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On subharmonic functions and differential geometry in the large*)

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1. Introduction

We consider an *open, two-dimensional* RIEMANN*ian manifold* M whose metric is defined by a positive definite quadratic form

$$ds^2 = E(\xi,\eta)d\xi^2 + 2F(\xi,\eta)d\xi d\eta + G(\xi,\eta)d\eta^2 , \qquad (1.1)$$

ξ and η denoting local parameters. If E, F and G are sufficiently regular, then it is possible to introduce (local) *isothermic parameters*, i. e. there exists a coordinate transformation $x = x(\xi,\eta)$, $y = y(\xi,\eta)$ such that $E = G > 0$, $F = 0$ in the (x,y)-parameter system. Then we can write

$$ds^2 = e^{2u(x,y)}(dx^2 + dy^2) = e^{2u(z)}|dz|^2 , \qquad (1.2)$$

putting $z = x + iy$. Such a transformation always exists, for example, when E, F and G are of class C^3, and in this case the corresponding function u is also of class C^3 (cf. A. WINTNER [34, p. 687]).

By the Theorema egregium the GAUSS*ian curvature* K can be calculated from the E, F, G and their partial derivatives up to the second order. In the isothermic parameter system (1.2) one obtains the particularly simple expression

$$K = -e^{-2u}\Delta u \qquad (\Delta \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2) . \qquad (1.3)$$

Hence, letting $dA = e^{2u}dxdy$ denote the area element on M we have

$$KdA = -\Delta u dx dy . \qquad (1.4)$$

Furthermore, one finds after some calculation (using e. g. [6, p. 175]) the following expression for the *geodesic curvature* k of a curve on M

$$k = e^{-u}\left(k_e + \frac{\partial u}{\partial n}\right) . \qquad (1.5)$$

Here k_e denotes the euclidean curvature of the corresponding curve $z = z(t)$ in the z-plane with the convention that $\operatorname{sign} k_e \equiv \operatorname{sign} \frac{d}{dt}\left[\arg \frac{dz}{dt}\right]$, and n

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designates the normal to $z(t)$ in the direction $\arg\left(-i\frac{dz}{dt}\right)$. (1.2) and (1.5) imply

$$kds = \left(k_e + \frac{\partial u}{\partial n}\right)|dz| \ . \tag{1.6}$$

In general, isothermic parameters can only be introduced in the small. In order to be able to treat problems pertaining to differential geometry in the large we have to consider the RIEMANN *surface* S which is determined by the conformal structure of M. (For a detailed discussion of this step the reader is referred to [31, pp. 2–5]. At this point we have to introduce the additional condition that M is *orientable*. However, if this should not be the case we simply replace M by an orientable, two-sheeted covering surface [33, p. 61]). The local uniformizers are then defined as functions which map a portion of M conformally onto a region in the z-plane. Hence their real and imaginary parts form a set of local isothermic parameters. Conversely, if x and y are local isothermic parameters, then either $x + iy$ or $y + ix$ constitutes a local uniformizer.

We thus are led to conceive of M as a RIEMANN surface on which a *conformal metric*

$$ds = e^{u(z)}|dz| \qquad (z = \text{local uniformizer}) \tag{1.7}$$

has been introduced. Thereby a change of uniformizers $z = \varphi(\zeta)$ implies the transformation

$$\tilde{u}(\zeta) = u(\varphi(\zeta)) + \log|\varphi'(\zeta)| \ , \tag{1.8}$$

due to the conformal invariance of $ds = e^{u(z)}|dz| = e^{\tilde{u}(\zeta)}|d\zeta|$.

We shall mainly be concerned with the relation between the surface integral of the GAUSSian curvature (curvatura integra) and the topological and conformal structures of complete, open, two-dimensional RIEMANNian manifolds. (According to the definition of H. HOPF and W. RINOW [17] the manifold M is called *complete* if every divergent path on M has infinite length. A path s is said to be *divergent* (or to *tend to the ideal boundary* of M) if (1) s is the topological image $p = p(t)$ of the half-open interval $0 \leqq t < 1$, (2) given an arbitrary subcompact K of M there always exists a number $t'(K) < 1$ such that $p(t)$ lies outside K for $t > t'$).

