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The spherical derivative of meromorphic functions in the neighbourhood of an isolated singularity

by Olli Lehto

1. Introduction

1. This paper deals primarily with the behaviour of meromorphic functions $f(z)$ in the neighbourhood of an isolated essential singularity. In a joint paper [4] K. I. Virtanen and I introduced the spherical derivative

$$\varrho(f(z)) = \frac{|f'(z)|}{1 + |f(z)|^2}$$

as a natural measure for the growth of $f(z)$ near the singularity. We proved that if $f(z)$ is meromorphic in the neighbourhood of the singularity $z = a$, an absolute constant $k > 0$ exists such that

$$\overline{\lim_{z \to a}} \, |z - a| \, \varrho(f(z)) \geqq k \, . \tag{1.1}$$

We found later that $k \geqq \frac{1}{2}$; a proof to this effect was reproduced in the survey lecture [2]. This numerical estimate of k became interesting as it appeared that $k = \frac{1}{2}$ is the best possible value in (1.1): there do exist meromorphic functions $f(z)$ for which

$$\overline{\lim_{z \to a}} \, |z - a| \, \varrho(f(z)) = \tfrac{1}{2} \, . \tag{1.2}$$

This result will be established in Theorem 1 below.

2. For meromorphic functions $f(z)$ omitting at least one value in a neighbourhood of the singularity $z = a$, we have always

$$\overline{\lim_{z \to a}} \, |z - a| \, \varrho(f(z)) = \infty \, .$$

In particular, this is true for regular functions $(f(z) \neq \infty)$. One might think that in this case, the slowest possible growth for $\varrho(f(z))$ would occur for functions $f(z)$ growing slowly in the classical sense, i. e. for $f(z)$ of order zero and with simple zeros very thinly distributed. This, however, is not the case: it will be shown (Theorem 2) that, on the contrary, for such functions $\varrho(f(z))$

is always of fairly rapid growth at the zeros of $f(z)$. This admits interesting conclusions regarding the distribution of values of such functions (Theorem 4).

3. In [4], particular attention was devoted to the class of meromorphic functions for which

$$|z-a|\,\varrho(f(z)) = O(1)\,.$$

It appeared that these functions are identical with functions *weakly normal* in a vicinity of $z = a$. This class also coincides with the functions *exceptional* in the sense of JULIA. This latter result was discovered already by MARTY [5], a fact which was unfortunately overlooked in [4].

Hence, JULIA's well-known modification of PICARD's Theorem can be stated as follows: If

$$\overline{\lim_{z\to a}}\,|z-a|\,\varrho(f(z)) = \infty\,, \tag{1.3}$$

there exists a sequence of circular discs $C_\nu : |z - z_\nu| < \varepsilon\,|z_\nu - a|$, $\lim z_\nu = a$, $\varepsilon > 0$ arbitrarily small, such that $f(z)$ takes all values, except perhaps two, in the union of every infinite subsequence of the discs C_ν.

In this paper, we shall show that quite a simple reasoning yields the following best possible improvement of JULIA's Theorem: Let $h(r)$ be an arbitrary function tending to zero with the positive variable r. If

$$\lim_{\nu\to\infty} h(|z_\nu - a|)\,\varrho(f(z_\nu)) = \infty\,,$$

then and only then PICARD's Theorem holds in the union of every infinite subsequence of the discs $C_\nu : |z - z_\nu| < \varepsilon h(|z_\nu - a|)$ with arbitrarily small positive ε[1]).

Hence, the more rapid is the maximal growth of the spherical derivative the smaller is the point set in which PICARD's Theorem already holds, and vice versa. The above-mentioned regular functions with a very small characteristic function but with a large spherical derivative at certain points show that it is in this connection essential to characterize the growth of $f(z)$ by means of the spherical derivative $\varrho(f(z))$ itself and not by integrated mean values of $\varrho(f(z))$ (characteristic function, spherical area of certain maps, etc.) as it is customary in the classical theory.

