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Autor: Heins, Maurice

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On the Principle of Harmonic Measure

by MAURICE HEINS

1. The principle of harmonic measure of R. NEVANLINNA [5] plays the important role of stating limitations which are imposed upon an analytic function which maps a plane region into a plane region and is subject to certain *boundary conditions*. It is assumed that the regions in question have a simple character (e.g. that they are JORDAN regions of finite connectivity). In the theory of the conformal mapping of RIEMANN surfaces we encounter the problem of finding a natural replacement for the NEVANLINNA principle of harmonic measure. The core of the question is, of course, to find a reasonable substitute for the classical harmonic measure of the NEVANLINNA theorem. In the present note we propose to show that the *generalized harmonic measure* which we have studied in [2] serves as a base for a principle of harmonic measure. We recall that a harmonic function u on a RIEMANN surface F satisfying $0 \leqslant u \leqslant 1$ is said to be a generalized harmonic measure (on F) provided that

$$\text{G.H.M.}\min\{u, 1-u\} = 0\,. \qquad (1.1)$$

Here "G.H.M." stands for "greatest harmonic minorant." Among the results established in the present paper is:

Theorem 1: *Let φ denote a conformal map (not necessarily univalent) of a* RIEMANN *surface F into a* RIEMANN *surface G. Suppose that u and v are nonconstant generalized harmonic measures on F and G respectively. Let*

$$\mu(\alpha) = \inf\{v(\varphi(p)) \mid u(p) > \alpha\}\,, \qquad 0 < \alpha < 1\,.$$

Then $v \circ \varphi \geqslant (\lim_{\alpha \to 1} \mu(\alpha))\, u\,.$

It is to be observed that from the hypothesis of the NEVANLINNA principle and Theorem 1 the conclusion of the NEVANLINNA principle follows. (This is not to say most elegantly.) Thus Theorem 1 is as effective as the original principle of harmonic measure in the classical case.

We shall discern in the foreground of the present study an *extremal principle* which holds for pairs consisting of a conformal map and a generalized harmonic measure (Theorem 2).

2. In this section we give an account of properties of generalized harmonic measure which will be of use in the present paper. The exposition will be independent of the summary indications of § 20 of [2].

Generalized harmonic measure in $\{|z| < 1\}$. Let E denote a measurable subset of $\{|z| = 1\}$. Let X_E denote the characteristic function of E. Let ω_E be defined by

$$\omega_E(z) = (2\pi)^{-1} \int_0^{2\pi} X_E(e^{i\alpha}) \Re \left[\frac{e^{i\alpha} + z}{e^{i\alpha} - z} \right] d\alpha , \quad |z| < 1 . \tag{2.1}$$

We have

(i) $E \to \omega_E$ *maps the family of admitted* E *onto the set of generalized harmonic measures (on* $\{|z| < 1\}$*). Further* $\omega_{E_1} = \omega_{E_2}$ *if and only if* $|E_1 \cap E_2| = |E_1| = |E_2|$. This result is easily established and we omit the details.

Another important preliminary result is the following. Let F denote a RIEMANN surface whose conformal universal coverings have hyperbolic domains. Let θ denote a conformal universal covering of F with domain $\{|z| < 1\}$. We have

ii) *A harmonic function* u *on* F *is a generalized harmonic measure on* F *if and only if* $u \circ \theta$ *is a generalized harmonic measure on* $\{|z| < 1\}$ [1]).

To establish this result, we first suppose that $u \circ \theta$ is a generalized harmonic measure on $\{|z| < 1\}$. Let $v = \text{G.H.M.}\min \{u, 1 - u\}$. From $0 \leqslant v \circ \theta \leqslant u \circ \theta,\ (1 - u) \circ \theta$, we conclude that $v \circ \theta = 0$ and thereupon that $v = 0$. To proceed in the opposite direction, we consider $V = \text{G.H.M.}\min \{u \circ \theta, 1 - u \circ \theta\}$ and note that for each conformal automorphism T of $\{|z| < 1\}$ which leaves θ invariant we have $V \circ T \leqslant V$. It follows that $V \circ T = V$ for each such T. Hence $V = v \circ \theta$ where v is harmonic on F. From $0 \leqslant v \circ \theta \leqslant u \circ \theta, (1 - u) \circ \theta$, we have $0 \leqslant v \leqslant u$, $1 - u$, so that $v = 0$. We conclude that $V = 0$.

A third property of generalized harmonic measure is

(iii) *Let* $(u_n)_0^\infty$ *denote a monotone sequence of generalized harmonic measures on* F *with limit* u. *Then* u *is also a generalized harmonic measure on* F.

