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Essential and inessential complexes

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In [1, Satz VII] H. HOPF and E. PANNWITZ have given a complete topological characterization of essentiality for a large class of complexes: an n-dimensional ($n \geqslant 3$) simply connected complex K is essential (im Grossen stabil) if and only if K is cyclic. In connection with this result, they raised the question whether there exist 2-dimensional simply connected complexes which are essential but not cyclic. An affirmative answer to this question is given by

Theorem 1. *There exists a finite homogeneous* 2*-dimensional complex* Q, *which is simply connected and essential but not cyclic.*

This theorem follows from the existence of a finitely generated group with particular properties (Lemma 4.2) and from

Theorem 2. *A finite homogeneous simply connected* 2*-dimensional complex* K *is essential if and only if for each proper subcomplex* $L \subset K$

A) $$H_2(\Pi, Z) \neq 0 \quad \textit{where} \quad \Pi = \pi_1(L)\,,$$

provided that the homomorphism

$$i_*\colon H_2(L, Z) \to H_2(K, Z)$$

induced by inclusion, is an isomorphism onto.

These theorems emphasize the particular behaviour of 2-dimensional complexes. In the last section, we show that the complex Q of Theorem 1 has the following rather striking property: $Q \times K$ is inessential for any complex K with $\dim K > 0$; in particular $Q \times Q$ is inessential.

As shown in [1, Satz V], for each $n > 0$ there are non-cyclic n-dimensional complexes which are essential but are not simply connected. Nevertheless, use of cohomology and of local coefficients leads to a suitable modification of the classical definition of cyclicity as given in [1]; we thus introduce the "cocyclicity" in Definition 1.1, and obtain the following generalization of Satz VII of [1]:

Theorem 3. *A finite homogeneous n-dimensional complex K with* $n \geqslant 3$ *is essential if and only if K is cocyclic.*

The condition of Theorem 3 can be expressed in another equivalent form, which shows that cocyclicity is in fact a topological invariant of the universal covering complex $\tilde{K}$ of K:

Theorem 4. *A finite homogeneous n-dimensional complex K with* $n \geqslant 3$ *is essential if and only if its universal covering complex is cocyclic with finite cochains.*

From Theorem 4 follows

Theorem 5. *A finite homogeneous n-dimensional complex K with* $n \geqslant 3$ *is essential if and only if its universal covering complex is essential with respect to proper maps.*

The author ignores whether Theorem 5 is also valid for $n = 2$. Moreover, the "unalgebraic" form of this last theorem leads one to raise the question of its validity for more general spaces than complexes.

1. Preliminaries

By an n-dimensional complex we mean a CW-complex as defined in [2], with the additional property that the closures of its cells are homeomorphic to closed simplexes. Without further mention, a complex will always be assumed to be *homogeneous*, i.e. such that every open subset meets some n-cell. No distinction is made between a complex and its underlying space.

We recall in a convenient form the definition of cohomology groups with local coefficients. Let K be a complex, $\tilde{K}$ its universal covering complex, and Π the group of covering transformations (Deckbewegungsgruppe); $\Pi \approx \pi_1(K)$. Assume that Π operates on an abelian group A. A q-cochain c is a function defined on the oriented q-cells of $\tilde{K}$, with values in A, such that $c(-\sigma^q) = -c(\sigma^q)$, $\sigma \subset \tilde{K}$. The cochain c is *equivariant* if $\xi(c(\sigma^q)) = c(\xi\sigma^q)$, for every $\xi \in \Pi$, $\sigma^q \subset \tilde{K}$; here $\xi\sigma^q$ is the image of σ^q under the homeomorphism ξ. The equivariant cochains form a group $C^q(K, A)$. The coboundary δc of $c \in C^q(K, A)$ is the equivariant $(q+1)$-cochain defined by

$$(\delta c)(\sigma^{q+1}) = \sum_i [\sigma^{q+1} : \sigma_i^q] c(\sigma_i^q), \tag{1.1}$$

where σ_i^q are all the q-cells of $\tilde{K}$ and $[\sigma^{q+1} : \sigma_i^q]$ are the incidence numbers. The cochain complex $(C^q(K, A), \delta)$ defines the cohomology groups $H^q(K, A)$ of K with local coefficients A. Considering only equivariant cochains which are zero on the cells of the subcomplex $p^{-1}(L) \subset \tilde{K}$, where $p : \tilde{K} \to K$ is the covering projection and $L \subset K$ is a subcomplex, leads one to define in a similar way the relative groups $H^q(K, L; A)$. If Π operates trivially on A, both $H^q(K, A)$ and $H^q(K, L; A)$ reduce to ordinary cohomology groups.

