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Slowly Growing Integral and Subharmonic Functions

by W. K. HAYMAN, London

1. G. PIRANIAN [3] recently proved the following

Theorem A. There exists a sequence $\{t_n, r_n\}$ such that the integral function

$$f(z) = \prod_{n=1}^{\infty} \left\{ 1 - \left(\frac{z}{r_n} \right)^n \right\}^{t_n}$$

has the property that each half-line contains infinitely many disjoint segments of length 1, on which |f(z)| < 1. Corresponding to each real-valued function h(r) satisfying the condition

$$\frac{h(r)}{(\log r)^2} \to \infty , \qquad (1.1)$$

the sequence $\{t_n, r_n\}$ can be so chosen that the inequality

$$\log |f(re^{i\theta})| < h(r)$$

holds for $r > r_0$ and all real θ .

Erdös conjectured that if on the other hand

$$\log |f(re^{i\theta})| < A(\log r)^2$$

as $r \to \infty$, uniformly in θ , then |f(z)| > K outside a set of bounded regions subtending angles at the origin whose sum is finite. It would follow that for almost every fixed θ , $|f(re^{i\theta})| \to \infty$ as $r \to \infty$.

In this paper the above conjecture will be proved and a little more.

We shall call an \mathcal{E} -set any countable set of circles not containing the origin, and subtending angles at the origin whose sum s is finite. The number s will be called the (angular) extent of the \mathcal{E} -set.

We make the following remarks

(i) For almost all fixed θ and $r > r_0(\theta)$, $z = re^{i\theta}$ lies outside the E-set.

In fact this is the case unless the ray $z = re^{i\theta}$, $0 < r < \infty$ meets infinitely many circles of the \mathcal{C} -set. We can write $\mathcal{C} = \mathcal{C}' \circ \mathcal{C}''$, where \mathcal{C}' contains only a finite number of circles and \mathcal{C}'' has extent less than ε . If the ray $z = re^{i\theta}$ meets infinitely many circles of \mathcal{C} , then this ray meets \mathcal{C}'' and the set of such θ has measure at most ε , i. e. measure zero.

(ii) The set E, of r for which the circle |z| = r meets the circles of an \mathcal{E} -set has finite logarithmic measure and à fortiori, zero density.

Let a circle C_n of an \mathcal{E} -set have radius r_n and centre distant d_n from the

origin. Then the logarithmic measure l_n of the set of r corresponding to circles |z| = r which C_n meets is given by

$$l_n = \int_{d_n-r_n}^{d_n+r_n} \frac{dr}{r} = \log \frac{d_n+r_n}{d_n-r_n} < 3 \frac{r_n}{d_n}, \quad \text{if} \quad r_n < \frac{1}{2}d_n.$$

The extent c_n of C_n is $2\sin^{-1}\frac{r_n}{d_n} > \frac{2r_n}{d_n}$. Thus for all but a finite number of values of n, $l_n < \frac{3}{2}c_n$, and so $\Sigma l_n < +\infty$. If c(t) is the characteristic function of the set E and $\int_0^\infty c(t) \, \frac{dt}{t}$

converges then

$$\int_{r_0}^{r} c(t) dt \leqslant \left[\int_{r_0}^{r} c(t) \frac{dt}{t} \int_{r_0}^{r} t dt \right]^{\frac{1}{2}} < \varepsilon^{\frac{1}{2}} r$$

if $r > r_0(\varepsilon)$, so that E has zero linear density, but the converse is false. Let u(z) be subharmonic and not constant in the plane and write

$$B(r) = B(r, u) = \sup_{|z|=r} u(z).$$

Then B(r) is a convex increasing function of $\log r$ and so tends to infinity with r. In the applications we may think of $u(z) = \log |f(z)|$ where f(z) is an integral function, but the more general case has some interest. We then have the following

Theorem 1. With the above hypotheses suppose that

$$B(r, u) = O(\log r)^2 \text{ as } r \to \infty;$$
 (1.2)

then

$$u(re^{i\theta}) \sim B(r)$$
 (1.3)

uniformly as $re^{i\theta} \to \infty$ outside an \mathcal{E} -set.

Corollary. The relation (1.3) holds as $r \to \infty$ for almost every fixed θ . It holds uniformly in θ as $r \to \infty$ outside a set of finite logarithmic measure.