It suffices to consider only non-decreasing sequences for, if v is a generalized harmonic measure, so is $1 - v$. Let $w = \text{G.H.M.}\min \{u, 1 - u\}$. From $w \leqslant u_n + (u - u_n)$ and $w \leqslant 1 - u_n$ we conclude with the aid of KJELLBERG's Lemma [4, 2] that $w \leqslant u - u_n$ and thereupon that $w = 0$.

Another useful property is:

(iv) *If* $u_1, \ldots, u_n$ *are generalized harmonic measures on* F, *then so are* $\text{G.H.M.}\min \{u_k\}_1^n$ *and* $\text{L.H.M.}\max \{u_k\}_1^n$.

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1) Actually this result is a special case of a more general theorem stated in § 20 of [2]. However use will not be made of this more general result. The present special case is easily established as we shall now see.

Here "L.H.M." stands for "least harmonic majorant." To see this let $u = \text{G.H.M.}\min\{u_k\}_1^n$ and let $v = \text{G.H.M.}\min\{u, 1-u\}$. For each k,

$$v \leqslant u_k, \quad v \leqslant (1-u_k) + (u_k - u).$$

By KJELLBERG's Lemma we have

$$v = v_1^{(k)} + v_2^{(k)} \qquad (k = 1, \ldots, n),$$

where $v_1^{(k)}$ and $v_2^{(k)}$ are non-negative harmonic functions on F satisfying

$$v_1^{(k)} \leqslant 1 - u_k, \quad v_2^{(k)} \leqslant u_k - u.$$

Since $v_1^{(k)}$ also satisfies $v_1^{(k)} \leqslant u_k$, we conclude that $v_1^{(k)} = 0$. From $v + u \leqslant u_k$, $k = 1, 2, \ldots, n$, we have $v + u \leqslant u$ and consequently $v = 0$. That $\text{L.H.M.}\max\{u_k\}_1^n$ is also a generalized harmonic measure on F may be established in a similar manner.

We shall also want to make use of:

(v) *Let u be a generalized harmonic measure and let α denote a real constant satisfying* $0 < \alpha < 1$. *Then*

$$\text{L.H.M.}\left(\frac{u-\alpha}{1-\alpha}\right)^+ = u. \tag{2.2}$$

The proof may be carried out as follows. We may dismiss the trivial cases $u = 0$, $u = 1$. If $F = \{|z| < 1\}$ then (2.2) is an easy consequence of (i). The general case may be reduced to this one. If u is a non-constant generalized harmonic measure on F and θ is a conformal universal covering of F with domain $\{|z| < 1\}$, on setting

$$U = \left(\frac{u-\alpha}{1-\alpha}\right)^+$$

we have

$$\text{L.H.M.}(U \circ \theta) \leqslant (\text{L.H.M.}\, U) \circ \theta.$$

Further $\text{L.H.M.}(U \circ \theta) = u \circ \theta$ since $u \circ \theta$ is a generalized harmonic measure in $\{|z| < 1\}$. The assertion (2.2) follows from

$$u \leqslant \text{L.H.M.}\, U \leqslant u.$$

Theorem 1 is an immediate consequence of (v). In fact it suffices to observe that

$$v \circ \varphi \geqslant \mu(\alpha)\left(\frac{u-\alpha}{1-\alpha}\right)^+.$$

3. Although we shall not require an answer to the question for the principal results of this paper, it is of interest to inquire what surfaces admit non-constant generalized harmonic measures. Obviously a surface admitting non-constant generalized harmonic measures admits non-constant bounded harmonic functions. The converse is also true. To see this, suppose that F is a RIEMANN surface admitting a non-constant bounded harmonic function b. We may assume that $0 < b < 1$. Let θ denote a conformal universal covering of F with domain $\{|z| < 1\}$. For some α, $0 < \alpha < 1$, the measure of the set

$$E = \{\eta \mid |\eta| = 1 , \quad \lim_{r\to 1} b(\theta(r\eta)) < \alpha\}$$

lies strictly between 0 and 2π. The harmonic measure ω_E is automorphic with respect to the group of conformal automorphisms of $\{|z| < 1\}$ which leave θ invariant. Hence by (ii) of § 2 F admits a non-constant generalized harmonic measure.

(This result may also be established directly on F without the aid of uniformization methods. The present argument is shorter.)