Definition 1.1. *An n-cell τ of an n-dimensional complex K will be called cocyclic if the homomorphism*

$$j^*: H^n(K, L; A) \to H^n(K, A), \qquad L = K - \tau, \tag{1.2}$$

induced by inclusion is non-trivial for at least a system of local coefficients A. An n-dimensional complex K will be called cocyclic if all of its n-cells are cocyclic.

Proposition 1.2. *A complex K which is cyclic in the sense of* ([1], p. 436) *is cocyclic. If K is simply connected and finite, cyclicity and cocyclicity are equivalent properties.*

Proof. HOPF and PANNWITZ called K cyclic if each n-cell τ of K belongs with a non-zero coefficient to an integral cycle or to a cycle mod. m for some m, i.e. if the homomorphism

$$j_*: H_n(K, G) \to H_n(K, L; G), \qquad L = K - \tau, \tag{1.3}$$

induced by inclusion, is non-trivial for $G = Z$ (the integers) or for $G = Z_m$ (integers mod. m). First notice that if j_* is trivial for Z and for all Z_m, it is trivial also for any abelian G. This is obvious when G is finitely generated (for, homology commutes with direct sums); for an arbitrary G this follows from the fact that homology commutes with direct limits.

In the general case, if (1.3) is non-trivial for some G, (1.2) is non-trivial for $A = D(G)$–the group of characters of G. If K is simply connected and finite, the coefficients in (1.2) are discrete abelian groups; by taking now $G = D(A)$, the non-triviality of (1.2) implies that of (1.3).

2. Proof of Theorem 3

By $\mu: (Y, B) \to (X, A)$, $B \subset Y$, $A \subset X$ we mean any continuous map $\mu: Y \to X$ satisfying $\mu(B) \subset A$. Two maps μ_0 and μ_1 are homotopic if there is a map $\lambda: (Y \times I, B \times I) \to (X, A)$ (I = unit interval) with

$$\lambda(x, 0) = \mu_0(x), \lambda(x, 1) = \mu_1(x).$$

Let X, $A \subset X$ both be arcwise connected and simply connected. Let further T^n be a complex homeomorphic to a (closed) n-simplex with a single n-cell σ^n; write T^k for its k-section (union of the cells of dimension $\leqslant k$). Introduce the notations

$$\begin{aligned} \overline{D}_j &= \bar{\sigma}_j \times I, \ \dot{D}_j = \dot{\sigma}_j \times I \cup \bar{\sigma}_j \times 0 \cup \bar{\sigma}_j \times 1, \\ \overline{E} &= T^n \times 0 \cup T^{n-1} \times I, \ \dot{E} = T^{n-1} \times 1, \end{aligned} \tag{2.1}$$

where $\sigma_j, j = 1, 2, \ldots, m$ are the $(n-1)$-cells of T^n and $\dot{\sigma}_j$ their boundaries; $\overline{D}_j$, $\overline{E}$ are closed n-cells and $\dot{D}_j$, $\dot{E}$ are their boundaries.

