}{|z-\zeta|}\right\} d\mu e_\zeta < n \log \frac{16h}{d} \,,$$

[1] This condition is not essential but simplifies the proof.

provided $d < \frac{1}{2}h$. The result holds also if $d \geqslant \frac{1}{2}h$ since we can choose for S the single circle $|z| < \frac{1}{2}h$. Since the circle $|z - \zeta| < \frac{1}{2}h$ includes the circle $|\zeta| < R$, the integral on the left-hand side may be taken over the whole plane. We deduce

$$\int \log \left| \frac{1}{z - \zeta} \right| d\mu e_\zeta \leqslant n \log \frac{32}{d}$$

for $|z| < R + \varrho$, outside the set of circles S the sum of whose radii is at most d, and setting $d = 32\varrho$ Lemma 2 follows.

Lemma 3. *Suppose that μ is a positive measure in the plane such that the measure of the whole plane outside the origin is n, where $0 < n < \infty$. Suppose also that $K \geqslant 7$. Then we have*

$$I_2(z) = \int_{\frac{1}{2}|z|<|\zeta|<2|z|} \log \frac{|\zeta - z|}{|\zeta| + |z|} d\mu e_\zeta > -nK$$

when $z \neq 0$ and z lies outside an $\mathcal{E}$-set S of angular extent at most $4000 e^{-K}$.

Set $R_\nu = 2^\nu$, $\nu = -\infty$ to ∞ and let $\mu_\nu = \mu[\zeta \mid R_{\nu-1} < |\zeta| \leqslant R_{\nu+2}]$.

Then $\sum_{\nu=-\infty}^{\infty} \mu_\nu = 3n$. Also we have by Lemma 2 for $R_\nu \leqslant |z| \leqslant R_{\nu+1}$

$$\int_{R_{\nu-1}<|\zeta|<R_{\nu+2}} \log |\zeta - z| \, d\mu e_\zeta \geqslant \mu_\nu \log \varrho_\nu$$

outside a set S_ν of circles the sum of whose radii is at most $32\varrho_\nu$. We assume $32\varrho_\nu < \frac{1}{4}R_\nu$. In this case each circle either lies entirely in $|z| < R_\nu$, in which case we ignore it, or in $|z| > \frac{1}{2}R_\nu$, in which case if h is its radius, the angle it subtends at the origin is at most $2 \sin^{-1} \frac{2h}{R_\nu} < \frac{2\pi h}{R_\nu}$. Hence the extent of all the circles of S_ν which meet the range $R_\nu \leqslant |z| \leqslant R_{\nu+1}$ is at most $\theta_\nu = \frac{64\pi\varrho_\nu}{R_\nu}$ provided $\varrho_\nu < \frac{R_\nu}{128}$. Since also $|z| + |\zeta| < 6R_\nu$ in the range we have outside these circles

$$\int_{R_{\nu-1}\leqslant|\zeta|\leqslant R_{\nu+2}} \log \frac{|\zeta - z|}{|\zeta| + |z|} d\mu e_\zeta > \mu_\nu \left[\log \varrho_\nu + \log \frac{1}{6R_\nu} \right].$$

Hence à fortiori

$$\int_{\frac{1}{2}|z|<|\zeta|<2|z|} \log \frac{|\zeta - z|}{|\zeta| + |z|} d\mu e_\zeta > \mu_\nu \log \frac{\varrho_\nu}{6R_\nu} = -nK$$

say. We have supposed $\varrho_\nu < \frac{R_\nu}{128}$, which is certainly satisfied if $K > \log 768 = 6.64$, since $\mu_\nu \leqslant n$. In this case

$$\theta_\nu = 64\pi \frac{\varrho_\nu}{R_\nu} = 384\pi \exp\left(-\frac{nK}{\mu_\nu} \right) \leqslant 384\pi \frac{\mu_\nu}{n} e^{-K},$$

since for $x \geqslant 1$, and $y \geqslant 1$, $e^{-xy} \leqslant \frac{1}{y} e^{-x}$. Thus we have in the whole plane

$$\int_{\frac{1}{2}|z|<|\zeta|<2|z|} \log \frac{|\zeta - z|}{|\zeta| + |z|} d\mu e_\zeta > -nK$$

outside an $\mathcal{E}$-set of extent at most

$$\sum_{\nu=-\infty}^{\infty} \theta_\nu < 3.384\pi e^{-K} < 4000 e^{-K}.$$

This proves Lemma 3.

5. Proof of Theorem 2. We can now prove our results. We start with the simpler Theorem 2. Suppose then that n is finite in (2.6) and that $n(t) > n - \frac{1}{p^2}$ for $r > r_p$. Then it follows from Lemma 3 that for $p \geqslant 7$ and $|z| > 2r_p$, we have

$$I_2 = \int_{\frac{1}{2}|z|<|\zeta|<2|z|} \log \frac{|\zeta - z|}{|\zeta| + |z|} d\mu e_\zeta > -\frac{1}{p} = -\frac{1}{p^2} \cdot p$$

outside an $\mathcal{E}$-set $\mathcal{E}_p$ of extent at most $4000 e^{-p}$. For in Lemma 3 we set $d\mu e_\zeta = 0$ for $|\zeta| \leqslant r_p$, and the total measure of the remainder of the plane is then at most p^{-2}. Thus we may take $n = p^{-2}$, $K = p$ in Lemma 3.