4. We now turn to the study of our principal problem. We suppose that we have given a conformal map φ of a RIEMANN surface F admitting non-constant generalized harmonic measures into a hyperbolic RIEMANN surface G and a non-constant generalized harmonic measure u on F. Consider the class of the positive superharmonic functions P on G which satisfy $u \leqslant P \circ \varphi$, the class of the harmonic functions h on G which satisfy $0 < h \leqslant 1$ and $u \leqslant h \circ \varphi$, and finally the class of the generalized harmonic measures v on G which satisfy $u \leqslant v \circ \varphi$. As a first step in our study we show

Theorem 2: *Each of the introduced classes has a least member. That is, there exist a least P, a least h and a least v.*

We first establish the existence of a least P. To that end, for each α satisfying $0 < \alpha < 1$, we introduce on G, the least positive superharmonic function which dominates 1 on $\varphi\{p \mid u(p) > \alpha\}$ and denote it by P_α. We observe that P_α is non-increasing in α. Also from

$$\left(\frac{u-\alpha}{1-\alpha}\right)^+ \leqslant P_\alpha \circ \varphi ,$$

we conclude that $u \leqslant P_\alpha \circ \varphi$. Hence with $\underline{P}$ the lower limit function of $\lim_{\alpha\to 1} P_\alpha$, not only is $\underline{P}$ superharmonic on G but also $u \leqslant \underline{P} \circ \varphi$. We assert that $\underline{P}$ is the least positive superharmonic function P on G satisfying $u \leqslant P \circ \varphi$ and that $\underline{P} = \lim_{\alpha\to 1} P_\alpha$. In fact, if we consider an arbitrary admitted P, we have $P \geqslant \alpha P_\alpha$ so that $P \geqslant \lim_{\alpha\to 1} P_\alpha \geqslant \underline{P}$. Hence $\underline{P}$ is the least admitted P. Setting $P = \underline{P}$, we conclude that $\underline{P} = \lim_{\alpha\to 1} P_\alpha$.

We now establish the existence of a least v. We note that we are presented with two alternatives. First, the only admitted v is 1. This case is trivial and we put it aside. Second, there exists v, $0 < v < 1$. From $u \leqslant v \circ \varphi$, we conclude that for α, $0 < \alpha < 1$, sufficiently near one, $\varphi\{p \mid u(p) > \alpha\}$ lies in the complement of a given compact subset of G. Hence it is easy to conclude that $\underline{P}$ is harmonic on G. This has important consequences, as we shall now see. For suppose that $v_1, \ldots, v_n$ are generalized harmonic measures on G satisfying $u \leqslant v_k \circ \varphi$, $k = 1, \ldots, n$. Then if $V = \text{G.H.M.min}\,\{v_k\}_1^n$, since $\underline{P}$ is harmonic and satisfies $\underline{P} \leqslant \min\,\{v_k\}_1^n$, $\underline{P} \leqslant V$. Further by (iv) of § 2, V is a generalized harmonic measure on G. Hence V is an admitted v.

It now follows by the standard reasoning of the PERRON method that the lower envelope $\underline{v}$ of the family of admitted v also belongs to the family. In fact, it suffices to fix a point $q \in G$ and to select a sequence $(v_k)_1^\infty$ of admitted v satisfying $\lim_{k\to\infty} v_k(q) = \underline{v}(q)$. Then $\lim_{n\to\infty} \text{G.H.M.min}\,\{v_k\}_1^n$ is an admitted v, the value of which at q is $\underline{v}(q)$. Given $q'\,(\neq q) \in G$, there exists a sequence $(v_k')_1^\infty$ of admitted v satisfying $\lim_{k\to\infty} v_k'(q') = \underline{v}(q')$. We now see that $\lim_{n\to\infty} \text{G.H.M.min}\,\{v_k, v_k'\}_1^n$ is an admitted v the values of which at q and q' are $\underline{v}(q)$ and $\underline{v}(q')$. Also

$$\lim_{n\to\infty} \text{G.H.M.min}\,\{v_k, v_k'\}_1^n \leqslant \lim_{n\to\infty} \text{G.H.M.min}\,\{v_k\}_1^n .$$

By the maximum principle equality must hold throughout. We conclude that $\underline{v} = \lim_{n\to\infty} \text{G.H.M.min}\,\{v_k\}_1^n$. It follows that $\underline{v}$ is an admitted v.

We now consider the family of admitted h and let $\underline{h}$ denote its lower envelope. Clearly $\underline{h} \leqslant 1$. We put aside the trivial case where $\underline{h} \equiv 1$. Hence there exists an admitted $h < 1$. Again we see that $\underline{P}$ is harmonic and we infer that $\underline{h} = \underline{P}$.

