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On the Ends of the Fundamental Groups of 3-Manifolds with Boundary

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§ 1. Introduction

Let M be any compact 3-manifold, closed[1]) or with boundary, whose components are *orientable* closed surfaces. According to[2]) H. HOPF [2], the number e of ends of $\pi_1(M)$ is either 0 or 1 or 2 or ∞, where $e = 0$ if and only if $\pi_1(M)$ is finite. The naturally arising problem is: *When is* $e = 1$ *or* 2 *or* ∞*?* This problem has been solved by E. SPECKER [10], p. 325, Satz VI, in case M is *closed.* Thus the remaining question is: What is the solution of this problem when M has non-vacuous boundary? We notice that, if some of the components of the boundary of M are 2-spheres, then there exists a 3-manifold M' closed or with boundary, whose components are orientable closed surfaces of positive genus, and such that[3]) $\pi_1(M) \simeq \pi_1(M')$. Thus the problem may be stated: *Let M be a compact 3-manifold with boundary, whose components are orientable closed surfaces of positive genus. When is*[4]) $e = 1$ *or* 2 *or* ∞*?* To the best knowledge of this author, some partial results have been obtained by E. SPECKER [10], pp. 326–327, Sätze VII and VIII, and this author [5], p. 296, theorems 1 and 2. In the present paper we solve this problem, and the solution is:

(1) *If M is aspherical and the injection* $\pi_1(F) \to \pi_1(M)$ *is an isomorphism for every component F of the boundary of M. then* $e = 1$.

(2) *If M is aspherical, the boundary F of M is connected of genus one, and the injection* $\pi_1(F) \to \pi_1(M)$ *is not an isomorphism, then* $e = 2$.

(3) *In any other case,* $e = \infty$.

These are provided us by the theorem 6 in § 4, which is the main theorem of this paper. The proof of theorem 6 is based on theorems 1, 2 and 3 4, 5. The theorems 3, 4 are lent from authors paper [5], and the theorem 5 is lent from E. SPECKER's paper [10]. The theorems 1 and 2 are explained in § 3, and their proofs are based on the lemmas 1, 2, 3, 4 and 5 of § 2.

In § 5 we give a short proof of theorems 1 and 2, using DEHN's lemma [7], p. 169, [8], p. 1, and theorem 1, [6], p. 281.

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1) *Closed* means compact without boundary.

2) Numbers in brackets refer to the bibliography at the end of the paper.

3) $\simeq$ means isomorphic to.

4) According to lemma 3, $\pi_1(M)$ is infinite, therefore $e > 0$.

All 3-manifolds under consideration in this paper will be considered as having a certain fixed triangulation. This is possible according to E. E. MOISE's work [4].

The paper is presented here in a form suggested by Professor H. HOPF to the autor, who would like to express his gratitude to Professor H. HOPF for his suggestions.

§ 2. Five Lemmas

1. In the Nos. 1–2, M will mean a 3-manifold with boundary, and F will mean a component of its boundary, but in No. 3, F will mean an abstract surface.

Lemma 1. *Let L_1, L_2 be loops on F such that*[5]) $s(Z_1, Z_2) \neq 0$, *where Z_1, Z_2 are the 1-cycles corresponding to L_1, L_2. Then, at most one of Z_1, Z_2 is*[6]) ~ 0 *in M.*

Proof. Let us suppose that $Z_1 \sim 0$ in M. Then there exists a 2-chain C in M, such that $\partial C = Z_1$. Let $M^* = M \cup M'$ be the duplication[7]) of M, and let C' be the copy of $-C$ in M', where M' is the second copy of M. Then $Z^* = C + C'$ is a 2-cycle in M^*, and[5])[8])

$$s^*(Z^*, Z_2) = \pm s(Z_1, Z_2) \neq 0 \ .$$

Thus $Z_2 \nsim 0$ in M^*, and hence $Z_2 \nsim 0$ in M. This proves lemma 1.

Lemma 2. *If M is simply connected, then F is homeomorphic to a region of a 2-sphere.*

Proof. By lemma 1, any simple[9]) loop on F decomposes F, i. e. F is "schlichtartig" [3], p. 140. Therefore F can be imbedded in a 2-sphere, according to [3], p. 165. This proves lemma 2.