We conclude the paper by certain remarks on the existence of a JULIA radius for functions meromorphic in the unit disc, which sharpen and complete a previous result of CONSTANTINESCU [1].

[1]) This result sharpens a recent generalization of JULIA's Theorem by CONSTANTINESCU [1].

2. Growth of the spherical derivative

4. As regards the growth of the spherical derivative, we shall now prove

Theorem 1. *Let* $f(z)$ *be meromorphic in a neighbourhood of the essential singularity* $z = a$. *Then*

$$\overline{\lim_{z \to a}} \, | z - a | \, \varrho(f(z)) \geqq \tfrac{1}{2} \, . \tag{2.1}$$

Equality holds for the WEIERSTRASS*ian products*

$$f(z) = \Pi \frac{z - a - a_\nu}{z - a + a_\nu} \, ,$$

where the numbers a_ν *satisfy the condition* $| a_{\nu+1} | = o(|a_\nu|)$.

Proof: For simplicity, we assume that $a = 0$, and recall first how the inequality (2.1) can be established.

Put $F(z) = f(z)\overline{f}(\overline{z}e^{i\vartheta})$ and choose ϑ so that $F(z)$ is singular at $z = 0$; all values of ϑ with one possible exception do for this purpose. By WEIERSTRASS' Theorem, there corresponds to every $\varepsilon > 0$ a sequence of points z_ν, converging towards $z = 0$, such that $| F(z_\nu) + 1 | < \varepsilon$. The points $f(z_\nu)$ and $f(\overline{z}_\nu e^{i\vartheta})$ lie "almost" diametrically opposite on the RIEMANN sphere, and hence the spherical length of the image of $| z | = | z_\nu |$ by $f(z)$ is greater than $\pi - \delta(\varepsilon)$, where $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. On the other hand, this length is at most equal to $2\pi \, | z_\nu | \max\limits_{|z| = |z_\nu|} \varrho(f(z))$, and (2.1) follows[2]).

In order to prove the latter part of the Theorem we construct a convergent WEIERSTRASSian product

$$f(z) = \prod_{\nu=0}^{\infty} \frac{z - a_\nu}{z + a_\nu} \, , \tag{2.2}$$

where $a_\nu > 0$, $a_\nu > a_{\nu+1}$, $\lim a_\nu = 0$. For this function, $f(-z) = 1/f(z)$ so that $\varrho(f(-z)) = \varrho(1/f(z)) = \varrho(f(z))$. Hence, in estimating $\varrho(f(z))$, we may assume that $\mathrm{Re}\,\{z\} \geqq 0$.

Differentiation yields

$$\varrho(f(z)) = \frac{| f(z) |}{1 + | f(z) |^2} \left| \sum \frac{2a_\nu}{z^2 - a_\nu^2} \right| . \tag{2.3}$$

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[2]) The reasoning applies also to functions $f(z)$, quasiconformal in a neighbourhood of the singularity $z = 0$. If $[a, b]$ denotes the chordal distance of a and b on the RIEMANN sphere, the result can be expressed as follows:

$$\overline{\lim_{r \to 0}} \, (\max_{|z_1| = |z_2| = r} [f(z_1), f(z_2)]) = 1 \, .$$

For $\operatorname{Re}\{z\} \geqq 0$,

$$|f(z)| \leqq \left|\frac{z-a_\nu}{z+a_\nu}\right| \leqq 1\,, \quad \nu = 0, 1, 2, \ldots \tag{2.4}$$