We shall see later (§ 7) that $\underline{h} = \underline{v}$ and that, if P is harmonic, then $\underline{P} = \underline{v}$. We also remark that Theorem 2 holds for $u = 1 : \underline{h} = \underline{v} = 1$ and $\underline{P}$ is the least positive superharmonic function on G which dominates 1 on $\varphi(F)$.

Suppose that u_1 and u_2 are given non-vanishing generalized harmonic measures on F and that $u = \text{L.H.M.max}\,\{u_1, u_2\}$. Let $\underline{P}_k$ and $\underline{v}_k$ denote the least P and the least v associated with $u_k (k = 1, 2)$ in the sense of Theorem 2. Then $\underline{P}$ *is the least superharmonic function on* G *which dominates* $\max\,\{\underline{P}_1, \underline{P}_2\}$ *and* $\underline{v} = \text{L.H.M.max}\,\{\underline{v}_1, \underline{v}_2\}$. Here $\underline{P}$ and $\underline{v}$ pertain to u.

To establish the first assertion, suppose that Q is a superharmonic function on G satisfying $Q \geqslant \max\{\underline{P}_1, \underline{P}_2\}$. Then from

$$u_k \leqslant \min\{\underline{P}, Q\} \circ \varphi, \qquad k = 1, 2,$$

we conclude that

$$u \leqslant \min\{\underline{P}, Q\} \circ \varphi .$$

Hence $\underline{P} \leqslant \min\{\underline{P}, Q\} \leqslant Q$.

To establish the second assertion we note that since $\underline{v} \geqslant \underline{v}_k (k = 1, 2)$, we have $\underline{v} \geqslant$ L.H.M.max $\{\underline{v}_1, \underline{v}_2\}$. Further since $u_k \leqslant$ L.H.M.max $\{\underline{v}_1, \underline{v}_2\} \circ \varphi$, $k = 1, 2$, we have $u \leqslant$ L.H.M.max $\{\underline{v}_1, \underline{v}_2\} \circ \varphi$, whence $\underline{v} \leqslant$ L.H.M.max $\{\underline{v}_1, \underline{v}_2\}$. The equality follows.

5. We now examine the influence of *boundary conditions* on the extremal $\underline{P}$. The first situation which we consider is that where $F = \{|z| < 1\}$. Let E denote a measurable subset of $\{|z| = 1\}$. The pair (φ, E) will be said to satisfy the *boundary condition I*, provided that for each $\eta \in E$, $\varphi(r\eta)$ tends to the ideal boundary of G as r tends to 1 or, in other words, for each $\eta \in E$ and each compact subset $K \subset G$, there exists r_0, $0 < r_0 < 1$, such that $\varphi(r\eta) \in G - K$ for $r_0 < r < 1$. The following theorem holds.

Theorem 3: *If (φ, E) satisfies I, then with $\underline{P}$ and $\underline{v}$ denoting respectively the minimal P and v for (φ, ω_E), $\underline{P} = \underline{v}$.*

We restrict our attention to the case: $0 < |E| < 2\pi$. The case where $|E| = 2\pi$ follows by an obvious limit argument. The method of proof will involve the factorization of φ into a conformal universal covering and an analytic function of modulus less than one. Before we turn to the details, it will be convenient to have available the following lemma.

Lemma 1: *Let f denote an analytic function in $\{|z| < 1\}$ of modulus less than 1. Let E_1 denote a measurable subset of $\{|z| = 1\}$ which has the property that for each $\eta \in E_1$, $\lim_{r \to 1} f(r\eta)$ exists and is of modulus 1. Let E_2 denote a G_δ subset of $\{|z| = 1\}$ which contains $f^*(E_1)$, the image of E_1 with respect to the* FATOU *radial limit function f^** of *f, and has measure equal to the outer measure of $f^*(E_1)$. Then*

$$\omega_{E_1} \leqslant \omega_{E_2} \circ f .$$

The lemma is easily established on observing that for each open subset O of $\{|z| = 1\}$ which contains $f^*(E_1)$, $\omega_O \circ f - \omega_{E_1} \geqslant 0$ as is readily seen on examining the radial limits.