The special case $u(z) = \log |f(z)|$ where f(z) is regular yields Erdös' conjecture and rather more, since Erdös only conjectured that u(z) > 0 outside an \mathcal{E} -set. In this case Valiron [4, p. 134] showed that (1.3) holds outside a set of linear density 0. As we have just noted an \mathcal{E} -set has linear density 0, but the converse is false, so that our result is stronger than that of Valiron.

We prove a further result generalizing the case $u(z) = \log |f(z)|$, when f(z) is a polynomial.

Theorem 2. Suppose that u(z) is subharmonic and not constant in the plane and that $B(r, u) = O(\log r)$, as $r \to \infty$.

Then $u(re^{i\theta}) = B(r, u) + o(1)$, uniformly as $re^{i\theta} \to \infty$ outside an \mathcal{E} -set.

Finally we note that if $e^{u(z)}$ is continuous it is not difficult to prove by means of the Heine-Borel theorem that we may select a subsystem \mathcal{C}' from our \mathcal{C} -set such that only a finite number of the circles of \mathcal{C}' meet any bounded set. In the general case this is not possible since $u(z) = -\infty$ may take place for a set of z which is dense in the plane.

2. Let u(z) be a subharmonic function satisfying u(0) = 0. If this condition is not satisfied we replace u(z) inside |z| < 1 by the Poisson integral of its values on |z| = 1 and leave u(z) unchanged for $|z| \ge 1$. The modified function is still subharmonic and is harmonic near z = 0, so that u(0) is finite. By subtracting a constant we may suppose that u(0) = 0.

It now follows (Heins [2]) that if the order

$$\varrho = \overline{\lim}_{r \to \infty} \frac{\log B(r, u)}{\log r} < 1$$

then u can be represented as

$$u(z) = \int \log \left| 1 - \frac{z}{\zeta} \right| d\mu \, e_{\zeta} \tag{2.1}$$

where $d\mu$ is a positive measure in the plane for which compact sets have finite measure, and the integral extends over the ζ plane. In our applications $\varrho = 0$, so that the above conditions are satisfied. The formula (2.1) reduces to the Weierstrass product expansion

$$\log |f(z)| = \sum_{1}^{\infty} \log \left| 1 - \frac{z}{\zeta_n} \right| \qquad (2.1')$$

when $u(z) = \log |f(z)|$ and f(z) is an integral function of order less than 1. Further let $n(t) = \mu[|z| < t]$,

$$N(r) = \int_{0}^{r} \frac{n(t)dt}{t}.$$

Then JENSEN's formula gives ([1], Lemma 1, p. 473 and (1.7) p. 474).

$$\frac{1}{2\pi}\int_{0}^{2\pi}u(re^{i\theta})d\theta=N(r)$$

so that in particular

$$N(r) \leqslant B(r) . \tag{2.2}$$

It follows from (2.1) that

$$u(z) \leqslant \int \log \left(1 + \left|\frac{z}{\zeta}\right|\right) d\mu \, e_{\zeta} = \int_{0}^{\infty} \log \left(1 + \frac{|z|}{t}\right) dn(t) .$$
 (2.3)

We suppose in all cases that

$$B(r) < C(\log r)^2, \quad r > r_0.$$
 (2.4)

Using (2.2) we deduce

$$n(r) \log r \leqslant \int_{r}^{r^2} n(t) \frac{dt}{t} \leqslant N(r^2) < 4C(\log r)^2, \quad r > r_0$$

i.e.

$$n(r) < 4C \log r$$
, $r > r_0$. (2.5)

Let

$$\lim_{t\to\infty} n(t) = n . \tag{2.6}$$

If n = 0, $u(z) \equiv 0$ which is contrary to our hypotheses. If $0 < n < \infty$

$$N(r) \sim n \log r$$
, as $r \to + \infty$. (2.7)

If
$$n = + \infty$$

$$\frac{N(r)}{\log r} \to +\infty$$
, as $r \to +\infty$. (2.8)

In the case (2.1'), (2.7) corresponds to the case when f(z) is a polynomial and (2.8) to the case when f(z) is transcendental. In this case Valiron [4, p. 132] noted that if (2.4) is satisfied then

$$B(r) \sim N(r) \tag{2.9}$$

as $r \to \infty$, and his argument extends at once to subharmonic functions. In fact from (2.3) we obtain

$$B(r) \leqslant \int_{0}^{\infty} \log \left(1 + \frac{r}{t}\right) dn(t) = r \int_{0}^{\infty} \frac{n(t) dt}{t(t+r)}$$