Because $x(1+x^2)^{-1}$ is increasing for $0 \leqq x < 1$, we get from (2.3), by replacing $f(z)$ by the linear majorants in (2.4),

$$\varrho(f(z)) \leqq \sum \frac{a_\nu}{|z|^2 + a_\nu^2}\,.$$

Let us now suppose that

$$a_{\nu+1} = o(a_\nu)\,. \tag{2.5}$$

If $a_{n+1} < |z| \leqq a_n$, it then follows readily that

$$\varrho(f(z)) \leqq \frac{1+o(1)}{2|z|} + \sum_{\nu \neq n,\, n+1} \frac{a_\nu}{|z|^2 + a_\nu^2}\,. \tag{2.6}$$

Further, by (2.5),

$$\sum_1^{n-1} \frac{a_\nu}{|z|^2 + a_\nu^2} = O\left(\frac{a_{n-1}}{a_{n-1}^2 + |z|^2}\right),$$

and hence,

$$\sum_1^{n-1} \frac{a_\nu}{|z|^2 + a_\nu^2} = \frac{O(1)}{a_{n-1}} = \frac{o(1)}{|z|}\,. \tag{2.7}$$

Similarly,

$$\sum_{n+2}^{\infty} \frac{a_\nu}{|z|^2 + a_\nu^2} = O\left(\frac{a_{n+2}}{|z|^2 + a_{n+2}^2}\right) = O\left(\frac{a_{n+2}}{|z|^2}\right) = \frac{o(1)}{|z|}\,. \tag{2.8}$$

Hence, by (2.6), (2.7) and (2.8),

$$|z|\,\varrho(f(z)) \leqq \tfrac{1}{2} + o(1)\,.$$

Thus we have equality in (2.1) for the functions (2.2) whenever $a_{\nu+1} = o(a_\nu)$; the assumptions $a = 0$ and a_ν real and positive are clearly unessential.

The above extremal functions satisfying (2.1) as an equality are meromorphic for all $z \neq 0$. Hence, there exist extremal functions meromorphic in every neighbourhood of the singularity.

5. Let us study the growth of $\varrho(f(z))$ for certain meromorphic and entire functions. In order to have the situation most familiar in the classical theory we assume, for the moment, that $f(z)$ is meromorphic in the whole plane except for the singularity at $z = \infty$. The inequality (2.1) then becomes

$$\overline{\lim_{z \to \infty}}\, |z|\,\varrho(f(z)) \geqq \tfrac{1}{2}\,.$$

It is known ([4], [3]) that

$$|z|\,\varrho(f(z)) = O(1) \tag{2.9}$$

is equivalent to $f(z)$ being an exceptional function in the sense of JULIA. The relation (2.9) implies that these functions have a slowly growing characteristic function: $T(r) = O(\log^2 r)$, and that the distribution of values is quite symmetric. Functions omitting a value cannot be exceptional, and it follows, therefore, that for all entire functions

$$\overline{\lim_{z\to\infty}}\,|z|\,\varrho(f(z)) = \infty\,.$$

We have, for instance, $\max \varrho(\cos z^{\frac{1}{2}}) \sim \frac{1}{2}\,|z|^{-\frac{1}{2}}$, $\max \varrho(e^{z^n}) \sim {}^n/_2\,|z|^{n-1}$, etc.

One might think that $\varrho(f(z))$ would be of quite a slow growth for entire functions of order zero with simple zeros very thinly distributed. Such functions possess a slowly growing characteristic function and maximum modulus. Oddly enough, almost the opposite is true: For such entire functions, $\varrho(f(z))$ is always fairly large at the zeros of the function.

We introduce the customary counting function $N(r, a)$ and establish the following result.

