

Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 32 (1957-1958)

Artikel: On the Ends of the Fundamental Groups of 3-Manifolds with Boundary.
Autor: Papakyriakopoulos, C.D.
DOI: <https://doi.org/10.5169/seals-25337>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 16.02.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

On the Ends of the Fundamental Groups of 3-Manifolds with Boundary

by C. D. PAPAKYRIAKOPOULOS, Princeton (N. J.)

§ 1. Introduction

Let M be any compact 3-manifold, closed¹⁾ or with boundary, whose components are *orientable* closed surfaces. According to²⁾ H. HOPF [2], the number e of ends of $\pi_1(M)$ is either 0 or 1 or 2 or ∞ , where $e = 0$ if and only if $\pi_1(M)$ is finite. The naturally arising problem is: *When is $e = 1$ or 2 or ∞ ?* This problem has been solved by E. SPECKER [10], p. 325, Satz VI, in case M is *closed*. Thus the remaining question is: What is the solution of this problem when M has non-vacuous boundary? We notice that, if some of the components of the boundary of M are 2-spheres, then there exists a 3-manifold M' closed or with boundary, whose components are orientable closed surfaces of positive genus, and such that³⁾ $\pi_1(M) \cong \pi_1(M')$. Thus the problem may be stated: *Let M be a compact 3-manifold with boundary, whose components are orientable closed surfaces of positive genus. When is⁴⁾ $e = 1$ or 2 or ∞ ?* To the best knowledge of this author, some partial results have been obtained by E. SPECKER [10], pp. 326–327, Sätze VII and VIII, and this author [5], p. 296, theorems 1 and 2. In the present paper we solve this problem, and the solution is:

- (1) *If M is aspherical and the injection $\pi_1(F) \rightarrow \pi_1(M)$ is an isomorphism for every component F of the boundary of M , then $e = 1$.*
- (2) *If M is aspherical, the boundary F of M is connected of genus one, and the injection $\pi_1(F) \rightarrow \pi_1(M)$ is not an isomorphism, then $e = 2$.*
- (3) *In any other case, $e = \infty$.*

These are provided us by the theorem 6 in § 4, which is the main theorem of this paper. The proof of theorem 6 is based on theorems 1, 2 and 3 4, 5. The theorems 3, 4 are lent from authors paper [5], and the theorem 5 is lent from E. SPECKER's paper [10]. The theorems 1 and 2 are explained in § 3, and their proofs are based on the lemmas 1, 2, 3, 4 and 5 of § 2.

In § 5 we give a short proof of theorems 1 and 2, using DEHN's lemma [7], p. 169, [8], p. 1, and theorem 1, [6], p. 281.

¹⁾ *Closed* means compact without boundary.

²⁾ Numbers in brackets refer to the bibliography at the end of the paper.

³⁾ \sim means isomorphic to.

⁴⁾ According to lemma 3, $\pi_1(M)$ is infinite, therefore $e > 0$.

All 3-manifolds under consideration in this paper will be considered as having a certain fixed triangulation. This is possible according to E. E. MOISE's work [4].

The paper is presented here in a form suggested by Professor H. HOPF to the autor, who would like to express his gratitude to Professor H. HOPF for his suggestions.

§ 2. Five Lemmas

1. In the Nos. 1-2, M will mean a 3-manifold with boundary, and F will mean a component of its boundary, but in No. 3, F will mean an abstract surface.

Lemma 1. *Let L_1, L_2 be loops on F such that⁵⁾ $s(Z_1, Z_2) \neq 0$, where Z_1, Z_2 are the 1-cycles corresponding to L_1, L_2 . Then, at most one of Z_1, Z_2 is⁶⁾ ~ 0 in M .*

Proof. Let us suppose that $Z_1 \sim 0$ in M . Then there exists a 2-chain C in M , such that $\partial C = Z_1$. Let $M^* = M \cup M'$ be the duplication⁷⁾ of M , and let C' be the copy of $-C$ in M' , where M' is the second copy of M . Then $Z^* = C + C'$ is a 2-cycle in M^* , and⁵⁾⁸⁾

$$s^*(Z^*, Z_2) = \pm s(Z_1, Z_2) \neq 0.$$

Thus $Z_2 \not\sim 0$ in M^* , and hence $Z_2 \not\sim 0$ in M . This proves lemma 1.