Suppose now first that n is finite in (2.6). Let η be a fixed small positive number and choose r so large that $n(t) > n - \eta$ for $t \ge \eta r$. Then

$$B(r) \leqslant \int_{0}^{\eta r} \frac{rn(t)dt}{t(t+r)} + \int_{\eta r}^{\infty} \frac{nrdt}{t(t+r)} \leqslant N(\eta r) + n \log \frac{r+\eta r}{\eta r}$$

$$= N(\eta r) + n \log \left(\frac{r}{\eta r}\right) + n \log (1+\eta)$$

$$\leqslant N(\eta r) + \int_{\eta r}^{r} (n(t) + \eta) \frac{dt}{t} + n \log (1+\eta)$$

$$= N(r) + \eta \log \frac{1}{\eta} + n \log (1+\eta).$$

.

Since η may be chosen as small as we please, we deduce in this case that

$$B(r) \leqslant N(r) + o(1)$$
, as $r \to \infty$.

In the case (2.8), when (2.4) holds we deduce from (2.5)

$$B(r) \leqslant N(r) + r \int_{r}^{\infty} \frac{O(\log t)}{t^2} dt \leqslant N(r) + O(\log r) \sim N(r)$$
.

Since (2.2) holds in all cases we deduce (2.9) and in the case (2.7) the stronger result

$$B(r) = N(r) + o(1)$$
, as $r \to \infty$. (2.10)

3. In order to prove our results we note that (2.1) and (2.3) give

$$u(z) - B(r) \geqslant \int \log \frac{|\zeta - z|}{|\zeta| + |z|} d\mu e_{\zeta} = I_1 + I_2 + I_3$$
 (3.1)

say, where I_1 is taken over the range $\mid \zeta \mid \leqslant \frac{1}{2} \mid z \mid$, I_2 over the range $\frac{1}{2} \mid z \mid < \mid \zeta \mid < 2 \mid z \mid$, and I_3 over the range $\mid \zeta \mid \geqslant 2 \mid z \mid$.

We note that $\log \frac{1+x}{1-x} < 3x$, for $0 < x < \frac{1}{2}$, so that for |z| = r

$$-\left|I_1\leqslant \int\limits_{|\zeta|\leqslant \frac{1}{2}|z|}\log\frac{1+\left|\frac{\zeta}{z}\right|}{1-\left|\frac{\zeta}{z}\right|}\,d\mu e_{\zeta}<\frac{3}{\mid z\mid}\int\limits_{|\zeta|\leqslant \frac{1}{2}|z|}\mid \zeta\mid d\mu e_{\zeta}=\frac{3}{r}\int\limits_{0}^{\frac{1}{2}r}t\,dn\left(t\right).$$

Similarly

$$-I_{3}<3r\int_{2\pi}^{\infty}\frac{1}{t}\,dn\left(t\right) \,.$$

In case n is finite in (2.6), suppose that $n(t) > n - \varepsilon$, $t > t_0$. Then if $t > 2t_0$, we have

$$\int_{0}^{\frac{1}{2}r}tdn(t)\leqslant \int_{0}^{t_{0}}tdn(t)+\int_{t_{0}}^{\frac{1}{2}r}tdn(t)\leqslant t_{0}n+\frac{1}{2}r\varepsilon,$$

so that

$$I_1 \to 0$$
, as $r \to \infty$.

Similarly we have for $r > t_0$

$$I_3 < rac{3r}{2r} \int\limits_{2r}^{\infty} dn(t) < rac{3}{2} \varepsilon$$
.

Thus in this case

$$I_1 \to 0$$
, $I_3 \to 0$, as $r \to \infty$. (3.2)

Consider next the case when (2.4) and hence (2.5) holds. In this case we have for $r > r_0$,

$$egin{align} I_1 \leqslant rac{3 \cdot rac{1}{2} r}{r} \int \limits_0^{rac{1}{2} r} dn(t) \leqslant 6 C \log r \; , \ I_3 \leqslant 3 r \int \limits_{2r}^{\infty} rac{1}{t} \, dn(t) = 3 r igg[-rac{n(2r)}{2r} + \int \limits_{2r}^{\infty} rac{n(t) dt}{t^2} igg] \leqslant 12 C r \int \limits_{2r}^{\infty} rac{\log t \, dt}{t^2} \ &= 6 C [\log(2r) + 1] \; . \end{split}$$

Thus in case (2.4) holds we have, uniformly as $z \to \infty$,

$$I_1 = O(\log |z|), \quad I_3 = O(\log |z|).$$
 (3.3)

4. It remains to estimate I_2 and this estimation is the crux of the paper. We need a form (Lemma 2) of the Boutroux-Cartan Lemma applicable to subharmonic functions.