The following lemma collects in a convenient form certain well known facts:

Lemma 2.1. *Let the map* $f : (T^n, T^{n-1}) \to (X, A)$ *determine the element* $\alpha \in \pi_n(X, A)$ (we assume T^n oriented and omit the base point for relative homotopy groups). *Consider arbitrary elements* $\alpha_j \in \pi_n(X, A)$, $j = 1, \ldots, m$. *Then*

i) *For each* j *there is a map* $f_j : (\overline{D}_j, \dot{D}_j) \to (X, A)$ which determines α_j and satisfies

$$f_j(x, t) = f(x) \quad \text{for} \quad (x, t) \in T^{n-2} \times I \cup T^{n-1} \times 0 . \tag{2.2}$$

ii) *The maps* f_j *and* f *yield a map*

$$F' : (\overline{E}, \dot{E}) \to (X, A)$$

which determines the element

$$\beta = \alpha + \Sigma[\sigma^n : \sigma_j]\alpha_j \in \pi_n(X, A) \tag{2.3}$$

for a suitable orientation of E.

iii) *If* $\beta = 0$, F' *can be extended to a map*

$$F : (T^n \times I,\ T^n \times 1) \to (X, A) . \tag{2.4}$$

Proof. i) Suppose $g_j : (\overline{D}_j, \dot{D}_j) \to (X, A)$ determines α_j. Define f_j on $\dot{\sigma}_j \times I \cap \sigma_j \times 0$ according to (2.2); since the set is contractible, g_j and f_j are homotopic. Extend first this homotopy over $\dot{D}_j$, with values in A, and further over $\overline{D}_j$, with values in X. Since the extension of f_j is homotopic to g_j, it determines α_j.

A proof of i), which is equivalent to the "addition theorem for relative homotopy groups", can be easily obtained by applying the relative HUREWICZ isomorphism theorem to the pair $(\overline{E}, T^{n-2} \times I \cup T^{n-1} \times 0 \cup \dot{E})$.

In order to derive iii), notice that $\beta = 0$ implies $F' \sim 0$, i.e. F' can be extended to yield a map

$$F'' : (\varkappa\overline{E}, \varkappa\dot{E}) \to (X, A) \tag{2.5}$$

where $\varkappa\overline{E}$, $\varkappa\dot{E}$ are cones over $\overline{E}$, $\dot{E}$ with the same vertex $\varkappa$. This already implies iii) since $(T^n \times I, T^n \times 1)$ and $(\varkappa\overline{E}, \varkappa\dot{E})$ are homeomorphic pairs.

Proof of Theorem 3. We recall that a complex K is essential (im Grossen stabil) if it cannot be deformed into a proper subset K_1. An (open) n-cell $\tau \subset K$ is essential if, whatever be the point $x \in \tau$, K cannot be deformed into $K - x$. Since $\dot{\tau} \subset K - \tau$ is a strong deformation retract of $\overline{\tau} - x$,

$x \in \tau$, it readily follows that τ is essential if and only if there is no deformation of K into $K - \tau$. If K is essential, all its n-cells are essential. Conversely, if K is a finite complex, hence compact, and if all of its n-cells are essential, then K is essential. For, the image K_1 of K under a deformation is then closed and the homogenity of K, which we always assume, implies that $K - K_1 \neq \emptyset$ yields $(K - K_1) \cap \tau \neq \emptyset$ for some n-cell τ. Theorem 3 is now a direct consequence of

Theorem 3'. *An n-cell* τ_0 *of an n-dimensional complex* K, $n \geqslant 3$, *is essential if and only if* τ_0 *is cocyclic.*

Proof. Let $L = K - \tau_0$. If τ_0 is inessential, there is a deformation φ_t of K into L. Since $\varphi_0 = j : (K, \emptyset) \to (K, L)$ is the inclusion and $\varphi_1(K) \subset L$, in (1.2) we have $j^* = 0$ and is non-cocyclic. This holds for n arbitrary.

Assume now that τ_0 is non-cocyclic and $n \geqslant 3$. In order to prove that τ_0 is inessential, we use a technique borrowed from obstructions theory, with relative homotopy groups instead of the absolute ones. This method of proof is similar to one used in the simply connected case by M. M. POSTNIKOV for deriving a theorem by PONTRJAGIN (see [3], 10 : 3).