Lemma 3. *Let M be a compact 3-manifold with boundary, where some one of its components, say F, is a closed orientable surface of positive genus $g(F)$. Then $\pi_1(M)$ is infinite.*

Proof. Let us suppose that $\pi_1(M)$ is finite. Let $p: \tilde{M} \to M$ be the universal covering of M, and let $\tilde{F}$ be a boundary surface of $\tilde{M}$ lying over F.

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[5]) s means intersection numbers on F.

[6]) $\sim$ or $\sim_R$ means homologous to ..., over the integers or rationals respectively.

[7]) *Duplication* = Verdoppelung [9], pp. 129, 223. Actually there the duplication is defined for a *solid torus* (= Henkelkörper), but the generalization to any 3-manifold with boundary is immediate.

[8]) s^* means intersection numbers in M^*.

[9]) *Simple* means without multiple points.

Then $\tilde{F}$ is closed, because $\tilde{M}$ is compact, and therefore $g(\tilde{F})>0$. But by lemma 2, $g(\tilde{F})=0$. We arrived at a contradiction. This proves lemma 3.

2. Let us start this No. with some remarks about the ends of F and those of M, where M is supposed to be simply connected. By lemma 2, F may be thought of as a region of a 2-sphere S. The ends of F are precisely the components of $S-F$, and $\bar{F}=F\cup(\text{ends of } F)=S$, see [2], p. 86, No. 4. An open subset of S is a neighbourhood of any of its points in $\bar{F}$. According to [2], p. 87, No. 6, to any end ε of F there corresponds a unique end η of M.

Lemma 4. *Let M be simply connected, ε_1, ε_2 be two ends of F, such that $\varepsilon_1 \neq \varepsilon_2$, and let η_1, η_2 be the ends of M corresponding to ε_1, ε_2. Then $\eta_1 \neq \eta_2$.*

Proof. Let L be a simple loop on F separating ε_1, ε_2, i. e. there exist open connected sets on $\bar{F}$, say U_1, U_2, which are disjoint from L, neighbourhoods of ε_1, ε_2, and such that the intersection number of L with any path on F connecting U_1 and U_2 is ± 1. There exists a 2-cell D with self-intersections, such that[10]) $\mathrm{bdD}=L$, and $D-L\subset \operatorname{int} M$, because M is simply connected. Let C be the 2-chain corresponding to the oriented D, and let Z be the 1-cycle corresponding to L. Then $\partial C=Z$. By [2], p. 84, Nos. 2–3, the lemma 3 will be proved if we have shown, that each path P in M with initial point $p_1 \in U_1$ and final point $p_2 \in U_2$ meets the compact set D.

Let Q be a path on F with initial point p_1 and final point p_2, and let us consider the loop $L_0=PQ^{-1}$. Let V, W and Z_0 be the 1-chains and 1-cycle corresponding to P, Q and L_0. Then $V-W=Z_0 \sim 0$ in M, because M is simply connected. Let $M^*=M\cup M'$ be the duplication[7]) of M, and let C' be the copy of $-C$ in M', where M' is the second copy of M. Then $Z^*=C+C'$ is a 2-cycle in M^*, and[5])[8])

$$0=s^*(Z_0,Z^*)=s^*(V,Z^*)-s^*(W,Z^*)$$
$$s^*(W,Z^*)=\pm s(W,Z)=\pm 1\ .$$

Thus $s^*(V,Z^*)=\pm 1 \neq 0$. Hence $P\cap D\neq \emptyset$. This completes the proof of lemma 4.

3. In the preceding Nos. 1–2, F was a component of the boundary of the 3-manifold M. In the present one F will be an abstract surface.

Lemma 5. *Let $q:\ \tilde{F}\to F$ be a regular[11]) covering, where F is a closed orientable surface, and $\tilde{F}$ is homeomorphic to a cylinder. Then the genus $g(F)=1$.*

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[10]) int = interior, cl = closure, bd = boundary.

[11]) [9], § 57, p. 195.

Proof. Let $G = \pi_1(F)$ and[3]) $K = \pi_1(\tilde{F}) \cong Z$. Then K is a normal subgroup of G, because the covering q: $\tilde{F} \to F$ is regular, and $H = G/K$ is the group of covering translations.

