

Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 33 (1959)

Artikel: An arithmetical property of quadratic forms.
Autor: Ledermann, Walter
DOI: <https://doi.org/10.5169/seals-26005>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 20.08.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

An arithmetical property of quadratic forms

By WALTER LEDERMANN, Manchester

In their paper [1] F. HIRZEBRUCH and H. HOPF have encountered an interesting arithmetical property possessed by certain symmetric bilinear forms

$$f(x, y) = \sum_{i, j=1}^n a_{ij} x_i y_j \quad (1)$$

that arise in algebraic topology. In the forms which they consider, the coefficients a_{ij} and the variables are integers and $\det a_{ij} = \pm 1$; and it is known that there exists an integral vector w such that

$$f(x, x) \equiv f(x, w) \pmod{2} \quad (2)$$

for all x . If τ is the signature of f , then it is a corollary of their topological investigations that

$$\tau \equiv f(w, w) \pmod{4}. \quad (3)$$

It is desirable to give a purely algebraic proof of (3), and I am greatly indebted to Professor HOPF for having drawn my attention to this question, which will be discussed in this note.

In fact, it will be shown that (3) is a special case of a result concerning forms (1) in which the coefficients and variables are rational numbers with odd denominators. This subset, Ω , of all rationals forms a ring, whose elements may be grouped into residue classes modulo any power of 2 by stipulating that

$$\frac{c_1}{d_1} \equiv \frac{c_2}{d_2} \pmod{2^\alpha}$$

whenever $c_1 d_2 - d_1 c_2 \equiv 0 \pmod{2^\alpha}$; since only odd denominators are allowed, this definition evidently does not depend on the representation of the fractions involved. In particular, a fraction is termed even or odd according as its numerator is even or odd; and we note that, if r is odd, $r^2 \equiv 1 \pmod{4}$.

The set, V , of n -tuples or "row-vectors" $x = (x_1, x_2, \dots, x_n)$ ($x_i \in \Omega$) is a Ω -module. A change of basis of V amounts to replacing x by the n -tuple $\tilde{x} = xP$, where P is a fixed n -rowed matrix in Ω with odd determinant.

Let f be a symmetric bilinear form which relative to the original basis is expressed as xAy' , where $A = (a_{ij})$. After the change of basis, f becomes $\tilde{x}B\tilde{y}'$, where

$$B = PAP' \quad (4)$$

We write $\Delta = \Delta_f = \det A$, and throughout this paper we restrict ourselves to forms with odd determinants, a property which is clearly preserved by the transformation (4).

For a given form f we can in many ways determine a constant vector w such that (2) holds for all x in Ω . Indeed, w may be taken as the solution of the vector equation

$$wA = (a_{11}, a_{22}, \dots, a_{nn}),$$

this solution being in Ω , because $\det A$ is odd. For since

$$f(x, x) \equiv \sum_i a_{ii} x_i^2 \equiv \sum_i a_{ii} x_i \pmod{2},$$

we have that

$$f(x, w) = wAx' = \sum_i a_{ii} x_i,$$

and (2) is satisfied. If \tilde{w} is another vector satisfying (2), then $f(x, \tilde{w} - w) \equiv 0 \pmod{2}$ for all x , so that $(\tilde{w} - w)A \equiv 0 \pmod{2}$. It follows that

$$\tilde{w} = w + 2z, \tag{5}$$

where z is a suitable vector in Ω . Conversely, any vector of the form (5) satisfies (2). We have that

$$f(\tilde{w}, \tilde{w}) = f(w, w) + 4f(w, z) + 4f(z, z).$$

Thus

$$f(\tilde{w}, \tilde{w}) \equiv f(w, w) \pmod{4},$$

that is, $f(w, w)$ (though not w itself) is an *invariant modulo 4* of f .

Our aim is to prove the following

Theorem. *Let f be a quadratic form in n variables in Ω with odd determinant Δ and with signature τ . Then¹⁾*

$$f(w, w) - \tau \equiv \Delta - \operatorname{sgn} \Delta \pmod{4}, \tag{6}$$

where w is a solution of (2).

We remark that, whilst Δ is not an invariant of f , both $\operatorname{sgn} \Delta$ and Δ are invariants mod 4. For in a transformation of the type (4), Δ is multiplied by $(\det P)^2$, which is congruent with 1 mod 4, since $\det P$ is odd.

In particular, when f is unimodular, whether integral or not, we have that $\Delta = \operatorname{sgn} \Delta$, so that (6) reduces to (3).