Let $\tilde{K}$ be the universal covering space of K with $p : \tilde{K} \to K$ as covering projection; $\tilde{K}$ also is a complex. Since $n \geqslant 3$, the subcomplex $\tilde{L} = p^{-1}(L)$ is connected and simply connected. A map $\tilde{\psi} : \tilde{K} \to \tilde{K}$ is *equivariant* if $\xi\tilde{\psi} = \tilde{\psi}\xi$ for each element ξ of the group Π of covering transformations (Deckbewegungsgruppe). The inclusion $(\bar{\sigma}^n, \dot{\sigma}^n) \to (\tilde{K}, \tilde{L})$ of each oriented n-cell $\sigma^n \subset \tilde{K}$ determines an element $o(\sigma^n) \in \pi_n(\tilde{K}, \tilde{L})$. Obviously Π operates on the group $\pi_n(\tilde{K}, \tilde{L})$ which is abelian since $n \geqslant 3$; we obtain thus an equivariant cocycle $o \in C^n(K, \pi_n(\tilde{K}, \tilde{L}))$. The non-cocyclity of τ_0 implies that

$$j^* : H^n(K, L; \pi_n(\tilde{K}, \tilde{L})) \to H^n(K, \pi_n(\tilde{K}, \tilde{L})) \tag{2.6}$$

is trivial; since it is clear that $o(\sigma^n) = 0$ for $\sigma^n \subset \tilde{L}$, we therefore have $o = \delta d$, $d \in C^{n-1}(K, \pi_n(\tilde{K}, \tilde{L}))$.

In $\tilde{K} \times I$, the group Π operates according to the rule $\xi(x, t) = (\xi x, t)$. For each $(n-1)$-cell $\tau^{n-1} \subset K$, select a cell $\sigma(x_j^{n-1}) \subset \tilde{K}$ satisfying $p(\sigma_j^{n-1}) = \tau_j^{n-1}$. Define $\tilde{\Theta} : \tilde{K} \times 0 \to \tilde{K}$ by $\tilde{\Theta}(x, t) = x$. According to 2.1, i), define maps

$$\tilde{f}_j : (\bar{\sigma}_j^{n-1} \times I, \dot{\sigma}_j^{n-1} \times I \cup \bar{\sigma}_j^{n-1} \times 0 \cup \bar{\sigma}_j^{n-1} \times 1) \to (\tilde{K}, \tilde{L}) \tag{2.7}$$

such that

$$\tilde{f}_j(x, t) = \tilde{\Theta}(x, 0) \quad \text{for} \quad (x, t) \in \dot{\sigma}_j^{n-1} \times I \cup \bar{\sigma}_j^{n-1} \times 0$$

and such that $\tilde{f}_j$ determines the element $-d(\sigma_j^{n-1}) \in \pi_n(\tilde{K}, \tilde{L})$; if $d(\sigma_j^{n-1}) = 0$, define $\tilde{f}_j$ by $\tilde{f}_j(x, t) = \tilde{\Theta}(x, 0)$ for $(x, t) \in \sigma_j^{n-1} \times I$. Upon noticing that each $(n-1)$-cell of K is uniquely represented as $\xi \sigma_j^{n-1}$ for some $\xi \in \Pi$, define an equivariant map

$$\tilde{F}' : \tilde{K} \times 0 \cup \tilde{K}^{n-1} \times I \to \tilde{K} \tag{2.8}$$

by $\tilde{F}' = \xi \tilde{f}_j \xi^{-1}$ on each cell $\xi \sigma_j^{n-1} \times I$ and $\tilde{F} = \tilde{\Theta}$ on $\tilde{K} \times 0$. Since the cochain $-d$ is itself equivariant, it is obvious that on $\xi \bar{\sigma}_j^{n-1} \times I$, F' determines the element $-\xi d(\sigma_j^{n-1}) = -d(\xi \sigma_j^{n-1})$.