It is also in order to observe that the FATOU theorem holds for φ (and in fact for an arbitrary LINDELÖFIAN conformal map with domain $\{|z| < 1\}$, cf. [3]) in the sense that *for almost all* η of $\{|z| = 1\}$, *$\varphi(z)$ tends to a point* *of G as z tends sectorially to η or else $\varphi(z)$ tends to the ideal boundary of G as z tends sectorially to η.*

We now turn to the proof of Theorem 3. By an argument of the EGOROFF type we conclude that for each c, $0 < c < 1$, there exists a closed subset $E(c)$ of E satisfying: (i) $|E(c)| > c|E|$, (ii) $\omega_E(r\eta) \to 1$ uniformly as $r \to 1$, $\eta \in E(c)$, (iii) $\varphi(r\eta)$ tends to the ideal boundary of G uniformly as $r \to 1$, $\eta \in E(c)$, i. e. for each compact $K \subset G$, there exists r_0, $0 < r_0 < 1$, such that $\varphi(r\eta) \in G - K$ for $r_0 < r < 1$, $\eta \in E(c)$. Let $P_{E(c),r}$ denote the least positive superharmonic function on G which dominates one on $\varphi\{\varrho\eta \mid r \leqslant \varrho < 1,\ \eta \in E(c)\}$. Let $P_{E(c)} = \lim_{r\to 1} P_{E(c),r}$. Now $P_{E(c)}$ is harmonic on G and from

$$\omega_{E(c)} \leqslant P_{E(c),r} \circ \varphi ,$$

it follows that

$$\omega_{E(c)} \leqslant P_{E(c)} \circ \varphi .$$

From $\omega_E \leqslant \underline{P} \circ \varphi$, it follows by (ii) that $\underline{P} \geqslant P_{E(c)}$. We take an increasing sequence of c, say $(c_k)_1^\infty$ with limit 1, such that $P_{E(c_k)}$ converges pointwise on G. Let h denote $\lim P_{E(c_k)}$. Then $\omega_E \leqslant h \circ \varphi$. Also $h \leqslant \underline{P}$ since $\underline{P} \geqslant P_{E(c)}$, and $h \geqslant \underline{P}$ by Theorem 2. Hence boundary condition I implies that *$\underline{P}$ is harmonic.* If $\underline{P} = 1$, then Theorem 3 is trivial. We suppose therefore that $\underline{P} \neq 1$.

To continue we introduce a conformal universal covering ψ of G with domain $\{|z| < 1\}$ and note that φ admits a factorization of the form $\psi \circ b$ where b is an analytic function in $\{|z| < 1\}$ of modulus less than one. Let b^* denote the FATOU radial limit function of b. Let W denote an analytic function in $\{|z| < 1\}$ satisfying $\Re W = \underline{P} \circ \psi$ and let $W_1 = W \circ b$, so that $\Re W_1 = \underline{P} \circ \varphi$. There exists a subset X of E which is an F_σ and satisfies: (i) $|X| = |E|$, (ii) W_1 possesses a finite sectorial limit with real part 1 at each point of X, (iii) $Y = b^*(X)$ is an F_σ. It is to be observed that b possesses a sectorial limit of modulus one at each point of X and that consequently by virtue of a classical function-theoretic lemma [4; p. 70] W possesses a finite sectorial limit with real part 1 at each point of Y.

Now let $Z = \bigcup_{\tau \in \mathfrak{G}} \tau(Y)$, where $\mathfrak{G}$ is the group of conformal automorphisms of $\{|z| \leqslant 1\}$ whose restrictions to $\{|z| < 1\}$ leave ψ invariant. We have

$$\lim_{r\to 1} \underline{P} \circ \psi(r\eta) = 1 , \qquad \eta \in Z .$$

Further since ω_Z is invariant with respect to the automorphisms belonging to $\mathfrak{G}$, $\omega_Z = v \circ \psi$ where v is a generalized harmonic measure on G. From

$\underline{P} \circ \psi \geqslant \omega_Z = v \circ \psi$, it follows that $\underline{P} \geqslant v$. On the other hand, using Lemma 1, we infer that

$$v \circ \varphi = \omega_Z \circ b \geqslant \omega_Y \circ b \geqslant \omega_E .$$

Hence $v \geqslant \underline{v}$. The equality of $\underline{P}$ and $\underline{v}$ follows.

Applications of Theorem 3. Suppose that E is a measurable subset of an open arc α of the unit circumference and that $|E| = |\alpha|$. Suppose further that $\underline{v} < 1$. What can be said about φ under these circumstances? In case $G = \{|z| < 1\}$, it is immediate that φ possesses a limit of modulus one at each point of α. Thus φ is the restriction to $\{|z| < 1\}$ of a function analytic at each point of α. We may also draw conclusions in the general case. Here we have

$$\underline{v} \circ \varphi = (\underline{v} \circ \psi) \circ b \geqslant \omega_\alpha$$

and it follows that b possesses a limit of modulus one at each point of α. It follows that $\mathfrak{G}$ is properly discontinuous at each point of $b^*(\alpha)$. Hence G is continuable and in fact has a free boundary arc Γ with the property that $\varphi(z)$ tends continuously to a point of Γ as z tends to a point of α. Of course, reference is made to an embedding of G.