In order to prove this we use the following result ([1], Lemma 4, p. 482).

Lemma 1. Suppose that $\mu[|z| < h] = n \ge 0$, and that $0 < d < \frac{1}{2}h$. Then there exists a set of circles S the sum of whose radii is at most d and such that for $|z| < \frac{1}{2}h$, and z outside S we have

$$\int_{|z-\zeta|<\frac{1}{h}} \log \left| \frac{h}{2(z-\zeta)} \right| d\mu e_{\zeta} < n \log \frac{16h}{d}.$$

We deduce

Lemma 2. Suppose that μ is a positive measure in the plane vanishing outside a compact set 1), and such that the measure n of the whole plane satisfies $0 < n < \infty$. Then we have

$$\int \log |z - \zeta| d\mu e_{\tau} \geqslant n \log \varrho$$

outside a set of circles the sum of whose radii is at most 320.

Suppose that $\mu[\mid \zeta \mid > R] = 0$. In this case we have for $\mid z \mid > R + \varrho$

$$\int \log |z - \zeta| d\mu e_{\zeta} \geqslant \int \log (|z| - R) d\mu e_{\zeta} = n \log (|z| - R) \geqslant n \log \varrho$$
.

Thus we may confine ourselves to points in the circle $|z| < R + \varrho$. In Lemma 1 choose $h = 4(R + \varrho)$. Then we have for $|z| < \frac{1}{4}h$ and z lying outside the set S of circles, the sum of whose radii is at most d

$$\int_{|z-\zeta|<\frac{1}{2}h}\left\{\log\frac{h}{2}+\log\frac{1}{|z-\zeta|}\right\}d\mu e_{\zeta}< n\log\frac{16h}{d},$$

¹⁾ This condition is not essential but simplifies the proof.

provided $d < \frac{1}{2}h$. The result holds also if $d \geqslant \frac{1}{2}h$ since we can choose for S the single circle $|z| < \frac{1}{2}h$. Since the circle $|z - \zeta| < \frac{1}{2}h$ includes the circle $|\zeta| < R$, the integral on the left-hand side may be taken over the whole plane. We deduce

$$\int \log \left| \frac{1}{z-\zeta} \right| d\mu e_{\zeta} \leqslant n \log \frac{32}{d}$$

for $|z| < R + \varrho$, outside the set of circles S the sum of whose radii is at most d, and setting $d = 32\varrho$ Lemma 2 follows.

Lemma 3. Suppose that μ is a positive measure in the plane such that the measure of the whole plane outside the origin is n, where $0 < n < \infty$. Suppose also that $K \geqslant 7$. Then we have

$$I_{\mathbf{2}}(z) = \int_{\frac{1}{2}|z| < |\zeta| < 2|z|} \log \frac{|\zeta - z|}{|\zeta| + |z|} d\mu e_{\zeta} > -nK$$

when $z \neq 0$ and z lies outside an \mathcal{E} -set S of angular extent at most $4000e^{-K}$. Set $R_{\nu} = 2^{\nu}$, $\nu = -\infty$ to ∞ and let $\mu_{\nu} = \mu[\zeta \mid R_{\nu-1} < \mid \zeta \mid \leqslant R_{\nu+2}]$.

Then $\sum_{\nu=-\infty}^{-}\mu_{
u}=3n$. Also we have by Lemma 2 for $\ R_{
u}\leqslant \mid z\mid\leqslant R_{
u+1}$

$$\int\limits_{R_{\nu-1}<|\zeta|< R_{\nu+2}} \log|\zeta-z| \, d\mu e_{\zeta} \geqslant \mu_{\nu} \log \varrho_{\nu}$$

outside a set S_{ν} of circles the sum of whose radii is at most $32\varrho_{\nu}$. We assume $32\varrho_{\nu} < \frac{1}{4}R_{\nu}$. In this case each circle either lies entirely in $|z| < R_{\nu}$, in which case we ignore it, or in $|z| > \frac{1}{2}R_{\nu}$, in which case if h is its radius, the angle it subtends at the origin is at most $2\sin^{-1}\frac{2h}{R_{\nu}} < \frac{2\pi h}{R_{\nu}}$. Hence the extent of all the circles of S_{ν} which meet the range $R_{\nu} \leqslant |z| \leqslant R_{\nu+1}$ is at most $\theta_{\nu} = \frac{64\pi\varrho_{\nu}}{R_{\nu}}$ provided $\varrho_{\nu} < \frac{R_{\nu}}{128}$. Since also $|z| + |\zeta| < 6R_{\nu}$ in the range we have outside these circles