If $\mathcal{E} = \bigcup_{p=7}^{\infty} \mathcal{E}_p$, then we have if z is outside $\mathcal{E}$ and $|z| > 2r_p$,

$$I_2 > -\frac{1}{p}.$$

In view of (2.10), (3.1) and (3.2) we deduce that

$$u(z) = B(r) + o(1) = N(r) + o(1)$$

as $z \to \infty$ outside $\mathcal{E}$, and this proves Theorem 2, since the extent of $\mathcal{E}$ is at most

$$\sum_{p=7}^{\infty} 4000 e^{-p} = \frac{4000 e^{-6}}{e - 1}.$$

6. Proof of Theorem 1. In view of Theorem 2, we may assume without loss of generality that $n(r) \to \infty$, as $r \to \infty$.

Let r_p be the upper bound of all numbers t such that $n(t) < p$. Then r_p is nondecreasing with increasing p and $r_p \to \infty$ as $p \to \infty$. In Lemma 3 take for $d\mu$ the mass distribution $d\mu e_\zeta$ of (2.1) for $|\zeta| < 2r_{p+1}^2$, and set $d\mu = 0$ otherwise. By (2.5), the total measure of the plane is then at most

$$4C \log (2r_{p+1}^2) = 8C \log r_{p+1} + O(1)$$

when p is large. Hence it follows from Lemma 3 that for large p, we have for $|z| < r_{p+1}^2$,

$$I_2(z) = \int_{\frac{1}{2}|z|<|\zeta|<2|z|} \log \frac{|\zeta - z|}{|\zeta| + |z|} d\mu e_\zeta > -8C\sqrt{p} \log r_{p+1} \tag{6.1}$$

outside an $\mathcal{E}$-set of extent $e^{-\frac{1}{2}\sqrt{p}}$.

We now distinguish two cases

(i) *Suppose that* $r_{p+1} < 2r_p^2$.

In this case we have for $r_p^2 \leqslant r < r_{p+1}^2$,

$$N(r) = \int_0^r \frac{n(t)}{t} dt \geqslant \int_{r_p}^{r_p^2} \frac{n(t)\,dt}{t} \geqslant p \log r_p \geqslant p \log\left(\frac{r_{p+1}}{2}\right)^{\frac{1}{2}} \geqslant \frac{p}{2} [\log r_{p+1} + O(1)] .$$

Thus in this case we have for $r_p^2 \leqslant |z| < r_{p+1}^2$, when p is large,

$$I_2(z) > -\frac{17C}{\sqrt{p}} N(|z|) , \tag{6.2}$$

outside an $\mathcal{E}$-set of extent at most $e^{-\frac{1}{2}\sqrt{p}}$.

(ii) *Suppose next that* $r_{p+1} \geqslant 2r_p^2$.

Then

$$\mu\{\zeta \mid \tfrac{1}{2}r_p^2 < |\zeta| < r_{p+1}\} \leqslant 1 ,$$

if $\frac{1}{2}r_p^2 > r_p$, i.e. $r_p > 2$ and so by Lemma 3 we have

$$I_2(z) = \int_{\frac{1}{2}|z|<|\zeta|<2|z|} \log \frac{|\zeta - z|}{|\zeta| + |z|} d\mu e_\zeta > -\sqrt{p} , \tag{6.3}$$

for $r_p^2 \leqslant |z| < \frac{1}{2}r_{p+1}$, outside an $\mathcal{E}$-set of extent at most $4000e^{-\sqrt{p}}$. Also in this range

$$N(|z|) \geqslant \int_{r_p}^{r_p^2} \frac{n(t)\,dt}{t} \geqslant p(\log r_p) .$$

Thus (6.3) implies

$$I_2(z) \geqslant -\frac{1}{\sqrt{p}\log r_p} N(|z|) . \tag{6.4}$$

Also for $\frac{1}{2}r_{p+1} \leqslant |z| < r_{p+1}^2$, we have

$$N(r) \geqslant \int_{r_p}^{\frac{1}{2}r_{p+1}} n(t) \frac{dt}{t} \geqslant p \log \frac{r_{p+1}}{2r_p} \geqslant p \log \left(\frac{r_{p+1}}{2}\right)^{\frac{1}{2}} = \frac{p}{2} \{\log r_{p+1} + O(1)\} .$$

Hence in view of (6.1) we deduce that for large p and $\frac{1}{2}r_{p+1} \leqslant |z| < r_{p+1}^2$

we have

$$I_2(z) > \frac{-17C}{\sqrt{p}} N(|z|)$$

outside an $\mathcal{E}$-set of extent at most $e^{-\frac{1}{2}\sqrt{p}}$. In view of (6.2) and (6.4) we see that in all cases we have for $p > p_0$ and $r_p^2 \leqslant |z| < r_{p+1}^2$

$$I_2(z) > -\frac{17C}{\sqrt{p}} N(|z|)$$

provided z lies outside an $\mathcal{E}$-set $\mathcal{E}_p$ of extent at most $2e^{-\frac{1}{2}\sqrt{p}}$. If $\mathcal{E} = \bigcup_{p=p_0}^{\infty} \mathcal{E}_p$. then the extent of $\mathcal{E}$ is finite and as $z \to \infty$ outside $\mathcal{E}$

$$I_2(z) = o\{N(|z|)\} = o\{B(|z|)\}$$

in view of (2.9). Using (2.8), (3.1) and (3.3) we deduce Theorem 1.

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