Lemma 2. *If M is simply connected, then F is homeomorphic to a region of a 2-sphere.*

Proof. By lemma 1, any simple⁹⁾ loop on F decomposes F , i. e. F is "schlicht-
artig" [3], p. 140. Therefore F can be imbedded in a 2-sphere, according to [3], p. 165. This proves lemma 2.

Lemma 3. *Let M be a compact 3-manifold with boundary, where some one of its components, say F , is a closed orientable surface of positive genus $g(F)$. Then $\pi_1(M)$ is infinite.*

Proof. Let us suppose that $\pi_1(M)$ is finite. Let $p: \tilde{M} \rightarrow M$ be the universal covering of M , and let \tilde{F} be a boundary surface of \tilde{M} lying over F .

⁵⁾ s means intersection numbers on F .

⁶⁾ \sim or \sim_R means homologous to ..., over the integers or rationals respectively.

⁷⁾ *Duplication* = Verdoppelung [9], pp. 129, 223. Actually there the duplication is defined for a *solid torus* (= Henkelkörper), but the generalization to any 3-manifold with boundary is immediate.

⁸⁾ s^* means intersection numbers in M^* .

⁹⁾ *Simple* means without multiple points.

Then \tilde{F} is closed, because \tilde{M} is compact, and therefore $g(\tilde{F}) > 0$. But by lemma 2, $g(\tilde{F}) = 0$. We arrived at a contradiction. This proves lemma 3.

2. Let us start this No. with some remarks about the ends of F and those of M , where M is supposed to be simply connected. By lemma 2, F may be thought of as a region of a 2-sphere S . The ends of F are precisely the components of $S - F$, and $\bar{F} = F \cup (\text{ends of } F) = S$, see [2], p. 86, No. 4. An open subset of S is a neighbourhood of any of its points in \bar{F} . According to [2], p. 87, No. 6, to any end ε of F there corresponds a unique end η of M .

Lemma 4. *Let M be simply connected, $\varepsilon_1, \varepsilon_2$ be two ends of F , such that $\varepsilon_1 \neq \varepsilon_2$, and let η_1, η_2 be the ends of M corresponding to $\varepsilon_1, \varepsilon_2$. Then $\eta_1 \neq \eta_2$.*

Proof. Let L be a simple loop on F separating $\varepsilon_1, \varepsilon_2$, i. e. there exist open connected sets on \bar{F} , say U_1, U_2 , which are disjoint from L , neighbourhoods of $\varepsilon_1, \varepsilon_2$, and such that the intersection number of L with any path on F connecting U_1 and U_2 is ± 1 . There exists a 2-cell D with self-intersections, such that¹⁰⁾ $\text{bd}D = L$, and $D - L \subset \text{int } M$, because M is simply connected. Let C be the 2-chain corresponding to the oriented D , and let Z be the 1-cycle corresponding to L . Then $\partial C = Z$. By [2], p. 84, Nos. 2–3, the lemma 3 will be proved if we have shown, that each path P in M with initial point $p_1 \in U_1$ and final point $p_2 \in U_2$ meets the compact set D .

Let Q be a path on F with initial point p_1 and final point p_2 , and let us consider the loop $L_0 = PQ^{-1}$. Let V, W and Z_0 be the 1-chains and 1-cycle corresponding to P, Q and L_0 . Then $V - W = Z_0 \sim 0$ in M , because M is simply connected. Let $M^* = M \cup M'$ be the duplication⁷⁾ of M , and let C' be the copy of $-C$ in M' , where M' is the second copy of M . Then $Z^* = C + C'$ is a 2-cycle in M^* , and⁵⁾⁸⁾

$$0 = s^*(Z_0, Z^*) = s^*(V, Z^*) - s^*(W, Z^*)$$

$$s^*(W, Z^*) = \pm s(W, Z) = \pm 1.$$

Thus $s^*(V, Z^*) = \pm 1 \neq 0$. Hence $P \cap D \neq \emptyset$. This completes the proof of lemma 4.