For each n-cell $\tau_k^n \subset K$ select now a cell $\sigma_k^n \subset \tilde{K}$, satisfying $p(\sigma_k^n) = \tau_k^n$. The restriction $\tilde{F}_k'$ of $\tilde{F}'$ maps $(\bar{\sigma}_k^n \times 0 \cup \dot{\sigma}_k^n \times I, \dot{\sigma}_k^n \times 1)$ into $(\tilde{K}, \tilde{L})$; according to its definition and to 2.1, i), it determines the element

$$\beta(\sigma_k^n) = o(\sigma_k^n) - \sum_j [\sigma_k^n : \sigma_j^{n-1}] d(\sigma_j^{n-1}) \in \pi_n(\tilde{K}, \tilde{L}) \tag{2.9}$$

which vanishes since $o = \delta d$. Therefore by 2.1, iii), $\tilde{F}_k'$ can be extended to a map $\tilde{F}_k : (\bar{\sigma}_k^n \times I, \bar{\sigma}_k^n \times 1) \to (\tilde{K}, \tilde{L})$; if $\tilde{F}_k'(x, t) = \tilde{\Theta}(x, 0)$, define $\tilde{F}_k$ by $\tilde{F}_k(x, t) = \tilde{\Theta}(x, 0)$ on $\bar{\sigma}_k \times I$. Extending equivariantly the maps $\tilde{F}_k$ as above, we obtain a map $\tilde{F} : \tilde{K} \times I \to \tilde{K}$.

A map $F : K \times I \to K$ is now well defined by $F(y, t) = p(F(p^{-1}y, t))$, $(y, t) \in K \times I$ and it satisfies $F(y, 0) = y$, $F(K \times 1) \subset L$, i. e. F is a deformation of K into L, which proves that τ_0 is inessential.

Remark 2.2. If we assume that K is locally finite and that the cochain d satisfies $d(\sigma^n) = 0$ for all but a finite number of values of j, the above construction yields a homotopy which is stationary on all but a finite number of n-cells of K.

3. Proof of Theorems 4 and 5

Proof of Theorem 4. Keeping the same notations as before, the group with operators $\pi_n(\tilde{K}, \tilde{L})$ is isomorphic to $H_n(\tilde{K}, \tilde{L}) \approx Z(\Pi)$, where $Z(\Pi)$ is the group algebra of Π over the integers Z. Since $\pi_n(\tilde{K}, \tilde{L})$ is the only system of local coefficients which was used in the proof of Theorem 3′, this theorem remains valid upon replacing in Definition 1.1, the phrase "for at least a system of local coefficients A" by "for $A = Z(\Pi)$".

On the other hand, if K and L are finite, we have natural isomorphisms [4, ch. XVI, § 10]

$$H^n(K, L; Z(\Pi)) \approx \overline{H}^n(K, L; Z) \approx H_f^n(\tilde{K}, \tilde{L}; Z) \tag{3.1}$$

and

$$H^n(K, Z(\Pi)) \approx \overline{H}^n(K, Z) \approx H_f^n(\tilde{K}, Z), \tag{3.2}$$

where $\overline{H}^n$ are groups in the "almost zero theory" of ECKMANN [5] and H_f^n are cohomology groups based on finite cochains. Thus an n-cell $\tau_0 \subset K$ is cocyclic if and only if the homomorphism

$$\tilde{j}_f^* : H_f^n(\tilde{K}, \tilde{L}; Z) \to H_f^n(\tilde{K}, Z) \tag{3.3}$$

induced by inclusion, is non-trivial (the notations are the same as in the proof of Theorem 3′).

The group Π operates on both groups in (3.3), which therefore are $Z(\Pi)$-modules, and $\tilde{j}_f^*$ is a $Z(\Pi)$-homomorphism. Let σ_0 be any cell satisfying $p(\sigma_0) = \tau_0$; since any generator of the group $H_f^n(\tilde{K}, \tilde{K} - \sigma_0; Z)$ generates the $Z(\Pi)$-module $H_f^n(\tilde{K}, \tilde{L}; Z)$ in (3.3), $\tilde{j}_f^*$ is non-trivial if and only if

$$j_f^* : H_f^n(\tilde{K}, \tilde{K} - \sigma_0; Z) \to H_f^n(\tilde{K}, Z) \tag{3.4}$$

is non-trivial, i.e. if and only if σ_0 is cocyclic with finite cochains (*f-cocyclic*). This obviously disposes of Theorem 4.