The present article originated from a suggestion of Professor H. HOPF. He drew our attention to the connection between differential geometry and potential theory which is revealed by relations (1.3) and (1.4). For example, the function $u(x,y)$ is subharmonic in a certain (x,y)-parameter region if and only if $K \leqq 0$ in the corresponding domain on M. (This fact had already been used by E. F. BECKENBACH and T. RADÓ [3] in their proof of the isoperimetric inequality on surfaces of negative curvature.) Analogously, u is superharmonic if and only if $K \geqq 0$. Furthermore, (1.4) discloses an even deeper connection: The surface integral of K, considered as a set function, is essentially the measure associated with u (i. e. the mass distribution of density $\Delta u/2\pi$, which appears when u is represented as a sum of a logarithmic potential and a harmonic function). Consequently, results of differential geometry in the large involving the curvatura integra, such as those due to S. COHN-VOSSEN [10], F. FIALA [12], CH. BLANC and F. FIALA [5] (see H. HOPF [16] for further references), have a potentialtheoretical meaning. It is therefore natural to apply functiontheoretical methods to this field in the hope that not only other (and eventually simpler) proofs of known results will be found, but also theorems which are new in both their differentialgeometrical and potentialtheoretical aspects. From this viewpoint [18], [19] and the present paper have been written.

Our geometrical results are contained in sections 4, 5 and 6. We consider manifolds which are given in the form (1.7), assuming merely that u can be represented as a difference of subharmonic functions[1]),

$$u(z) = u_1(z) - u_2(z) \qquad (z = \text{local uniformizer}). \tag{1.9}$$

Of course, neither the functions u_1 and u_2 nor their associated measures μ_1 and μ_2 are uniquely determined. However, the difference $\mu = \mu_1 - \mu_2$ does not depend on the choice of decomposition or uniformizer. Consequently, μ is defined as a measure on S. Let $\mu = \mu^+ - \mu^-$ denote the JORDAN decomposition [30, p. 11] of μ, which can be characterized by the property that

$$\mu^+(e) \leqq \mu_1(e) \quad \text{and} \quad \mu^-(e) \leqq \mu_2(e)$$

for all BOREL sets e and any representation $\mu = \mu_1 - \mu_2$ of μ as a difference of positive measures. Further, let $C^+ = 2\pi\mu^-(S)$ and $C^- = 2\pi\mu^+(S)$. The difference $C = C^+ - C^- = -2\pi\mu(S)$, defined whenever C^+ and C^- are not both infinite, will be called the *curvatura integra* of the metric. This terminology is justified, since for sufficiently regular u we have indeed, by (1.4),

$$C^+ = \iint_S \Delta^- u\,dx\,dy = \iint_M K^+ dA \ ,$$

$$C^- = \iint_S \Delta^+ u\,dx\,dy = \iint_M K^- dA$$

and

$$C = -\iint_S \Delta u\,dx\,dy = \iint_M K\,dA \ ,$$

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[1]) Such a representation always exists, for example, when u is of class C^2. For the definition and properties of subharmonic functions the reader may consult the book of T. RADÓ [25].

where $\Delta^+ u = \max[\Delta u, 0]$, $\Delta^- u = \max[-\Delta u, 0]$, $K^+ = \max[K, 0]$ and $K^- = \max[-K, 0]$.

From the viewpoint of potential theory (1.9) is the most natural condition to impose on u. (We mention that such a metric has already been considered by A. BEURLING [4]. He restricted himself to the case of negative curvature, i. e. subharmonic u.) The following remark may illustrate that this generality is also useful for differentialgeometrical purposes: The theory presented here applies to all RIEMANNian manifolds whose coefficients E, F and G in (1.1) are of class C^1 and which possess a continuous GAUSSian curvature in the sense of H. WEYL [32, pp. 43–44]. This is a consequence of results due to S. S. CHERN, P. HARTMAN and A. WINTNER [9] who demonstrated that under these hypotheses isothermic parameters can be introduced, the corresponding function u being of class C^1. Furthermore, one deduces easily from [9] that u is representable in the form (1.9) and that the curvatura integra C (in the above definition) is equal to the surface integral of the GAUSSian curvature.