3. In the preceding Nos. 1–2, F was a component of the boundary of the 3-manifold M . In the present one F will be an abstract surface.

Lemma 5. *Let $q: \tilde{F} \rightarrow F$ be a regular¹¹⁾ covering, where F is a closed orientable surface, and \tilde{F} is homeomorphic to a cylinder. Then the genus $g(F) = 1$.*

¹⁰⁾ $\text{int} = \text{interior}$, $\text{cl} = \text{closure}$, $\text{bd} = \text{boundary}$.

¹¹⁾ [9], § 57, p. 195.

Proof. Let $G = \pi_1(F)$ and³⁾ $K = \pi_1(\tilde{F}) \cong \mathbb{Z}$. Then K is a normal subgroup of G , because the covering $q: \tilde{F} \rightarrow F$ is regular, and $H = G/K$ is the group of covering translations.

According to [2], p. 96, No. 16¹²⁾, $e(H) = e(\tilde{F}) = 2$, because \tilde{F} is a cylinder. By [2], p. 97, Satz V, the group H has an infinite cyclic subgroup B with finite index in H . Let¹³⁾

$$G' = G/[G, G], \quad H' = H/[H, H], \quad K' = K/(K \cap [G, G]).$$

Then $G'/K' \cong H'$, and because G', H', K' are abelian we have by [1], p. 573, Satz¹⁴⁾

$$r(G') = r(K') + r(H').$$

$r(K') \leq 1$, because $K \cong \mathbb{Z}$. Abelianizing H we obtain the group H' which has an infinite cyclic subgroup B' with finite index in H' , where B' is obtained from the subgroup B of H . Thus

$$r(H') = r(B') + r(H'/B') = 1.$$

Hence $r(G') \leq 2$. But G' is the 1-homology group, and $r(G')$ is the 1-Betti number of F . Thus $g(F) \leq 1$. On the other hand $g(F) > 0$, because \tilde{F} is infinite. Hence $g(F) = 1$. This proves lemma 5.

§ 3. Two Theorems

4. The conjecture in [5], p. 298, § 5, is a special case of the following

Theorem 1. *Let M be a compact 3-manifold with boundary, and let F be a component of its boundary, where F is an orientable surface of genus $g(F) > 1$, and the injection $j: \pi_1(F) \rightarrow \pi_1(M)$ is not an isomorphism. Then $\pi_1(M)$ has infinitely many ends.*

Proof. Let $p: \tilde{M} \rightarrow M$ be the universal covering of M , where \tilde{M} has the induced triangulation, and let \tilde{F} be a component of $p^{-1}(F)$. Then \tilde{F} may be considered as a region of a 2-sphere S , by lemma 2. The number¹²⁾ $e(\tilde{F})$ is equal to the number of the components of $S - \tilde{F}$, by No. 2. It is easily seen that

$$\pi_1(\tilde{F}) \cong j^{-1}(1) \neq 1$$

where the isomorphism is induced by the projection map p . Thus \tilde{F} is not

¹²⁾ e means the number of ends of a group or a space.

¹³⁾ $[,]$ means the commutator subgroup of.

¹⁴⁾ r means the rank of an abelian group.

simply connected, and therefore $e(\tilde{F}) > 1$. The covering $q: \tilde{F} \rightarrow F$ is regular, where $q = p|_{\tilde{F}}$, because $j^{-1}(1)$ is a normal subgroup of $\pi_1(F)$, [9], p. 195. Therefore, by lemma 5 and because $g(F) > 1$, \tilde{F} is not homeomorphic to a cylinder. Thus $e(\tilde{F}) > 2$. Hence $e(\tilde{F}) = \infty$, by [2], p. 93, Satz II. Thus $e(\tilde{M}) = \infty$, by lemma 4. Hence $e(\pi_1(M)) = e(\tilde{M}) = \infty$, by [2], p. 96, No. 16. This proves theorem 1.