$$\int\limits_{R_{\nu-1}\leqslant \, |\zeta|\leqslant R_{\nu+2}}\!\!\!\log\frac{\mid \zeta-z\mid}{\mid \zeta\mid +\mid z\mid}\,d\mu e_{\zeta}>\mu_{\nu}\!\left[\log \varrho_{\nu}+\log\frac{1}{6\,R_{\nu}}\right].$$

Hence à fortiori

$$\int_{rac{1}{2}|z|<|\zeta|<2|z|}\lograc{\mid\zeta-z\mid}{\mid\zeta\mid+\midz\mid}\,d\mu e_{\zeta}>\mu_{
u}\lograc{arrho_{
u}}{6R_{
u}}=-nK$$

say. We have supposed $\varrho_{\nu} < \frac{R_{\nu}}{128}$, which is certainly satisfied if $K > \log 768 = 6.64$, since $\mu_{\nu} \leqslant n$. In this case

$$heta_{
u} = 64\pi rac{arrho_{
u}}{R_{
u}} = 384\pi \exp\left(-rac{nK}{\mu_{
u}}
ight) \leqslant 384\pi rac{\mu_{
u}}{n} e^{-K}$$

since for $x \geqslant 1$, and $y \geqslant 1$, $e^{-xy} \leqslant \frac{1}{y}e^{-x}$. Thus we have in the whole plane

$$\int_{rac{1}{2}|z|<|\zeta|<2\,|z|} \lograc{\mid \zeta-z\mid}{\mid \zeta\mid +\mid z\mid} d\mu e_{\zeta}>-nK$$

outside an E-set of extent at most

$$\sum_{\nu=-\infty}^{\infty} \theta_{\nu} < 3.384 \pi e^{-K} < 4000 e^{-K}$$
.

This proves Lemma 3.

5. Proof of Theorem 2. We can now prove our results. We start with the simpler Theorem 2. Suppose then that n is finite in (2.6) and that $n(t) > n - \frac{1}{p^2}$ for $r > r_p$. Then it follows from Lemma 3 that for $p \ge 7$ and $|z| > 2r_p$, we have

$$I_2 = \int_{\frac{1}{2}|z| < |\zeta| < 2|z|} \log \frac{\mid \zeta - z \mid}{\mid \zeta \mid + \mid z \mid} d\mu e_{\zeta} > -\frac{1}{p} = -\frac{1}{p^2} \cdot p$$

outside an \mathcal{E} -set \mathcal{E}_p of extent at most $4000e^{-p}$. For in Lemma 3 we set $d\mu e_{\zeta} = 0$ for $|\zeta| \leqslant r_p$, and the total measure of the remainder of the plane is then at most p^{-2} . Thus we may take $n = p^{-2}$, K = p in Lemma 3.

If $\mathcal{E} = \bigcup_{p=7}^{\infty} \mathcal{E}_p$, then we have if z is outside \mathcal{E} and $|z| > 2r_p$,

$$I_2>-rac{1}{p}$$
 .

In view of (2.10), (3.1) and (3.2) we deduce that

$$u(z) = B(r) + o(1) = N(r) + o(1)$$

as $z \to \infty$ outside \mathcal{E} , and this proves Theorem 2, since the extent of \mathcal{E} is at most

$$\sum_{p=7}^{\infty} 4000 e^{-p} = \frac{4000 e^{-6}}{e-1} .$$

6. Proof of Theorem 1. In view of Theorem 2, we may assume without loss of generality that $n(r) \to \infty$, as $r \to \infty$.

Let r_p be the upper bound of all numbers t such that n(t) < p. Then r_p is nondecreasing with increasing p and $r_p \to \infty$ as $p \to \infty$. In Lemma 3 take for $d\mu$ the mass distribution $d\mu e_{\zeta}$ of (2.1) for $|\zeta| < 2r_{p+1}^2$, and set $d\mu = 0$ otherwise. By (2.5), the total measure of the plane is then at most

$$4C \log (2r_{p+1}^2) = 8C \log r_{p+1} + O(1)$$

when p is large. Hence it follows from Lemma 3 that for large p, we have for $|z| < r_{p+1}^2$,

$$I_{2}(z) = \int_{\frac{1}{2}|z| < |\zeta| < 2|z|} \log \frac{|\zeta - z|}{|\zeta| + |z|} d\mu e_{\zeta} > -8C \sqrt{p} \log r_{p+1}$$
 (6.1)

outside an \mathcal{E} -set of extent $e^{-\frac{1}{2}\sqrt{p}}$.