An interesting special case of the metric (1.7) is given by the modulus of an analytic differential (cf. R. NEVANLINNA [24, p. 103]), $ds = |dw| = |\varphi(z)|\,|dz|$. Thereby we allow dw to be multiple-valued as long as $|dw|$ is single-valued. Furthermore, we admit isolated singularities a_k in whose neighborhoods φ is representable in the form $\varphi(z) = (z - a_k)^{p_k}\Psi(z)$, where p_k denotes an arbitrary real number and Ψ is a function regular at a_k $(k = 1, 2, 3, \ldots)$. The p_k's are conformal invariants, and we have $C = -2\pi\Sigma p_k$, this quantity being defined whenever $C^+ = 2\pi\Sigma\max(-p_k, 0)$ and $C^- = 2\pi\Sigma\max(p_k, 0)$ are not both infinite.

Throughout section 4 we suppose that S is finitely connected. Theorem 10 states that $C \leqq 2\pi\chi$ for any complete metric (1.7) whose curvatura integra exists, χ denoting the EULER-POINCARÉ characteristic of S. This result has already been proved by S. COHN-VOSSEN (Satz 6, p. 79 in [10]) under more restricted regularity conditions. (He admitted manifolds whose metrics were defined by positive definite quadratic forms (1.1), the coefficients E, F and G being of class C^2. In this case the GAUSSian curvature is defined and continuous. Hence, by a previous remark, our theorems can be applied.) Our proof is different from the one given by COHN-VOSSEN, although the central idea of this author has rather been transformed than altogether eliminated. We believe that our reasoning is simpler, at least if one disregards the complications needed for getting rid of unnatural regularity assumptions. It is function-theoretical. No use is made of the theorem of H. HOPF and W. RINOW [17] which states that on a (sufficiently regular) complete manifold any two points can be joined by a geodesic whose length is equal to their distance.

The remainder of section 4 is devoted to sufficient conditions for equality,

$C = 2\pi\chi$. One of these results (Theorem 11) implies a statement due to S. COHN-VOSSEN (Satz 7, p. 79 in [10]).

In section 5 infinitely connected, complete manifolds are investigated. Finally, in section 6 theorems of F. FIALA [12] and of CH. BLANC and F. FIALA [5] are extended.

Theorems 10 to 14, as formulated, apply only to orientable manifolds. However, it is easy to consider also the non-orientable case. The completeness of the metric is not destroyed if we pass to a two-sheeted, orientable covering manifold. Furthermore, in this process curvatura integra, EULER-POINCARÉ characteristic, total area and length of closed curves are all multiplied by 2. Consequently, an application of the above mentioned theorems to the covering manifold yields immediately the corresponding results in the non-orientable case. Hence in these theorems the metric (1.7) need not necessarily be defined on a RIEMANN surface. It is sufficient to suppose that S is a generalized RIEMANN surface in the sense that S is defined like a RIEMANN surface (cf. R. NEVANLINNA [24, p. 53]), but that both directly and indirectly conformal neighborhood relations are admitted.

In section 2 some theorems on conformal metrics defined in doubly connected, schlicht regions are demonstrated. These results are needed for subsequent applications, but they are also of interest by themselves. In particular statements concerning integrals of moduli of analytic functions along certain curves are implied. Such integrals have been the object of previous investigations – we mention the work of L. FEJÉR and F. RIESZ [11], R. M. GABRIEL [13], F. CARLSON [8], M. RIESZ [29] and B. ANDERSSON [1] – but, to our knowledge, problems of the type treated here have not been considered.

Section 3 contains a special result (Theorem 7) whose possible generalizations[2]) might warrant further investigation.