Theorem 2. *Let M be a 3-manifold with boundary, which is not connected if M is compact, and let F be a component of its boundary, where F is an orientable closed surface of genus $g(F) = 1$, and the injection $j: \pi_1(F) \rightarrow \pi_1(M)$ is not an isomorphism. Then $\pi_2(M) \neq 0$.*

Proof. Let us suppose that $\pi_2(M) = 0$. We are going to prove that⁶⁾ $\Phi \sim_R 0$, where Φ is the basic 2-cycle of F .

There exists on F a loop L which is¹⁵⁾ $\simeq 0$ in M but $\neq 0$ on F , because j is not an isomorphism. Let X, Y be the 1-cycles of two simple loops A, B on F , having only one point in common, and such that X, Y form a generating system of¹⁶⁾ $H_1(F)$. Then there exist integers a, b, t such that¹⁷⁾

$$V \sim t(aX + bY) \text{ on } F, \quad (|a|, |b|) = 1, \quad t \neq 0, \quad (1)$$

where V is the 1-cycle of L . Thus there exist integers c, d such that

$$ad - bc = 1, \quad (|c|, |d|) = 1.$$

Hence the 1-cycles

$$X_1 = aX + bY, \quad Y_1 = cX + dY \quad (2)$$

from a generating system of $H_1(F)$, and moreover there exist on F two simple loops A_1, B_1 , having only one point in common, and such that their 1-cycles are X_1, Y_1 respectively.

Let F_0 be a torus and let X_0, Y_0 be the 1-cycles of two simple loops A_0, B_0 on F_0 , having only one point in common, and such that X_0, Y_0 form a generating system of $H_1(F_0)$. Let $f: F_0 \rightarrow F$ be a t -sheeted covering, such that $f(A_0) = A_1^t, f(B_0) = B_1$.

Let now C be a 2-cell such that $\text{bd}C = C \cap F_0 = X_0$. Then the map f can be extended to a map $f': C \cup F_0 \rightarrow M$, because $f(A_0) \simeq 0$ in M , by $L \simeq 0$ in M , and

$$f(A_0) = A_1^t \simeq L \text{ on } F,$$

where the last relations hold by (1) and (2).

¹⁵⁾ \simeq means homotopic to.

¹⁶⁾ H_1 means 1-homology group.

¹⁷⁾ (,) means greatest common divisor.

Let $h: S \rightarrow C \cup F_0$ be a map, where S is a 2-sphere defined in the following way: two disjoint 2-cells C_1, C_2 on S are mapped homeomorphically on C such that¹⁸⁾ $h(C_1) = C^+$, $h(C_2) = C^-$, and $S - C_1 - C_2$ is mapped homeomorphically on $F_0 - X_0$. So we have the following map $h' = f'h: S \rightarrow M$.

Let now Φ_0 and Ψ be the basic 2-cycles of F_0 and S . Then

$$f'(\Phi_0) = f(\Phi_0) = t\Phi, \quad \text{and} \quad h(\Psi) = \Phi_0.$$

Therefore $h'(\Psi) = t\Phi$. But $h' \simeq 0$, because $\pi_2(M) = 0$. Hence

$$t\Phi = h'(\Psi) \sim 0 \quad \text{in } M.$$

Thus $\Phi \sim_R 0$ in M , as we asserted. This contradicts the fact that M is a 3-manifold, which is either infinite, or in case it is finite $\text{bd}M$ is not connected. Hence $\pi_2(M) \neq 0$. This proves theorem 2.

§ 4. The main Theorem

5. The following three theorems will be used in the proof of the main theorem.

Theorem 3. *Let M be an aspherical compact 3-manifold with boundary, where the components are orientable closed surfaces of positive genus, such that the injection $\pi_1(F) \rightarrow \pi_1(M)$ is an isomorphism, for any component F of the boundary. Then $e(\pi_1(M)) = 1$.*

Proof. According to lemma 3, $\pi_1(M)$ is infinite. Hence $e(\pi_1(M)) = 1$, by [5], p. 296, theorem 1.