The theorem is proved by an induction with respect to n which is based on the following simple matrix formula. Consider a partitioning of A , say

$$A = \begin{pmatrix} K & L' \\ L & M \end{pmatrix},$$

¹⁾ As usual, we define $\operatorname{sgn} \Delta$ to be +1 or -1 according as $\Delta > 0$ or $\Delta < 0$.

where K is non-singular and of dimension less than n . Put

$$P = \begin{pmatrix} I & O \\ -LK^{-1} & I \end{pmatrix}$$

where the identity matrices on the diagonal are of dimensions (in general distinct) equal to those of K and M respectively. Then

$$PAP' = \begin{pmatrix} K & O \\ O & M - LK^{-1}L' \end{pmatrix}. \quad (7)$$

When $\det K$ is odd, this transformation is admissible, since P then lies in \mathfrak{Q} . Now if not all diagonal elements of A are even, we may, without loss of generality, assume that a_{11} is odd and then put $K = (a_{11})$. If, on the other hand, all diagonal elements are even, then each row of A must contain at least one odd element, or else $\det A$ could not be odd. We may then assume that a_{12} is odd and that K is the leading 2-rowed submatrix; for in that case $\det K = a_{11}a_{22} - a_{12}^2 \equiv -1 \pmod{4}$, which is certainly odd. Thus, when $n > 2$, we can always apply a transformation of the type (7), in which the dimension of K is either 1 or 2.

When V is referred to the new basis, f splits and we write

$$f(x, x) = g(x^{(1)}, x^{(1)}) + h(x^{(2)}, x^{(2)}),$$

where $x = (x^{(1)}, x^{(2)})$ and the dimensions of the vectors $x^{(1)}$ and $x^{(2)}$ are those of K and M respectively²). Evidently

$$\Delta_f = \Delta_g \Delta_h, \quad \tau_f = \tau_g + \tau_h,$$

where suffixes are used to distinguish quantities corresponding to different forms. Also, if $w^{(1)}$ and $w^{(2)}$ are such that

$$g(x^{(1)}, x^{(1)}) \equiv g(x^{(1)}, w^{(1)}) \pmod{2}$$

for all $x^{(1)}$ and

$$h(x^{(2)}, x^{(2)}) \equiv h(x^{(2)}, w^{(2)}) \pmod{2}$$

for all $x^{(2)}$, then $w = (w^{(1)}, w^{(2)})$ satisfies (2).

Leaving aside for the present the cases in which $n = 1$ or $n = 2$, we may assume, by induction, that the theorem holds for the forms g and h . Then, since

$$f(w, w) - \tau_f = (g(w^{(1)}, w^{(1)}) - \tau_g) + (h(w^{(2)}, w^{(2)}) - \tau_h),$$

we have that

$$f(w, w) - \tau_f \equiv \Delta_g - \text{sgn} \Delta_g + \Delta_h - \text{sgn} \Delta_h, \quad (8)$$

²) A somewhat similar method of reduction, but in a different context, has been employed by MINKOWSKI ([2], 16–20).

with the convention that henceforth all congruences are mod 4. Now, if r and s are odd, $(1 - r)(1 - s)$ is divisible by 4, so that

$$r + s \equiv 1 + rs.$$

Hence, in particular,

$$\Delta_g + \Delta_h \equiv 1 + \Delta_g \Delta_h = 1 + \Delta_f$$

and

$$\text{sgn} \Delta_g + \text{sgn} \Delta_h \equiv 1 + \text{sgn}(\Delta_g \Delta_h) = 1 + \text{sgn} \Delta_f.$$

Substituting in (8) we immediately obtain (6).

It only remains to verify the theorem for the two lowest dimensions. When $n = 1$, $f = a_{11}x_1^2$, where a_{11} is odd. We may then put $w_1 = 1$ to satisfy (2). Thus $f(w, w) = a_{11} = \Delta$. Since $\tau = \text{sgn} a_{11} = \text{sgn} \Delta$, the relation (6) is certainly true. When $n = 2$, that is when $f = a_{11}x_1^2 + a_{22}x_2^2 + 2a_{12}x_1x_2$, we have to distinguish two cases.

(i) Assume that a_{11} and a_{22} are not both even, so that we may assume that a_{11} is odd. The transformation (7) can then be applied with $K = (a_{11})$, and f splits into two unary forms. The induction argument is therefore available as before.

(ii) If a_{11} and a_{22} are both even, a_{12} is necessarily odd and $\Delta = a_{11}a_{22} - a_{12}^2 \equiv -1$. Evidently, $f(x, x)$ is even for all x , so that the vector $w = 0$ satisfies (2). We have therefore to show that

$$-\tau \equiv -1 - \text{sgn} \Delta. \quad (9)$$

When $\text{sgn} \Delta = -1$, the form is indefinite, that is $\tau = 0$, and (9) is true. On the other hand, when $\text{sgn} \Delta = 1$, then $\tau = 2$ or $\tau = -2$ according as $a_{11} > 0$ or $a_{11} < 0$. But $2 \equiv -2$, and again (9) holds in each case.

REFERENCE

- [1] F. HIRZEBRUCH and H. HOPF, *Felder von Flächenelementen in 4-dimensionalen Mannigfaltigkeiten*. Math. Annalen 136 (1958).
- [2] H. MINKOWSKI, *Gesammelte Abhandlungen I* (Leipzig 1911).

(Received April 14, 1958)