The reader is assumed to be familiar with some properties of subharmonic functions (cf. T. RADÓ [25]), in particular the theory of F. RIESZ [27].

We express our sincere gratitude to Professor H. HOPF for suggesting the problem. We are very much indebted to Professor A. PFLUGER for encouragement.

2. Some theorems on conformal metrics defined in schlicht annular regions

In the following let Ω be a doubly connected region in the z-plane which does not contain the point at infinity in its interior. We denote by Γ and Δ the outer and inner boundaries of Ω, respectively. (We make no regularity

[2]) Such as the subharmonic analogue or the extension from the plane to more general RIEMANN surfaces. See also the remarks to Theorem 7.

assumptions about Γ and Δ. In particular we allow Γ to consist of only the point at infinity.) Let Ω_0 designate the interior region of Γ.

Let $u(z)$ be a function defined in Ω and locally representable as a difference of two subharmonic functions

$$u(z) = u_1(z) - u_2(z) \ . \tag{2.1}$$

$u(z)$ can assume the values $+\infty$ (u_1 finite, $u_2 = -\infty$) and $-\infty$ ($u_1 = -\infty$, u_2 finite). It may be left undefined at those points where $u_1 = u_2 = -\infty$. This point set is of no concern to us since it is a set of measure zero with respect to all occurring integrations.

In a well known way (F. RIESZ [27]) measures $\mu_1(e)$ and $\mu_2(e)$ are associated with the functions $u_1(z)$ and $u_2(z)$, respectively. We define

$$\mu(e) = \mu_1(e) - \mu_2(e) \ . \tag{2.2}$$

If, for a given function $u(z)$, there exists one decomposition of the form (2.1), then there are an infinite number. Of course, the corresponding measures μ_1 and μ_2 depend on the choice of decomposition. However, the difference μ is the same for every such representation. Furthermore, for every prescribed decomposition (2.2) there exists a corresponding representation (2.1). If (2.2) is the JORDAN decomposition [30, p. 11] of μ, then (2.1) is called canonical. We may assume without loss of generality that (2.1) is canonical and valid throughout Ω (cf. M. G. ARSOVE [2, p. 331]).

Let γ be a JORDAN curve in Ω which encloses Δ. We introduce the *flux* of u through γ in accordance with the theory of F. RIESZ [27]. If u is of class C^2 and if γ is analytic, then we simply define

$$\Phi(u,\gamma;\Gamma) = \frac{1}{2\pi}\int_\gamma \frac{\partial u}{\partial n}\,|dz| \ , \tag{2.3}$$

n denoting the outer normal. In the case of general u and γ we introduce a sequence $\{\delta_k\}$ of JORDAN curves such that the annular regions (Δ, δ_k), bounded by Δ and δ_k, tend increasingly to (Δ, γ) as $k \to \infty$. Let

$$h_k(z) = h_{1k}(z) - h_{2k}(z) \ ,$$

where h_{1k} and h_{2k} are the best harmonic majorants of u_1 and u_2, respectively, in (δ_k, γ). h_k is independent of the choice of the decomposition (2.1). Let now δ'_k denote an analytic JORDAN curve in (δ_k, γ) which encloses δ_k. Then $\Phi(h_k, \delta'_k; \gamma)$ is well defined and its value is the same for every such δ'_k, since h_k is harmonic. We define

$$\Phi(u, \gamma; \Gamma) = \lim_{k\to\infty} \Phi(h_k, \delta'_k; \gamma) \ . \tag{2.4}$$

F. RIESZ [27] proved that if u is subharmonic then this limit always exists, being finite and independent of the choice of $\{\delta_k\}$. It is easy to conclude from this that the same is true in our case.