Theorem 2. *Let*

$$f(z) = \prod_{\nu=1}^{\infty}(1 - z/a_\nu)$$

be an entire function the zeros of which satisfy the conditions

$$|a_{n+1}/a_n| \geqq q > 1\,. \tag{2.10}$$

Then

$$\underline{\lim_{\nu\to\infty}}\,|a_\nu|\,e^{-N(|a_\nu|,0)}\,\varrho(f(a_\nu)) > 0\,. \tag{2.11}$$

Proof: At a zero $z = a_n$ we have

$$|a_n|\,\varrho(f(a_n)) = \prod_{\nu\neq n}|1 - a_n/a_\nu|\,.$$

This can be written in the form

$$|a_n|\,\varrho(f(a_n)) = \prod_{\nu<n}\left|\frac{a_n}{a_\nu}\right|\prod_{\nu<n}|1 - a_\nu/a_n|\prod_{\nu>n}|1 - a_n/a_\nu|\,.$$

By the condition (2.10) we thus have

$$|a_n|\,\varrho(f(a_n)) \geqq C_q \prod_{\nu<n}|a_n/a_\nu|\,,$$

where $C_q = \prod_{1}^{\infty}(1 - q^{-\nu})^2 > 0$. Since

$$\log \prod_{\nu<n}|a_n/a_\nu| = N(|a_n|, 0)\,,$$

the Theorem follows.

6. It is clear that if a function $f(z)$ takes a value a infinitely often, then

$$\varliminf_{\nu \to \infty} \frac{N(r, a)}{\log r} = \infty .$$

Hence, we obtain the following less accurate but more striking version of Theorem 2:

For the functions of Theorem 2,

$$\lim_{\nu \to \infty} | a_\nu |^{-p} \varrho(f(a_\nu)) = \infty$$

for arbitrarily large p.

Hence, for the functions of Theorem 2, which from the classical point of view represent non-rational functions of slowest possible growth, the maximal growth of the spherical derivative is more rapid than e. g. for any function e^{z^n}, $n = 1, 2, \ldots$

In certain cases it is possible to state (2.11) in a more explicit form. For instance, if $| a_{n+1}/a_n | = q > 1$, it follows by an easy computation that

$$\varliminf_{\nu \to \infty} \varrho(f(a_\nu)) \, | a_\nu |^{-\frac{\log | a_\nu |}{2 \log q}} > 0 .$$

In contrast to the Julia exceptional functions, the distribution of values of the entire functions of Theorem 2 is quite "non-uniform". If the zeros $z = a_\nu$ are isolated with small circles, the function tends uniformly to ∞ outside the discs, whereas the behaviour inside the discs is quite different: for large ν, the set of values in the disc covers the whole Riemann sphere, except perhaps for two small islands. (See Theorem 4 below.)

3. Spherical derivative and Picard's Theorem

7. In this section we shall prove, in exact terms, that if $f(z)$ is of rapid spherical growth in the neighbourhood of the isolated singularity, then $f(z)$ takes all values, with two possible exceptions, already in a small point set, and conversely. We shall say in the following that Picard's Theorem holds for $f(z)$ in a point set E if $f(z)$ omits at most two values in E.

The proof is based on the following simple remark (cf. also [1], [3]). Let us consider all meromorphic functions $f(z)$, omitting three given values in a simply-connected domain G. Denoting by $d\sigma$ the element of length in the hyperbolic metric of G and fixing a point z in G, we obviously have at this point

$$\sup \frac{\varrho(f(z)) \, | dz |}{d\sigma(z)} = C < \infty , \tag{3.1}$$

since the existence of the extremal functions is clear by virtue of SCHWARZ's Lemma. Now (3.1) is a conformal invariant, and it follows immediately that (3.1) holds with the same constant C, irrespective of the simply-connected domain G and the special point z.

Denoting by $h(r)$ an arbitrary positive function of the positive variable r, with the property $h(r) = O(r)$ as $r \to 0$, we shall now prove

Theorem 3. *Let $f(z)$ be meromorphic in a neighbourhood of the singularity $z = a$. If for a sequence z_ν, $\lim z_\nu = a$,*

$$\lim_{\nu \to \infty} h(|z_\nu - a|)\varrho(f(z_\nu)) = \infty , \tag{3.2}$$

PICARD's *Theorem holds for $f(z)$ in the union of any infinite subsequence of the discs*

$$C_\nu: \quad | z - z_\nu | < \varepsilon h(|z_\nu - a|) \tag{3.3}$$

for each $\varepsilon > 0$.