We now distinguish two cases

(i) Suppose that $r_{p+1} < 2r_p^2$. In this case we have for $r_p^2 \leqslant r < r_{p+1}^2$,

$$N(r) = \int_{\mathbf{0}}^{r} \frac{n(t)}{t} dt \geqslant \int_{r_{\mathbf{p}}}^{r_{\mathbf{p}}^2} \frac{n(t) dt}{t} \geqslant p \log r_{\mathbf{p}} \geqslant p \log \left(\frac{r_{\mathbf{p}+1}}{2}\right)^{\frac{1}{2}} \geqslant \frac{p}{2} \left[\log r_{\mathbf{p}+1} + O(1)\right].$$

Thus in this case we have for $r_p^2 \leqslant |z| < r_{p+1}^2$, when p is large,

$$I_2(z) > -\frac{17C}{V\overline{p}} N(|z|),$$
 (6.2)

outside an \mathcal{E} -set of extent at most $e^{-\frac{1}{2}\sqrt{p}}$.

(ii) Suppose next that $r_{p+1} \geqslant 2r_p^2$.

Then

$$\mu\left\{\zeta\mid \frac{1}{2}r_p^2<\mid \zeta\mid < r_{p+1}
ight\}\leqslant 1$$
 ,

if $\frac{1}{2}r_p^2 > r_p$, i.e. $r_p > 2$ and so by Lemma 3 we have

$$I_{2}(z) = \int_{\frac{1}{2}|z| < |\zeta| < 2|z|} \log \frac{|\zeta - z|}{|\zeta| + |z|} d\mu e_{\zeta} > - V_{p}, \qquad (6.3)$$

for $r_p^2 \leqslant |z| < \frac{1}{2}r_{p+1}$, outside an \mathcal{E} -set of extent at most $4000e^{-1/p}$. Also in this range

$$N(\mid z\mid) \geqslant \int_{r_p}^{r_p^2} \frac{n(t)dt}{t} \geqslant p(\log r_p)$$
.

Thus (6.3) implies

$$I_2(z) \geqslant -\frac{1}{\sqrt{p}\log r_p} N(|z|). \tag{6.4}$$

Also for $\frac{1}{2}r_{p+1} \leq |z| < r_{p+1}^2$, we have

$$N(r) \geqslant \int_{r_p}^{\frac{1}{2}r_{p+1}} n(t) \frac{dt}{t} \geqslant p \log \frac{r_{p+1}}{2r_p} \geqslant p \log \left(\frac{r_{p+1}}{2}\right)^{\frac{1}{2}} = \frac{p}{2} \left\{ \log r_{p+1} + O(1) \right\}.$$

Hence in view of (6.1) we deduce that for large p and $\frac{1}{2}r_{p+1} \leqslant |z| < r_{p+1}^2$

we have

$$I_{\mathbf{2}}(z) > \frac{-17C}{Vp} N(\mid z\mid)$$

outside an \mathcal{E} -set of extent at most $e^{-\frac{1}{2}\sqrt{p}}$. In view of (6.2) and (6.4) we see that in all cases we have for $p > p_0$ and $r_p^2 \leqslant |z| < r_{p+1}^2$

$$I_{\mathbf{2}}(z)>-rac{17C}{V\overline{p}}\,N(\mid z\mid)$$

provided z lies outside an \mathcal{E} -set \mathcal{E}_p of extent at most $2e^{-\frac{1}{2}\sqrt{p}}$. If $\mathcal{E} = \bigcup_{p=p_0}^{\infty} \mathcal{E}_p$. then the extent of \mathcal{E} is finite and as $z \to \infty$ outside \mathcal{E}

$$I_{\mathbf{z}}(z) = o\{N(|z|)\} = o\{B(|z|)\}$$

in view of (2.9). Using (2.8), (3.1) and (3.3) we deduce Theorem 1.

I am greatly indebted to Professor PIRANIAN for letting me see the M. S. of his paper and to Dr. Erdős for suggesting the problem to me.

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