Theorem 4. *Let M be an aspherical 3-manifold whose boundary F is an orientable closed surface of genus one, and suppose that the injection $\pi_1(F) \rightarrow \pi_1(M)$ is not an isomorphism. Then M is compact and orientable and $\pi_1(M)$ is free cyclic, therefore $e(\pi_1(M)) = 2$.*

See [5], p. 296, theorem 2.

Theorem 5. *Let M be a compact 3-manifold with boundary, where the components are orientable surfaces of positive genus, such that $e(\pi_1(M)) = 1$ or 2. Then M is aspherical.*

This is a slight modification of Satz VII, [10], p. 326, and the proof of this is precisely the proof of Satz VII. The following corollary refines Satz VIII, [10], p. 327.

Corollary 1. *Let M be a compact 3-manifold with non-connected boundary,*

¹⁸⁾ Here we suppose that S and C have an arbitrary orientation, and that C_1 and C_2 have the induced orientation. C^+ and C^- mean the two orientations of C .

where the components are orientable surfaces of genus one, then $e = e(\pi_1(M)) = 1$ or ∞ , i. e. always $\neq 2$.

Proof. If M is not aspherical, then $e = \infty$, by theorem 5. If M is aspherical, then the injection $\pi_1(F) \rightarrow \pi_1(M)$ is an isomorphism for any component F of ∂M , by theorem 2. Hence $e = 1$, by theorem 3.

6. The following is the main theorem of the paper.

Theorem 6. *Let M be a compact 3-manifold with boundary, where the components are orientable closed surfaces of positive genus. Then¹⁹⁾ $e = e(\pi_1(M)) \neq 0$, and the following hold*

(1) *If M is aspherical and the injection $\pi_1(F) \rightarrow \pi_1(M)$ is an isomorphism, for any component F of the boundary of M , then $e = 1$.*

(2) *If M is aspherical, its boundary F is connected and has genus $g(F) = 1$, and the injection $\pi_1(F) \rightarrow \pi_1(M)$ is not an isomorphism, then $e = 2$.*

(3) *In any other case (i. e. if either M is not aspherical; or M is aspherical, and there is a component F of the boundary, of genus $g(F) > 1$, such that the injection $\pi_1(F) \rightarrow \pi_1(M)$ is not an isomorphism) $e = \infty$.*

Proof. If M is not aspherical, then $e = \infty$, by theorem 5. Let us from now on suppose that M is aspherical. If the hypotheses of (1) hold, then $e = 1$, by theorem 3. Let us from now on suppose that the hypotheses of (1) do not hold, i. e. there is a component F of the boundary of M , such that the injection $\pi_1(F) \rightarrow \pi_1(M)$ is not an isomorphism. We have to consider the following two cases: (i) The boundary of M is not connected. (ii) The boundary of M is connected.

Case (i): By theorem 2, the genus $g(F) > 1$, because M is aspherical. Hence $e = \infty$, by theorem 1.

Case (ii): In this case F is the whole boundary of M . If the genus $g(F) = 1$, then $e = 2$, by theorem 4. If $g(F) > 1$, then $e = \infty$, by theorem 1.

From the above we conclude the truth of theorem 6.

§ 5. Applications of DEHN's Lemma

7. We now are going to give a short proof of theorems 1 and 2, using DEHN's lemma [7], p. 169, [8], p. 1, and theorem 1, [6], p. 281, which we shall call the *loop theorem* for convenience.

