Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 34 (1960)

Artikel: Paths of Rapid Growth of Entire Functions.
Autor: Kaplan, Wilfred
DOI: https://doi.org/10.5169/seals-26624

Nutzungsbedingungen
Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation
L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use
The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 15.04.2026

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

Paths of Rapid Growth of Entire Functions

by WILFRED KAPLAN, Ann Arbor, Mich. (USA)

In 1957 A. HUBER published a paper in which he deduced the following theorem ([1], p. 52):

Theorem. *Let $f(z)$ be an entire function, not a polynomial. Let $\lambda > 0$. Then there exists a locally rectifiable path C_λ tending to infinity, such that*

$$\int_{C_\lambda} |f(z)|^{-\lambda} |dz| < \infty . \tag{1}$$

HUBER's proof depends on a deep study of subharmonic functions and is quite involved. Because of the simplicity of the result, I have been seeking a simple proof. This I have succeeded in obtaining only for special values of λ: $\lambda \geqq 1$ or $\lambda = 1 - (1/n)$, $n = 2, 3, \ldots$ In this note I present the proof for these values of λ.

As remarked by HUBER, there is no difficulty if $f(z)$ has only a finite number of zeros, so that $f(z) = P(z) \exp[g(z)]$, where P is a polynomial and g is entire. The function

$$w = \Phi(z) = \int_0^z e^{-\lambda g(z)} dz$$

is then entire and without critical points. If the inverse function $\Phi^{-1}(w)$ has no singular points, then it is also entire, so that $\Phi(z)$ has form $az + b$ and $f(z)$ is a polynomial; hence $\Phi^{-1}(w)$ must have singularities. In particular there must be a functional element of $\Phi^{-1}(w)$ which can be continued from $w = 0$ along a finite segment ending at a singularity w_0. The segment is mapped by $\Phi^{-1}(w)$ on a path C_λ in the z-plane, on which $z \to \infty$ as $w \to w_0$. Then

$$|w_0| = \int_{C_\lambda} \left| \frac{dw}{dz} \right| |dz| = \int_{C_\lambda} |e^{g(z)}|^{-\lambda} |dz| .$$

Thus C_λ is the desired path if $P(z) \equiv 1$; by removing a finite portion of C_λ, one can ensure that $|P(z)| \geqq 1$ on the remaining portion C'_λ, so that C'_λ is the desired path.

Now let us suppose that f has infinitely many zeros and let λ have form $1 - (1/n)$, $n = 2, 3, \ldots$ We can then assume without loss of generality that $f(z)$ is expressible as $z^{2n} G(z)$, where $G(z)$ is entire and $G(0) \neq 0$. For moving a zero of f from z_1 to the origin, or from the origin to z_1, is equivalent to multiplying f by $z/(z - z_1)$, or by $(z - z_1)/z$, a factor which approaches 1

as z approaches infinity and which has therefore no effect on the integral in (1). We select k such that $f(k) \neq 0$ and introduce

$$w = \Phi(z) = \int_k^z [f(z)]^{-\lambda} dz = \int_k^z z^{2-2n}[G(z)]^{\frac{1}{n}-1} dz \ . \tag{2}$$

This equation defines $\Phi(z)$ as a multiple-valued function of z. However, we remark that one branch (in fact, every branch) has a pole of order $2n - 3$ at $z = 0$. The inverse function $\Phi^{-1}(w)$ can be considered as the solution of the differential equation

$$\frac{dz}{dw} = z^{2n-2}[G(z)]^{1-\frac{1}{n}} \ , \tag{3}$$

such that $z = k$ when $w = 0$. We consider the solution along rays $\arg w = \text{const.}$, starting with a given analytic branch at $w = 0$. By the theory of differential equations, the solution continues to exist as long as the value of z remains within the domain of analyticity of the right-hand member of (3). Trouble can arise as $w \to w_0$ $(w_0 \neq \infty)$ only if, as $w \to w_0$, z approaches a zero of G or z approaches infinity. If $z \to z_0$, $G(z_0) = 0$, then z_0 must be a zero of first order of G, for by (2) at a multiple zero $w \to \infty$ as $z \to z_0$. Near a first order zero we obtain series expansions

$$w - w_0 = (z - z_0)^{1/n}[b_0 + b_1(z - z_0) + \ldots] \ , \ b_0 \neq 0 \ ,$$
$$z - z_0 = b_0^{-n}(w - w_0)^n + \ldots \ ;$$

that is, $\Phi^{-1}(w)$ is a single-valued analytic function in a neighborhood of w_0. [An illustration is provided by $z = \sin w$ as a solution of the differential equation $dz/dw = (1 - z^2)^{\frac{1}{2}}$].

Therefore continuation of $\Phi^{-1}(w)$ can be interrupted at a finite value w_0 only if, as $w \to w_0$, $z \to \infty$. If indefinite continuation were possible along all rays, then $\Phi^{-1}(w)$ would be an entire function of w. But we know that one branch of $\Phi^{-1}(w)$ approaches 0 as $w \to \infty$, because of the pole of $\Phi(z)$ at $z = 0$. Therefore $\Phi^{-1}(w) \to 0$ as $w \to \infty$. Accordingly, $\Phi^{-1}(w) \equiv 0$, and there is a contradiction. Hence continuation must be interrupted at at least one value w_0, and we obtain the path C_λ as in the first part of the proof.