Proof of Theorem 5. A map $\varphi : (Y, B) \to (X, A)$ is called *proper* if the inverse images of compact sets under φ are compact. We shall now restrict ourselves to the category of locally finite (i.e. locally compact) complexes and of proper maps and homotopies. Theorem 5 follows from Theorem 4 and from

Proposition 3.1. *A simply connected locally finite n-dimensional complex K, with $n \geqslant 3$, is essential with respect to proper maps and homotopies (p-essential) if and only if K is f-cocyclic.*

Proof. Notice first that if each n-cell of K is p-essential, K is p-essential. For, the same argument which was used to derive Theorem 3 from 3′, applies for locally finite complexes K and proper maps, since the image of K under a proper map is closed. It is also obvious that a f-cocyclic cell $\sigma_0 \subset K$ is p-essential (because any two proper maps which are connected by a proper homotopy induce the same homomorphisms of cohomology groups based on finite cochains). The converse is an immediate consequence of Remark 2.2, since a homotopy which is stationary on all but a finite number of cells is obviously proper.

4. Proof of Theorems 2 and 1

A procedure similar to that used in deriving Theorem 3 from 3′ yields Theorem 2 as a consequence of

Theorem 2′. *A simply connected homogeneous 2-dimensional complex K can be deformed into a proper subcomplex $L \subset K$ if and only if*

$$i_* : H_2(L, Z) \to H_2(K, Z) \tag{4.1}$$

is an isomorphism onto and

$$\text{A)} \qquad H_2(\Pi, Z) \neq 0 \quad \textit{where} \quad \Pi = \pi_1(L)\,.$$

Proof. First assume that K can be deformed into L, i.e. there exists $\varphi : K \to L$, with $i\varphi \sim$ identity ($i : L \to K$ is the inclusion map). Then

$$i_*\varphi_* : H_2(K, Z) \to H_2(K, Z) \tag{4.2}$$

is an isomorphism onto and therefore i_* is onto. Noticing that i^* is $1-1$ (since $\dim K = 2$), i_* is an isomorphism onto and it follows that

$$\varphi_* : H_2(K, Z) \to H_2(L, Z) \tag{4.3}$$

is onto. This proves that the elements of $H_2(L, Z)$ are represented by spherical cycles, since this holds for $H_2(K, Z) \approx H_2(L, Z)$. Applying now to L the well known result of [6], we obtain $H_2(\Pi, Z) = 0$.

Conversely, if we assume that i_* is an isomorphism onto and that $H_2(\Pi, Z) = 0$, then by [6] all cycles of $H_2(L, Z)$ are spherical. On the other hand, it is well known that a simply connected 2-dimensional complex K has the same homotopy type as an union $X = \underset{\alpha}{V} S_\alpha$ of 2-spheres S_α with a single common point x_0. Consider maps $\psi : K \to X$, $\chi : X \to K$ with $\psi\chi \approx$ identity and $\chi\psi \sim$ identity. Let a_α generate the groups $H_2(S_\alpha, Z)$. For each α define a map $\lambda_\alpha : S_\alpha \to L$ such that $\lambda_{\alpha *}(a_\alpha) = (i_*^{-1}\chi_*)(a_\alpha)$ and $\lambda_\alpha(x_0) = Y_0 \in L$. This is possible since $(i_*^{-1}\chi_*)(a_\alpha)$ are spherical cycles. We thus obtain a map $\lambda : X \to L$. Since $H_2(X, Z) = \underset{\alpha}{\Sigma} H_2(S_\alpha, Z)$, we have $\lambda_* = i_*^{-1}\chi_*$. Consider the composite maps $\varphi = \lambda\psi : K \to L$ and $i\varphi : K \to K$ and observe that $i_*\varphi_* = i_*\lambda_*\psi_* = i_* i_*^{-1}\chi_*\psi_* =$ identity. By the classification theorem of Whitney this amounts to $i\varphi \sim$ identity, which shows that K can be deformed into L.