The second application which we have in mind is the following. Let E now denote the set of η, $|\eta| = 1$, for which $\varphi(r\eta)$ tends to the ideal boundary of G as $r \to 1$. Let g denote the GREEN's function for G with pole at $q \in G$. Then $\lim_{r \to 1} g(\varphi(r\eta)) = 0$ p. p. on E.

Suppose that this were not the case. Then there would exist a measurable subset E_1 of E, $|E_1| > 0$, such that $\omega_{E_1} \leqslant \beta g \circ \varphi$ where β is a positive constant. By Theorem 3, $\underline{v}$ associated with (φ, E_1) satisfies $\underline{v} \leqslant \beta g$. This implies $\underline{v} = 0$ which is impossible.

6. It is now possible to correlate the boundary behavior of a conformal map φ of $\{|z| < 1\}$ into a hyperbolic RIEMANN surface G with the behavior of the $\underline{P}$ associated with φ and a generalized harmonic measure in $\{|z| < 1\}$. Specifically suppose that E is a measurable subset of $\{|z| = 1\}$. Let X denote a measurable subset of E of positive measure and let $\underline{P}^X$ denote the $\underline{P}$ associated with (φ, ω_X). Then we have

Theorem 4: *A necessary and sufficient condition that $\varphi(r\eta)$ tends to the ideal boundary of G as $r \to 1$ for almost all $\eta \in E$ is that for each X, $\underline{P}^X$ be harmonic.*

The necessity is immediate. Suppose that the stated condition were not sufficient. Then there would exist an X, say X_0 with $|X_0| > 0$ enjoying the following properties: (i) X_0 is closed, (ii) $\varphi^*(\eta) = \lim_{r \to 1} \varphi(r\eta)$ exists $(\in G)$ for each $\eta \in X_0$ and $\varphi^*(X_0)$ is compact. It follows that $\underline{P}^{X_0}$ would not be harmonic. This is contrary to the stated condition.

7. We now turn to conformal maps of RIEMANN surfaces. Here we prove

Theorem 5: *Let φ denote a conformal map of a RIEMANN surface F into a hyperbolic RIEMANN surface G. Suppose that u is a non-trivial generalized harmonic measure on F which has the property that $\varphi(p)$ tends to the ideal boundary of G as $u(p) \to 1$ in the sense that for each compact $K \subset G$ there exists α, $0 < \alpha < 1$, such that*

$$\varphi\{u > \alpha\} \subset G - K .$$

Then $\underline{P} = \underline{v}$.

We show that we may conclude this theorem from Theorem 3. We put aside the trivial case: $\underline{P} = 1$. It is to be observed that the imposed boundary condition assures the *harmonicity* of $\underline{P}$ (cf. § 4. $\lim_{\alpha \to 1} P_\alpha$ is harmonic here).

Let θ denote as above a conformal universal covering of F with domain $\{|z| < 1\}$. It is to be observed that $\underline{P}$ and $\underline{v}$ are also respectively the least P and v associated with $(\varphi \circ \theta,\, u \circ \theta)$. We have: $u \circ \theta = \omega_E$ where E is a measurable subset of $\{|z| = 1\}$ for which $\lim_{r \to 1} \omega_E(r\eta) = 1$, $\eta \in E$. Hence $\varphi \circ \theta(r\eta)$ tends to the ideal boundary of G as $r \to 1$, $\eta \in E$. Theorem 3 is applicable and we conclude that $\underline{P} = \underline{v}$.

It is an immediate corollary of Theorem 5 that in the unrestricted situation of Theorem 2, $\underline{h} = \underline{v}$ and that the harmonicity of $\underline{P}$ implies that $\underline{P} = \underline{v}$.

8. A question pertaining to Theorem 2 which merits attention is the following. Given a generalized harmonic measure $u(> 0)$ on F, does the class of generalized harmonic measures $h(\leqslant u)$ on F which satisfy:

(*) For each generalized harmonic measure w on F satisfying $0 < w \leqslant h$, the least P associated with w in the sense of Theorem 2 is harmonic,

have a largest member? We shall see that *this is indeed the case* and that *with $\bar{h}$ denoting this largest h, if the generalized harmonic measure $u - \bar{h} > 0$, then the least P associated with $u - \bar{h}$ in the sense of Theorem 2 is the limit of a non-decreasing sequence of equilibrium GREEN's potentials*[2]) *on G associated with compact sets of positive capacity.*

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[2]) Specifically, we define this notion as follows (we put aside unicity questions which are not essential for our purposes (cf. [1]). Let K denote a compact subset of G. Let Q_O denote the least positive superharmonic function on G which dominates 1 on the open set $O(\neq \emptyset) \subset G$. By Π_K, the *equilibrium* GREEN's *potential* associated with K, we shall understand the lower limit function of $\inf_{K \subset O} Q_O$.