For $\lambda \geqq 1$ we consider two cases: λ rational, equal to m/n; λ irrational. In the rational case the proof for the case $\lambda = 1 - (1/n)$ can be repeated with the simplification that, at each zero of $G(z)$, $w \to \infty$ as $z \to z_0$.

If λ is irrational, we do not need to normalize f at $z = 0$. The previous argument can be repeated with slight modification; the differential equation (3) is replaced by the equation $dz/dw = [f(z)]^\lambda$ and a solution $z(w)$ can be continued along a ray $\arg w = \text{const.}$ unless z approaches the boundary of

the domain of analyticity of $[f(z)]^\lambda$, a RIEMANN surface over the z-plane. Since $|f(z)|^\lambda$ has the same value on all sheets of this surface, we conclude that continuation can be interrupted for finite w_0 only if, as $w \to w_0$, z approaches ∞ or a zero of f. But since $\lambda > 1$, $w \to \infty$ as z approaches a zero of f. Hence, if $\Phi^{-1}(w)$ has no singularity at which $z \to \infty$, then $\Phi^{-1}(w)$ is single-valued, an entire function $\psi(w)$, and

$$\frac{dz}{dw} = \frac{d\psi}{dw} = [f(z)]^\lambda = [f(\psi(w))]^\lambda = [g(w)]^\lambda ,$$

where $g(w)$ is entire. Therefore $[g(w)]^\lambda$ is also entire. This is possible with λ irrational only if $g(w)$ has no zeros—hence only if $f(z)$ has at most one zero. Again we have a contradiction. Therefore HUBER's theorem is proved for $\lambda \geqq 1$ and for $\lambda = 1 - (1/n)$ $(n = 2, 3, \ldots)$.

Remark 1. The theorem can be strengthened for functions having no zeros. For then $\log f(z)$ can be defined as an entire function; if $\log f(z)$ is not a polynomial, there exists a path C_λ on which

$$\int_{C_\lambda} |\log f(z)|^{-1} |dz| < \infty .$$

Remark 2. In his paper ([1], p. 52) HUBER raises the question: Suppose $f(z)$ is entire and that there exists $\lambda > 0$ such that

$$\int_1^\infty |f(re^{i\theta})|^{-\lambda} dr = \infty$$

for all θ, $0 \leqq \theta < 2\pi$; does this imply that $f(z)$ is a polynomial? In other words, in the preceding theorem, can C_λ be chosen to be a ray?

This question we answer negatively as follows. A theorem of KELDYS and MERGELYAN ([2], p. 37) implies that, if $g(z)$ is continuous on a closed set E and analytic on the interior of E, then for each $\epsilon > 0$ there exists an entire function $f(z)$ such that $|f(z) - g(z)| < \epsilon$ on E, provided the complement E' of E is locally connected at infinity. In particular, E can be chosen to be the closure of a domain bounded by a simple path γ which approaches infinity in both directions. On such a set E we can easily construct $g(z)$, not identically constant, such that $|g(z)| < \frac{1}{2}$ on E (for example, $g(z)$ can be obtained with the aid of conformal mapping from the function $\frac{1}{4}e^z$ in the left half-plane). Let $g(z_0) = a$, $g(z_1) = b \neq a$. We choose $\epsilon = |b - a|/2$ and $f(z)$ entire, so that $|f - g| < \epsilon$ on E. Then f is not identically constant and $|f| < 1$ on E. Since f is bounded on such a set, f cannot be a polynomial. By proper choice of γ, we can force every ray $C_\theta : \theta = \text{const.}$ to meet E in a set of infinite length; for example, γ can be formed of two spirals which approach

each other as $|z| \to \infty$, and E' as the set between the spirals. Then

$$\int_{C_\theta} |f(z)|^{-\lambda} |dz| \geqq \int_{C_\theta \cap E} |dz| = \infty .$$

For such a function $f(z)$ it is clear that the path C_λ of HUBER's theorem must either lie between the spirals (that is, in E') or be asymptotic to E' in the sense that the length of the part of C_λ outside of E' must be finite; hence effectively there is only one path.

Remark 3. Although the paths $\arg z = \text{const.}$ are not generally allowable as a choice of C_λ, it appears reasonable that the paths $\arg w = \text{const.}$ can serve. For on such a path, not passing through a zero of f, $|f(z)|$ grows steadily in one direction. I conjecture that, for each $f(z)$, a path $\arg f(z) = c$ can serve as C_λ for almost all values of c. For a similar reason, it appears probable that the paths $\mathrm{Re}[f(z)] = c$, $\mathrm{Im}[f(z)] = c$ can also serve as C_λ for almost all c.

REFERENCES

[1] A. HUBER, *On subharmonic functions and differential geometry in the large*, Comment. Math. Helv., 32 (1957), 13–72.

[2] S. N. MERGELYAN, *Uniform approximations to functions of a complex variable* (in Russian), Uspehi Mat. Nauk (N.S.) 7, No. 2 (48) (1952), 31–122. Amer. Math. Soc. Transl. No. 101, Providence, 1954.

(Received April 3, 1959)