According to [2], p. 96, No. 16[12]), $e(H) = e(\tilde{F}) = 2$, because $\tilde{F}$ is a cylinder. By [2], p. 97, Satz V, the group H has an infinite cyclic subgroup B with finite index in H. Let[13])

$$G' = G/[G,G],\ H' = H/[H,H],\ K' = K/(K \frown [G,G]) .$$

Then $G'/K' \cong H'$, and because G', H', K' are abelian we have by [1], p. 573, Satz[14])

$$r(G') = r(K') + r(H') .$$

$r(K') \leqq 1$, because $K \cong Z$. Abelianizing H we obtain the group H' which has an infinite cyclic subgroup B' with finite index in H', where B' is obtained from the subgroup B of H. Thus

$$r(H') = r(B') + r(H'/B') = 1 .$$

Hence $r(G') \leqq 2$. But G' is the 1-homology group, and $r(G')$ is the 1-Betti number of F. Thus $g(F) \leqq 1$. On the other hand $g(F) > 0$, because $\tilde{F}$ is infinite. Hence $g(F) = 1$. This proves lemma 5.

§ 3. Two Theorems

4. The conjecture in [5], p. 298, § 5, is a special case of the following

Theorem 1. *Let M be a compact 3-manifold with boundary, and let F be a component of its boundary, where F is an orientable surface of genus $g(F) > 1$, and the injection j: $\pi_1(F) \to \pi_1(M)$ is not an isomorphism. Then $\pi_1(M)$ has infinitely many ends.*

Proof. Let p: $\tilde{M} \to M$ be the universal covering of M, where $\tilde{M}$ has the induced triangulation, and let $\tilde{F}$ be a component of $p^{-1}(F)$. Then $\tilde{F}$ may be considered as a region of a 2-sphere S, by lemma 2. The number[12]) $e(\tilde{F})$ is equal to the number of the components of $S - \tilde{F}$, by No. 2. It is easily seen that

$$\pi_1(\tilde{F}) \cong j^{-1}(1) \neq 1$$

where the isomorphism is induced by the projection map p. Thus $\tilde{F}$ is not

[12]) e means the number of ends of a group or a space.

[13]) [,] means the commutator subgroup of.

[14]) r means the rank of an abelian group.

simply connected, and therefore $e(\tilde{F})>1$. The covering $q\colon \tilde{F}\to F$ is regular, where $q=p|\tilde{F}$, because $j^{-1}(1)$ is a normal subgroup of $\pi_1(F)$, [9], p. 195. Therefore, by lemma 5 and because $g(F)>1$, $\tilde{F}$ is not homeomorphic to a cylinder. Thus $e(\tilde{F})>2$. Hence $e(\tilde{F})=\infty$, by [2], p. 93, Satz II. Thus $e(\tilde{M})=\infty$, by lemma 4. Hence $e(\pi_1(M))=e(\tilde{M})=\infty$, by [2], p. 96, No. 16. This proves theorem 1.

Theorem 2. *Let M be a 3-manifold with boundary, which is not connected if M is compact, and let F be a component of its boundary, where F is an orientable closed surface of genus $g(F)=1$, and the injection $j\colon \pi_1(F)\to\pi_1(M)$ is not an isomorphism. Then $\pi_2(M)\neq 0$.*

Proof. Let us suppose that $\pi_2(M)=0$. We are going to prove that[6]) $\Phi\sim_R 0$, where Φ is the basic 2-cycle of F.

There exists on F a loop L which is[15]) $\simeq 0$ in M but $\not\simeq 0$ on F, because j is not an isomorphism. Let X, Y be the 1-cycles of two simple loops A, B on F, having only one point in common, and such that X, Y form a generating system of[16]) $H_1(F)$. Then there exist integers a, b, t such that[17])

$$V\sim t(aX+bY) \text{ on } F,\ (|a|,|b|)=1,\quad t\neq 0\ , \tag{1}$$

where V is the 1-cycle of L. Thus there exist integers c, d such that

$$ad-bc=1\ ,\quad (|c|,|d|)=1\ .$$

Hence the 1-cycles

$$X_1=aX+bY\ ,\quad Y_1=cX+dY \tag{2}$$

from a generating system of $H_1(F)$, and moreover there exist on F two simple loops A_1, B_1, having only one point in common, and such that their 1-cycles are X_1, Y_1 respectively.