Lemma 4.1. *Let Π be a group with a finite number of generators and relations, such that*

B) $\Pi/[\Pi, \Pi] = Z$ (where $[\Pi, \Pi]$ is the commutator subgroup);

C) *there exists a $\xi \in \Pi$ satisfying $N(\xi) = \Pi$, where $N(\xi)$ is the least normal subgroup generated by ξ.*

Under these conditions there exists a finite homogeneous 2-dimensional complex Q and a subcomplex $R \subset Q$, such that

1) $\pi_1(Q) = 0$, $\pi_1(R) \approx \Pi$;

2) R is *cyclic*;

3) $i_* : H_2(R, Z) \to H_2(Q, Z)$ *is an isomorphism onto;*

4) $Q - R$ is *homeomorphic to an open* 2-*cell.*

Proof. It is well known (see for instance [7, § 45]) that for a given Π we can construct a finite simplicial complex R' satisfying $\pi_1(R') = \Pi$. Adding, if needed, trivial relations of the form $\xi\xi^{-1} = e$, we can assume that each generator belongs to some relation and this readily implies that we can assume R' to be homogeneous.

Take two disjoint copies R'_1 and R'_2 of R' and identify their corresponding 1-cells. This yields a complex R; it is easy to check (e.g. by applying [7, § 52]) that $\pi_1(R) \approx \Pi$. The complex R is cyclic; for, any two corresponding 2-cells $\tau'_1 \in R'_1$ and $\tau'_2 \in R'_2$ have in R a common boundary, hence determine the 2-cycle $\tau'_1 - \tau'_2$.

Consider now a simplicial path in R, representing the element $\xi \in \Pi$ of condition C). By the procedure described in [7, § 45] attach to this path a 2-cell τ which adds the relation $\xi = e$. By C), the resulting complex Q is simply connected; obviously, Q is homogenous and admits a simplicial subdivision. Consider the exact sequence

$$0 \to H_2(R, Z) \xrightarrow{i_*} H_2(Q, Z) \xrightarrow{j_*} H_2(Q, R; Z) \xrightarrow{\delta} H_1(R, Z) \to 0. \qquad (4.4)$$

By B), $H_1(R, Z) \approx \Pi/[\Pi, \Pi] \approx Z$ and since $H_2(Q, R; Z) \approx Z$ (Q is an open cell!), δ is an isomorphism. This implies that j_* is trivial and i_* is an isomorphism.

Lemma 4.2. *There exists a finitely generated and finitely related group* Π_0 *which satisfies conditions* A), B) *and* C).

Proof. We define Π_0 as a particular split extension (semi-direct product) over Z with kernel $\pi = Z + Z$.

More precisely, Π_0 is the group with generators a, b, c and relations

$$\begin{aligned} aba^{-1}b^{-1} &= e \qquad &(\alpha)\\ ac &= cb \qquad &(\beta)\\ bc &= cab \qquad &(\gamma) \end{aligned}$$

In order to prove B), observe that the homomorphism φ, defined by $\varphi(a) = \varphi(b) = e$, $\varphi(c) = c$, maps Π_0 onto the infinite cyclic subgroup generated by c and has as kernel the subgroup π generated by a and b. Let $\psi : \Pi_0 \to G$ be any homomorphism onto an abelian group. Then (β) implies

$\psi(a) = \psi(b)$ and (γ) implies $\psi(a) = e$, whence $\pi \subset \operatorname{Ker}\psi$, i. e. $\pi = [\Pi_0, \Pi_0]$ and thus $\Pi_0/[\Pi_0, \Pi_0] = e$.

Consider now an homomorphism $\chi : \Pi_0 \to \Pi$ (Π arbitrary) such that $\chi(c) = e$. Relation (β) yields $\chi(a) = \chi(b)$ and from (γ) we obtain $\chi(a) = e$, i.e. $\chi(\Pi_0) = e$. This obviously proves C).