Conversely, if there exist discs (3.3) *such that for any $\varepsilon > 0$* PICARD's *Theorem is valid in every $\bigcup\limits_1^\infty C_{\nu_i}$, then* (3.2) *is true.*

Proof: Supposing first that (3.2) holds, we make the antithesis that $f(z)$ omits three values in a set $\bigcup C_{\nu_i}$. A fortiori, $f(z)$ omits the same three values in every C_{ν_i}. Hence, by (3.1),

$$\varrho(f(z)) \mid dz \mid \leqq C d\sigma$$

in C_{ν_i}. If especially $z = z_{\nu_i}$ ($=$ centre of C_{ν_i}), $d\sigma / |dz|$ equals the reciprocal value of the radius of C_{ν_i} so that

$$h(|z_{\nu_i} - a|)\varrho(f(z_{\nu_i})) \leqq C/\varepsilon < \infty .$$

This, however, contradicts the assumption (3.2), and the sufficiency of the condition (3.2) is proved.

In order to prove the necessity of (3.2), we suppose that for any $\varepsilon > 0$, PICARD's Theorem is valid in the union of every infinite subsequence of the discs (3.3).

Let us consider the function family

$$f_\nu(w) = f(z_\nu + w h(|z_\nu - a|)) , \qquad \nu = 1, 2, \ldots \tag{3.4}$$

By hypothesis, the functions of any infinite subsequence $\{f_{\nu_i}(w)\}$ take all values, with two possible exceptions, in every disc $| w | < \varepsilon$. The family $\{f_\nu(w)\}$ cannot, therefore, be normal at $w = 0$. Hence, by MARTY's condition,

$$\lim_{\nu \to \infty} \varrho(f_\nu(0)) = \infty . \tag{3.5}$$

By (3.4),

$$\varrho(f_\nu(0)) = \varrho(f(z_\nu))\, h(|z_\nu - a|)$$

so that (3.5) is equivalent to (3.2). The Theorem is thus completely proved.

If the singularity lies at infinity, we must put in (3.2) and (3.3) $a = 0$. The particular choice $h(r) = r$ then yields JULIA's Theorem.

8. CONSTANTINESCU ([1], Theorem 5) proved the validity of PICARD's Theorem in $\cup\, C_{\nu_i}$ under a condition which in our notations assumes the form

$$\overline{\lim_{z \to a}}\, h(|z-a|)\Big(\int_{|z-a| = \text{const.}} \varrho^2(f(z))\, d\vartheta\Big)^{\frac{1}{2}} = \infty \qquad (z - a = r e^{i\vartheta})\,. \tag{3.6}$$

This is sometimes a much stronger requirement than (3.2). For instance, for the function

$$f(z) = \prod_{\nu=1}^{\infty} (1 - z e^{-\nu})$$

we have the striking difference

$$\Big(\int_{|z| = r} \varrho^2(f(z))\, d\vartheta\Big)^{\frac{1}{2}} = O(1/\sqrt{r})$$

as $r \to \infty$, while (cf. n° 6)

$$\varrho(f(r)) \neq o(r^{\log\sqrt{r} - 3/2})\,.$$

9. Applying Theorem 3 to the entire functions of Theorem 2, slowly growing in the classical sense, we see that in the neighbourhood of the zeros the functions are extremely active as regards taking many values.

Theorem 4. *Let the zeros of the entire function*

$$f(z) = \prod_{\nu=1}^{\infty}(1 - z/a_\nu)$$

satisfy the conditions $|a_{n+1}/a_n| \geqq q > 1$. *Then* PICARD's *Theorem holds in the union of every infinite subsequence of the discs*

$$C_\nu: \quad |z - a_\nu| < \varepsilon\, |a_\nu|^{-p}$$

where p may be chosen arbitrarily large.