In both theorems 1 and 2, there exists a loop L on F which is $\simeq 0$ in M , and $\not\simeq 0$ on F , because j is not an isomorphism. Thus there exists a simple²⁰⁾

¹⁹⁾ By lemma 3, $\pi_1(M)$ is infinite, therefore $e \neq 0$.

loop L_0 on F which is $\simeq 0$ in M , and $\not\simeq 0$ on F , by the loop theorem. Hence there exists a 2-cell D , such that¹⁰⁾ $\text{bd}D = L_0$, and $D - L_0 \subset \text{int } M$, by DEHN's lemma. Let M_0 be the 3-manifold with boundary we obtain cutting M along D , and let F_0 be the component of $\text{bd}M_0$, which comes from F , and on which there lie the two copies D'_0 and D''_0 of D . Let $p_0: \tilde{M}_0 \rightarrow M_0$ be the universal covering of M_0 , and let D'_{0j}, D''_{0j} , $j = 1, 2, \dots$, be the 2-cells on $\text{bd}\tilde{M}_0$ lying over D'_0, D''_0 . The universal covering $p: \tilde{M} \rightarrow M$ of M is composed of a (denumerable) number of copies of \tilde{M}_0 , and is obtained glueing the D'_{0j} of a copy with the D''_{0j} of another copy, in such a way that we obtain a simply connected complex. Let us denote by \tilde{D}_k , $k = 1, 2, \dots$, the 2-cells lying over D .

Let us now pass to the proof of theorem 1: By lemma 1, there is a simple loop L_1 on F_0 , which is $\not\simeq 0$ in M_0 , because the genus $g(F_0) > 0$. As we can easily see, using the loop theorem, $L_1^s \not\simeq 0$ in M_0 for any natural number s ²⁰⁾. This implies that $\pi_1(M_0)$ is infinite, and thus \tilde{M}_0 is not compact. As we can easily see, using the cells \tilde{D}_k , $k = 1, 2, \dots$, \tilde{M} has infinitely many ends. This proves theorem 1.

Let us now pass to the proof of theorem 2: F_0 is a 2-sphere, therefore $p_0^{-1}(F_0)$ consists of 2-spheres. But \tilde{M}_0 is at least either not compact, or $\text{bd}\tilde{M}_0 - p_0^{-1}(F_0) \neq \emptyset$. Thus, if \tilde{X}_0 is a basic 2-cycle of a component \tilde{F}_0 of $p_0^{-1}(F_0)$, $\tilde{X}_0 \not\simeq 0$ in \tilde{M} . Hence, by standard Hurewicz theorems, $\pi_2(M) \neq 0$. This proves theorem 2.

BIBLIOGRAPHY

- [1] P. ALEXANDROFF and H. HOPF, *Topologie*. Springer, Berlin 1935.
- [2] H. HOPF, *Enden offener Räume und unendliche diskontinuierliche Gruppen*. Comment. Math. Helv. 16 (1944) 81–100.
- [3] B. v. KERÉKJÁRTÓ, *Vorlesungen über Topologie*. Springer, Berlin 1923.
- [4] E. E. MOISE, *Affine structures in 3-manifolds V and VIII*. Ann. Math. 56 (1952) 96–114 and 59 (1954) 159–170.
- [5] C. D. PAPAKYRIAKOPOULOS, *On the ends of knot groups*. Ann. Math. 62 (1955) 293–299.
- [6] C. D. PAPAKYRIAKOPOULOS, *On solid tori*. Proc. London Math. Soc. (3) 7 (1957) 281–299.
- [7] C. D. PAPAKYRIAKOPOULOS, *On Dehn's lemma and the asphericity of knots*. Proc. Nat. Acad. Sci. U. S. A. 43 (1957) 169–172.
- [8] C. D. PAPAKYRIAKOPOULOS, *On Dehn's lemma and the asphericity of knots*. Ann. Math. 66 (1957) 1–26.
- [9] H. SEIFERT and W. THRELFALL, *Lehrbuch der Topologie*. Teubner, Leipzig-Berlin 1934.
- [10] E. SPECKER, *Die erste Cohomologiegruppe von Überlagerungen und Homotopieeigenschaften dreidimensionaler Mannigfaltigkeiten*. Comment. Math. Helv. 23 (1949) 303–332.

(Received July 28, 1956, in revised form August 22, 1957)

²⁰⁾ Compare also [6], p. 287, lemma (9.3).