Now, let $\{\gamma_l\}$, $l = 1, 2, 3, \ldots$, be an arbitrary sequence of JORDAN curves, enclosing Δ, whose interior regions tend increasingly to Ω_0. In Theorems 1, 2 and 3 (below) we make the hypothesis

(A) *For any such sequence the limit*

$$\Phi(\Gamma) = \lim_{l \to \infty} \Phi(u, \gamma_l; \Gamma) \tag{2.5}$$

exists, admitting the values $+\infty$ *and* $-\infty$. Of course, $\Phi(\Gamma)$ is necessarily independent of the sequence $\{\gamma_l\}$.

The theory of F. RIESZ implies that

$$\Phi(u_1, \gamma_{l+1}; \Gamma) - \Phi(u_1, \gamma_l; \Gamma) = \mu_1[\gamma_l \cup (\gamma_l, \gamma_{l+1})]$$

for all l. Hence the sequence $\{\Phi(u_1, \gamma_l; \Gamma)\}$ is non-decreasing. The same is true for $\{\Phi(u_2, \gamma_l; \Gamma)\}$. Consequently, the limits

$$\Phi_1(\Gamma) = \lim_{l \to \infty} \Phi(u_1, \gamma_l; \Gamma)$$

and

$$\Phi_2(\Gamma) = \lim_{l \to \infty} \Phi(u_2, \gamma_l; \Gamma)$$

always exist, being finite or $+\infty$.

Hypothesis (A) *is equivalent to the assumption:*

(B) $\Phi_1(\Gamma)$ *and* $\Phi_2(\Gamma)$ *are not both infinite. Furthermore*

$$\Phi(\Gamma) = \Phi_1(\Gamma) - \Phi_2(\Gamma)\ . \tag{2.6}$$

Let us briefly indicate a proof of this statement. If (B) is fulfilled, then (A) and (2.6) follow immediately from the relation

$$\Phi(u, \gamma_l; \Gamma) = \Phi(u_1, \gamma_l; \Gamma) - \Phi(u_2, \gamma_l; \Gamma) \qquad (l = 1, 2, 3, \ldots)\ ,$$

which is an obvious consequence of the definition of Φ. The second half of the equivalence proof has to be based on the fact that $\mu = \mu_1 - \mu_2$ is a JORDAN decomposition. (We have supposed that the representation (2.1) is canonical.) Without entering into details we mention that the assumption $\Phi_1(\Gamma) = \Phi_2(\Gamma) = +\infty$ makes it possible to construct two sequences $\{\gamma_l\}$ and $\{\gamma_l'\}$ of the above mentioned type for which $\Phi(\Gamma) = +\infty$ and $\Phi(\Gamma) = -\infty$, respectively. Clearly, this yields a contradiction to (A).

A *path* σ in Ω will be said to *tend to* Γ if the following conditions are fulfilled: (1) σ is the topological image $z = z(t)$ of the half-open interval $0 \leqq t < 1$;

(2) given an arbitrary subcompact K of Ω_0 there exists a number $t'(K)<1$, such that $z(t)$ lies outside K for $t>t'$.

Theorem 1. *If* $\Phi(\Gamma)<-1$, *then there exists a locally rectifiable path* σ *in* Ω, *tending to* Γ, *such that*

$$\int_\sigma e^{u(z)}|dz|<+\infty\ . \tag{2.7}$$

Remarks. There is no value of $\Phi(\Gamma)$ outside the above mentioned interval for which the theorem is also true. This can be seen from the following counterexamples:

(I) $\Omega_0 =$ finite z-plane, $u_1 = |z|^2$, $u_2 \equiv 0$. Then $\Phi(\Gamma) = +\infty$.

(II) $\Omega_0 =$ finite z-plane, $u_1 = \alpha \log |z|$, $u_2 \equiv 0$. Then $\Phi(\Gamma) = \alpha$ $(0 \leqq \alpha<+\infty)$.

(III) $\Omega_0 =$ finite z-plane, $u_1 \equiv 0$, $u_2 = -\alpha \log |z|$. Then $\Phi(\Gamma) = \alpha$ $(-1 \leqq \alpha \leqq 0)$.