Further, if $u = 1$, *then* $\bar{h} = 1$ *if and only if* φ *is of type-Bl*[3]).

It will be convenient to establish the above assertions for the case where $F = \{|z| < 1\}$ and thereupon to reduce the general case to this one.

We start then with $u = \omega_E$ and let E_1 denote the set of points $\eta \in E$ such that $\varphi(z)$ tends to the ideal boundary of G as z tends radially to η. We claim that $\bar{h} = \omega_{E_1}$. Clearly $h = \omega_{E_1}$ satisfies (*). For if w is a generalized harmonic measure satisfying $0 < w \leqslant \omega_{E_1}$, then by Theorem 3 the least P associated with w is harmonic.

On the other hand, if w is a generalized harmonic measure satisfying $u \geqslant w > \omega_{E_1}$, then $w = \omega_{E_2}$ where $E_1 \subset E_2 \subset E$ and $|E_2 - E_1| > 0$. There exists a closed subset C of $E_2 - E_1$ where $|C| > 0$ and the restriction of φ to $\{r\eta \mid 0 \leqslant r \leqslant 1, \eta \in C\}$ is continuous, $\varphi(\eta)$ being taken as the radial limit of φ at η. For each $O \supset \varphi(C)$, $\omega_C \leqslant Q_O \circ \varphi$. It follows that $\omega_C \leqslant \Pi_{\varphi(C)} \circ \varphi$. Clearly the least P associated with ω_C is not harmonic.

Suppose now that $h (\leqslant u)$ is a generalized harmonic measure satisfying (*). If w is a positive generalized harmonic measure satisfying $w \leqslant \text{L.H.M.max} \{h, \omega_{E_1}\}$, then $w = w' + w''$, where w' and w'' are generalized harmonic measures satisfying $\text{G.H.M.min} \{w', w''\} = 0$ and $w' \leqslant h$, $w'' \leqslant \omega_{E_1}$. We may put aside the case where either $w' = 0$ or $w'' = 0$. Since the least P associated with w' and w'' are harmonic and $w = \text{L.H.M.max} \{w', w''\}$, it follows from the concluding remarks of § 4 that the least P associated with w is harmonic. Hence by the preceding paragraph $\omega_{E_1} = \text{L.H.M.max} \{h, \omega_{E_1}\}$. It follows that $\omega_{E_1} = \bar{h}$.

We now show that, if $h^* = u - \bar{h} > 0$, then the least P associated with h^* is the limit of a non-decreasing sequence of equilibrium GREEN's potentials. The proof is quite simple. It suffices to note that there exists a non-decreasing sequence $(C_k)_1^\infty$ of closed subsets of $E - E_1$ satisfying: (i) $\lim |C_k| = |E - E_1|$, (ii) for each k, the restriction of φ to $\{r\eta \mid 0 \leqslant r \leqslant 1, \eta \in C_k\}$ is continuous, the conventions indicated above prevailing, (iii) for each k, $h^*(r\eta) \to 1$ uniformly as $r \to 1$, $\eta \in C_k$.

As above, we have for each k,

$$\omega_{C_k} \leqslant \Pi_{\varphi(C_k)} \circ \varphi . \tag{8.1}$$

Further $\Pi_{\varphi(C_k)} \leqslant \underline{P}$, where $\underline{P}$ is the least P associated with h^*. In fact, for each α, $0 < \alpha < 1$, each point of $\varphi(C_k)$ either belongs to $\varphi\{h^* > \alpha\}$ or is

[3]) cf. [2]. We recall that φ is of type-Bl, provided that for each $q \in G$, $\mathfrak{G}_G(\varphi(p), q)$ does not dominate a positive bounded harmonic function on F. Here $\mathfrak{G}_G$ ist the GREEN's function for G. We shall make use later of the following theorem: Let φ_1 denote a conformal map of a hyperbolic RIEMANN surface F_1 into a hyperbolic RIEMANN surface F_2 and let φ_2 denote a conformal map of F_2 into a hyperbolic RIEMANN surface F_3. Then $\varphi_2 \circ \varphi_1$ is a map of type-Bl if and only if φ_1 and φ_2 are maps of type-Bl.

an accessible boundary point of a component of $\varphi\{h^* > \alpha\}$. It follows from the Carleman-Milloux-Beurling inequality that $\alpha^{-1}\underline{P}(q) \geqslant 1$, $q \in \varphi(C_k)$. We conclude that $\alpha^{-1}\underline{P} \geqslant \Pi_{\varphi(C_k)}$ and consequently that $\underline{P} \geqslant \Pi_{\varphi(C_k)}$.