Let F_0 be a torus and let X_0, Y_0 be the 1-cycles of two simple loops A_0, B_0 on F_0, having only one point in common, and such that X_0, Y_0 form a generating system of $H_1(F_0)$. Let $f\colon F_0\to F$ be a t-sheeted covering, such that $f(A_0)=A_1^t$, $f(B_0)=B_1$.

Let now C be a 2-cell such that bdC $=C\frown F_0=X_0$. Then the map f can be extended to a map $f'\colon C\smile F_0\to M$, because $f(A_0)\simeq 0$ in M, by $L\simeq 0$ in M, and

$$f(A_0)=A_1^t\simeq L \quad\text{on } F\ ,$$

where the last relations hold by (1) and (2).

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15) $\simeq$ means homotopic to.

16) H_1 means 1-homology group.

17) (,) means greatest common divisor.

Let $h\colon S \to C \cup F_0$ be a map, where S is a 2-sphere defined in the following way: two disjoint 2-cells C_1, C_2 on S are mapped homeomorphically on C such that[18]) $h(C_1) = C^+$, $h(C_2) = C^-$, and $S - C_1 - C_2$ is mapped homeomorphically on $F_0 - X_0$. So we have the following map $h' = f'h\colon S \to M$.

Let now Φ_0 and Ψ be the basic 2-cycles of F_0 and S. Then

$$f'(\Phi_0) = f(\Phi_0) = t\Phi, \quad \text{and} \quad h(\Psi) = \Phi_0 .$$

Therefore $h'(\Psi) = t\Phi$. But $h' \simeq 0$, because $\pi_2(M) = 0$. Hence

$$t\Phi = h'(\Psi) \sim 0 \quad \text{in } M .$$

Thus $\Phi \sim_R 0$ in M, as we asserted. This contradicts the fact that M is a 3-manifold, which is either infinite, or in case it is finite bdM is not connected. Hence $\pi_2(M) \neq 0$. This proves theorem 2.

§ 4. The main Theorem

5. The following three theorems will be used in the proof of the main theorem.

Theorem 3. *Let M be an aspherical compact 3-manifold with boundary, where the components are orientable closed surfaces of positive genus, such that the injection $\pi_1(F) \to \pi_1(M)$ is an isomorphism, for any component F of the boundary. Then $e(\pi_1(M)) = 1$.*

Proof. According to lemma 3, $\pi_1(M)$ is infinite. Hence $e(\pi_1(M)) = 1$, by [5], p. 296, theorem 1.

Theorem 4. *Let M be an aspherical 3-manifold whose boundary F is an orientable closed surface of genus one, and suppose that the injection $\pi_1(F) \to \pi_1(M)$ is not an isomorphism. Then M is compact and orientable and $\pi_1(M)$ is free cyclic, therefore $e(\pi_1(M)) = 2$.*

See [5], p. 296, theorem 2.

Theorem 5. *Let M be a compact 3-manifold with boundary, where the components are orientable surfaces of positive genus, such that $e(\pi_1(M)) = 1$ or 2. Then M is aspherical.*

This is a slight modification of Satz VII, [10], p. 326, and the proof of this is precisely the proof of Satz VII. The following corollary refines Satz VIII, [10], p. 327.

Corollary 1. *Let M be a compact 3-manifold with non-connected boundary,*

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18) Here we suppose that S and C have an arbitrary orientation, and that C_1 and C_2 have the induced orientation. C^+ and C^- mean the two orientations of C.

where the components are orientable surfaces of genus one, then $e = e(\pi_1(M)) = 1$ *or* ∞, *i. e. always* $\neq 2$.

Proof. If M is not aspherical, then $e = \infty$, by theorem 5. If M is aspherical, then the injection $\pi_1(F) \to \pi_1(M)$ is an isomorphism for any component F of bdM, by theorem 2. Hence $e = 1$, by theorem 3.

6. The following is the main theorem of the paper.

Theorem 6. *Let M be a compact 3-manifold with boundary, where the components are orientable closed surfaces of positive genus. Then*[19]) $e = e(\pi_1(M)) \neq 0$, *and the following hold*

(1) *If M is aspherical and the injection* $\pi_1(F) \to \pi_1(M)$ *is an isomorphism, for any component F of the boundary of M, then* $e = 1$.