4. Functions in the unit disc

10. Let $f(z)$ be meromorphic in the unit disc $|z| < 1$. If $f(z)$ omits at most two values in the angle $\vartheta - \varepsilon < \arg z < \vartheta + \varepsilon$, no matter how small $\varepsilon > 0$ has been chosen, $\arg z = \vartheta$ is called a JULIA *radius*. CONSTANTINESCU [1] proved that $f(z)$ possesses a JULIA radius if

$$\overline{\lim_{r \to 1}}\,(1 - r)A(r) = \infty\,, \tag{4.1}$$

where $A(r)$ denotes the spherical area of the map of $|z| < r$ by $f(z)$.

It seems to be difficult to give a necessary and sufficient metrical condition for the existence of a JULIA radius. However, simple necessary and sufficient conditions, in terms of the spherical derivative, can be given which are not very far apart from each other.

Theorem 5. *A function* $f(z)$, *meromorphic in* $|z| < 1$, *possesses a* JULIA *radius if* $f(z)$ *is not normal, i. e. if*

$$\overline{\lim_{|z| \to 1}}\,(1 - |z|)\varrho(f(z)) = \infty\,^{3)}. \tag{4.2}$$

This condition is not necessary; on the other hand, the condition

$$\overline{\lim_{|z| \to 1}}\,(1 - |z|)\varrho(f(z)) > 0 \tag{4.3}$$

is not sufficient.

A necessary condition for the existence of a JULIA *radius is that*

$$\overline{\lim_{|z| \to 1}}\,\varphi(1 - |z|)\varrho(f(z)) = \infty \tag{4.4}$$

for all positive functions φ *with the property*

$$\int\limits_0 \frac{dr}{\varphi(r)} < \infty\,. \tag{4.5}$$

Proof: Let us first suppose that $f(z)$ does not possess any JULIA radius. Then every radius $\arg z = \vartheta$ has an angular neighbourhood A_ϑ: $\vartheta - \varepsilon_\vartheta < \arg z < \vartheta + \varepsilon_\vartheta$ in which $f(z)$ omits three values. Thus $f(z)$ is normal in every A_ϑ, and by HEINE-BOREL's covering theorem, $f(z)$ is normal in the whole unit disc. Hence (4.2) implies the existence of a JULIA radius.

In order to prove that (4.2) is not a necessary condition, let us consider an arbitrary angle $\vartheta - \varepsilon < \arg z < \vartheta + \varepsilon$. If $f(z)$ omits three values in this angle, then by a well-known theorem, it possesses radial limits on a set dense on the arc $(e^{i(\vartheta - \varepsilon)}, e^{i(\vartheta + \varepsilon)})$. Hence, for a function with no radial limits on $|z| = 1$, every radius is a JULIA radius.

Now it was proved in [3] (p. 58) that there exist normal functions with no radial limits on $|z| = 1$. In other words, there exist functions for which $(1 - |z|)\varrho(f(z)) = O(1)$ and for which every radius is a JULIA radius. Hence, (4.2) is certainly not a necessary condition.

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[3]) The condition (4.1) implies of course (4.2).

For the elliptic modular function the condition (4.3) is fulfilled. On the other hand, since this function omits three values in the whole disc $|z| < 1$, no radius is a JULIA radius. Thus (4.3) is not a sufficient condition.

The necessity of the condition (4.3) can be proved by a direct computation. If (4.4) does not hold, we get the estimate

$$s(f(r_1 e^{i\vartheta}), f(r_2 e^{i\vartheta})) = O\left(\int_{r_1}^{r_2} \frac{|dr|}{\varphi(r)}\right)$$

for the spherical distance of the points $f(r_1 e^{i\vartheta})$ and $f(r_2 e^{i\vartheta})$. By (4.5), this implies the existence of continuous radial limits on $|z| = 1$. Hence, $f(z)$ cannot possess any JULIA radius.

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