In order to prove A), let K_0 be an aspherical complex such that $\pi_1(K_0, x) = \Pi_0$. Is is known that $H_n(K_0, G) \approx H_n(\Pi_0, G)$. Let (K_0^*, p) be a covering space of K_0 corresponding to the normal subgroup π. Since the group of covering transformations of (K_0^*, p) is $\Pi_0/\pi \approx Z$, the exact sequence

$$0 \to H_2(K_0^*, Z)_{\Pi_0/\pi} \to H_2(K_0, Z) \tag{4.5}$$

applies (see [8], Appendice). But K_0^* is aspherical and $H_2(K_0^*, Z) \approx H_2(\pi, Z) \approx Z$, since π is isomorphic to the direct product two infinite cyclic groups. The group Π_0/π operates on K_0^* and therefore on $H_2(K_0^*, Z) \approx Z$; by definition, $H_2(K_0^*, Z)_{\Pi_0/\pi}$ is the factor group of $H_2(K_0^*, Z)$ by the subgroup generated by the elements $a - \xi a$, $a \in H_2(K_0^*, Z)$, $\xi \in \Pi_0/\pi$. Since the only automorphisms of Z are the identity and $a \to -a$, we have $a - \xi a = 0$ or $a - \xi a = 2a$. Thus

$$H_2(K_0^*, Z)_{\Pi_0/\pi} \approx Z \quad \text{or} \quad \approx Z_2 . \tag{4.6}$$

In both cases (4.2) yields $H_2(K_0, Z) \approx H_2(\Pi_0, Z) \neq 0$.

Proof of Theorem 1. Since Π_0 of Lemma 4.2 satisfies both B) and C), construct, according to Lemma 4.1, a simply connected complex Q and a subcomplex $R \subset Q$ satisfying 1), 2) and 3). Q is not cyclic; indeed for any 2-cell $\tau \subset Q - R$, 3) readily implies that

$$H_2(Q - \tau, Z) \to H_2(Q, Z) \tag{4.7}$$

is an isomorphism and therefore

$$H_2(Q, Z) \to H_2(Q, Q - \tau; Z) \tag{4.8}$$

is trivial. This is also true for any abelian group G, since $H_2(Q, G) = H_2(Q, Z) \otimes G$. If we now assume that it is possible to deform Q into a proper subcomplex $Q_1 \subset Q$, we must necessarily have $Q_1 \supset R$, since R is cyclic. This would imply that Q can be deformed into R, since $Q - R$ is homeomorphic to an open 2-cell. According to Theorem 2 this is impossible because R satisfies 3) and A), whence Q is essential.

5. Further properties of Q

Let Q be the complex of Theorem 1 and K an arbitrary complex with $\dim K = n > 0$. Consider the product $Q \times K$. From the "Künneth" theorem for cohomology, there results that for any system A of local coefficients on $Q \times K$, $H^{n+2}(Q \times K, A)$ is naturally isomorphic to $\operatorname{Hom}(H_2(Q, Z), H^n(K, A))$. This is obtained by applying [4, ch. VI, Th. 3.1a], using the isomorphism of [4, Ch. II, Prop. 5.2] and noticing that $H_1(Q, Z) = 0$ and $H_2(Q, Z)$ is abelian free.

Let τ be any 2-simplex of $Q - R$, σ any n-cell of K; $\tau \times \sigma$ is a $(n+2)$-cell of $Q \times K$. There results the following commutative diagram

$$\begin{tikzcd}
H^{n+2}(Q \times K, A) \arrow[r, leftrightarrow] \arrow[d, "i_1^*"] & \operatorname{Hom}(H_2(Q, Z), H^n(K, A)) \arrow[dd] \\
H^{n+2}(Q \times K - \tau \times \sigma, A) \arrow[d, "i_2^*"] & \\
H^{n+2}(R \times K, A) \arrow[r, leftrightarrow] & \operatorname{Hom}(H_2(R, Z), H^n(K, A)) .
\end{tikzcd}$$

Since $H_2(R, Z) = Z$ and $H_2(R, Z)$ are free abelian groups, the lower horizontal map is an isomorphism onto; all the vertical arrows are induced by inclusions.

Since $i_* : H_2(R, Z) \to H_2(Q, Z)$ is an isomorphism onto, the second vertical map and $i_2^* i_1^*$ are also isomorphisms onto. Since i_1^* is onto (for, $\dim Q \times K = n + 2$), i_1^* is an isomorphism onto. The exactness implies that $Q \times K$ is non-cyclic and application of Theorem 3 shows that $Q \times K$ is inessential.

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