(2) *If M is aspherical, its boundary F is connected and has genus* $g(F) = 1$, *and the injection* $\pi_1(F) \to \pi_1(M)$ *is not an isomorphism, then* $e = 2$.

(3) *In any other case (i. e. if either M is not aspherical; or M is aspherical, and there is a component F of the boundary, of genus* $g(F) > 1$, *such that the injection* $\pi_1(F) \to \pi_1(M)$ *is not an isomorphism)* $e = \infty$.

Proof. If M is not aspherical, then $e = \infty$, by theorem 5. Let us from now on suppose that M is aspherical. If the hypotheses of (1) hold, then $e = 1$, by theorem 3. Let us from now on suppose that the hypotheses of (1) do not hold, i. e. there is a component F of the boundary of M, such that the injection $\pi_1(F) \to \pi_1(M)$ is not an isomorphism. We have to consider the following two cases: (i) The boundary of M is not connected. (ii) The boundary of M is connected.

Case (i): By theorem 2, the genus $g(F) > 1$, because M is aspherical. Hence $e = \infty$, by theorem 1.

Case (ii): In this case F is the whole boundary of M. If the genus $g(F) = 1$, then $e = 2$, by theorem 4. If $g(F) > 1$, then $e = \infty$, by theorem 1.

From the above we conclude the truth of theorem 6.

§ 5. Applications of Dehn's Lemma

7. We now are going to give a short proof of theorems 1 and 2, using Dehn's *lemma* [7], p. 169, [8], p. 1, and theorem 1, [6], p. 281, which we shall call the *loop theorem* for convenience.

In both theorems 1 and 2, there exists a loop L on F which is $\simeq 0$ in M, and $\not\simeq 0$ on F, because j is not an isomorphism. Thus there exists a simple[9])

[19]) By lemma 3, $\pi_1(M)$ is infinite, therefore $e \neq 0$.

loop L_0 on F which is $\simeq 0$ in M, and $\not\sim 0$ on F, by the loop theorem. Hence there exists a 2-cell D, such that[10]) $\mathrm{bd}D = L_0$, and $D - L_0 \subset \mathrm{int} M$, by DEHN's lemma. Let M_0 be the 3-manifold with boundary we obtain cutting M along D, and let F_0 be the component of bdM_0, which comes from F, and on which there lie the two copies D_0' and D_0'' of D. Let p_0: $\tilde{M}_0 \to M_0$ be the universal covering of M_0, and let D_{0j}', D_{0j}'', $j = 1,2,\ldots,$ be the 2-cells on $bd\tilde{M}_0$ lying over D_0', D_0''. The universal covering p: $\tilde{M} \to M$ of M is composed of a (denumerable) number of copies of $\tilde{M}_0$, and is obtained glueing the D_{0j}' of a copy with the D_{0j}'' of another copy, in such a way that we obtain a simply connected complex. Let us denote by $\tilde{D}_k$, $k = 1,2,\ldots,$ the 2-cells lying over D.

Let us now pass to the proof of theorem 1: By lemma 1, there is a simple loop L_1 on F_0, which is $\not\sim 0$ in M_0, because the genus $g(F_0) > 0$. As we can easily see, using the loop theorem, $L_1^s \not\sim 0$ in M_0 for any natural number s[20]). This implies that $\pi_1(M_0)$ is infinite, and thus $\tilde{M}_0$ is not compact. As we can easily see, using the cells $\tilde{D}_k$, $k = 1,2,\ldots,$ $\tilde{M}$ has infinitely many ends. This proves theorem 1.

Let us now pass to the proof of theorem 2: F_0 is a 2-sphere, therefore $p_0^{-1}(F_0)$ consists of 2-spheres. But $\tilde{M}_0$ is at least either not compact, or $bd\tilde{M}_0 - p_0^{-1}(F_0) \neq \emptyset$. Thus, if $\tilde{X}_0$ is a basic 2-cycle of a component $\tilde{F}_0$ of $p_0^{-1}(F_0)$, $\tilde{X}_0 \not\sim 0$ in $\tilde{M}$. Hence, by standard Hurewicz theorems, $\pi_2(M) \neq 0$. This proves theorem 2.

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[20]) Compare also [6], p. 287, lemma (9.3).