From $\lim \Pi_{\varphi(C_k)} \leqslant \underline{P}$ and $h^* \leqslant (\lim \Pi_{\varphi(C_k)}) \circ \varphi$, it follows that $\lim \Pi_{\varphi(C_k)} = \underline{P}$.

Suppose that $u = 1$. If φ is of type-Bl, then from the fact that for almost all η of $\{|z| = 1\}$, $\mathfrak{G}_G(\varphi(z), q_0)$ tends to zero as z tends to η sectorially, q_0 being a point of G, we conclude that $\bar{h} = 1$. If φ is not of type-Bl, then on considering the factorization of § 5, $\varphi = \psi \circ b$, we see that b is not of type-Bl relative to the unit disk by virtue of the cited theorem concerning maps of type-Bl and we conclude that $\bar{h} < 1$.

There now remains the problem of reducing the general case to the one just treated. With θ having the same meaning as above, let $\bar{H}$ and H^* denote the counterparts of $\bar{h}$ and h^* respectively relative to the generalized harmonic measure $u \circ \theta$ and the conformal map $\varphi \circ \theta$ of $\{|z| < 1\}$ into G. It is to be observed that $\bar{H}$ and H^* are automorphic with respect to the group of conformal automorphisms of $\{|z| < 1\}$ leaving θ invariant. Hence $\bar{H} = u_1 \circ \theta$ and $H^* = u_2 \circ \theta$ where u_1 and u_2 are generalized harmonic measures on F. We assert that u_1 is the desired $\bar{h}$ and that $u_2 = h^*$.

To see this, we note that, condition (*) is fulfilled by u_1 replacing h, for if w is a generalized harmonic measure on F satisfying $0 < w \leqslant u_1$, the least P for (φ, w) is harmonic since the least P for $(\varphi \circ \theta, w \circ \theta)$ and for (φ, w) are the same.

Suppose that h satisfies (*). We wish to show that $h \leqslant u_1$. We consider $h \circ \theta$ relative to $\varphi \circ \theta$. Suppose that W is a generalized harmonic measure in $\{|z| < 1\}$ satisfying $0 < W \leqslant h \circ \theta$. Let $\underline{P}$ denote the least P associated with $(\varphi \circ \theta, W)$. It suffices to show that $\underline{P}$ is harmonic, for then $h \circ \theta \leqslant u_1 \circ \theta$ and consequently $h \leqslant u_1$. The case where F is simply-connected is immediate. In the remaining case let $(\tau_n)_1^\infty$ denote a univalent enumeration of the conformal automorphisms of $\{|z| < 1\}$ which leave θ invariant. Let $\bar{W} = \lim_{n \to \infty} \text{L.H.M.} \max \{W \circ \tau_k\}_1^n$. From $W \circ \tau_k \leqslant \underline{P} \circ \varphi \circ \theta$, $k = 1, 2, \ldots$, it follows that $\bar{W} \leqslant \underline{P} \circ \varphi \circ \theta$. On the other hand, $\bar{W} \circ \tau_k = \bar{W}$, $k = 1, 2, \ldots$ This follows from $W \circ \tau_j \circ \tau_k \leqslant \bar{W} \circ \tau_k$, $j = 1, 2, \ldots$ which implies $\bar{W} \leqslant \bar{W} \circ \tau_k$. Further $\bar{W} \leqslant h \circ \theta$. Hence $\bar{W} = \bar{w} \circ \theta$, where $\bar{w}$ is a generalized harmonic measure on F satisfying $\bar{w} \leqslant h$. Now $\underline{P}$ is the least P associated with $(\varphi \circ \theta, \bar{W})$ and hence is the least P associated with $(\varphi, \bar{w})$. Consequently $\underline{P}$ is *harmonic* since h satisfies (*).

It is immediate now that the second assertion holds.

Finally, suppose that $u = 1$. If $\bar{h} = 1$, then $\bar{H} = 1$ and $\varphi \circ \theta$ is a map of type-Bl of $\{|z| < 1\}$ into G. Hence by the cited composition theorem, φ is a map of type-Bl of F into G. Conversely, if φ is a map of type-Bl of F into G, $\varphi \circ \theta$ is a map of type-Bl of $\{|z| < 1\}$ into G and $\overline{H} = 1$. Hence $\overline{h} = 1$.

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