(IV) Let Γ contain at least two points. Then there exists a conformal mapping $w = \varphi(z)$ of Ω_0 onto $|w|<1$. We define

$$u_1 = \frac{1}{1-|\varphi(z)|} + \log|\varphi'(z)|\ , \qquad u_2 \equiv 0\ .$$

Then $\Phi(\Gamma) = +\infty$.

The reader will easily verify that in each of these examples there exists no path σ having the properties postulated in Theorem 1. (It should be observed that the choice of Δ is irrelevant.)

It is natural to ask whether the hypothesis $\Phi(\Gamma)<-1$ can be weakened if more restrictive conditions are imposed on Ω. Examples (I), (II) and (III) show that the condition $\Phi(\Gamma)<-1$ cannot be replaced by a weaker inequality for those domains Ω whose boundary component Γ consists of only the point at infinity. We shall now demonstrate that for all other regions the hypothesis $\Phi(\Gamma)<-1$ in Theorem 1 can be replaced by $\Phi(\Gamma)<+\infty$. Since, on the other hand, for the case $\Phi(\Gamma) = +\infty$ we have given counterexamples for any Ω, this settles the question completely.

Theorem 2. *If* Γ *contains more than one point, and if* $\Phi(\Gamma)<+\infty$, *then there exists a locally rectifiable path* σ, *tending to* Γ, *such that* (2.7) *is fulfilled.*

Proof of Theorem 2. From (2.6) and the hypothesis $\Phi(\Gamma)<+\infty$ we infer that $\Phi_1(\Gamma)<+\infty$. We choose an arbitrary positive number $\alpha>\Phi_1(\Gamma)+1$ and consider the function $u^* = u_1^* - u_2^*$, where $u_1^*(z) \equiv u_1(z)$ and $u_2^*(z) = u_2(z) - \alpha g_0(z,z_0)$. Here g_0 denotes GREEN's function of Φ_0 and z_0 is an arbitrary but fixed point in this region. Throughout Ω, $u < u^*$. The functions u_1^* and u_2^* are obviously subharmonic. Because of the choice of α, $\Phi^*(\Gamma) < -1$. Therefore, Theorem 1 may be applied to $u^*(z)$.[3]) There exists a locally rectifiable path σ, tending to Γ, such that

$$\int_\sigma e^u |dz| < \int_\sigma e^{u^*} |dz| < +\infty .$$

This completes the proof of Theorem 2.

It is also possible to weaken the hypothesis $\Phi(\Gamma) < -1$ in Theorem 1 by making further assumptions about the function u. We shall now discuss the effect of a condition which is natural from the point of view of both the theory of functions and the applications to differential geometry.

A sequence of curves $\{\gamma_n\}$, $n = 1, 2, 3, \ldots$, will be said to *come arbitrarily near to* Γ, if the point set $\bigcup_n \gamma_n$ is not contained in any subcompact of Ω_0.

Theorem 3. *Suppose there exists a sequence* $\{\gamma_n\}$, $n = 1, 2, 3, \ldots$, *of rectifiable* Jordan *curves, enclosing* Δ, *in* Ω *and a number* M *such that*

(a) $\{\gamma_n\}$ *comes arbitrarily near to* Γ,

(b) $\int_{\gamma_n} e^u |dz| < M$ *for all* n.

Then, if $\Phi(\Gamma) \neq -1$, *there exists a locally rectifiable path* σ, *tending to* Γ, *such that* (2.7) *is fulfilled.*

Remarks. The hypothesis $\Phi(\Gamma) \neq -1$ cannot be dropped. This follows from the example

(V) $\Omega_0 =$ finite z-plane, $u_1 = 0$, $u_2 = \log |z|$, $\gamma_n = [|z| = n]$, $\Phi(\Gamma) = -1$.

By Theorem 2 no such counterexamples exist if Ω_0 is of hyperbolic type. In this case Theorem 3 actually gives new information only for $\Phi(\Gamma) = +\infty$.

If $\Phi(\Gamma) < -1$, then Theorem 3 is, of course, superseded by Theorem 1 for any Ω.

We are left to prove Theorems 1 and 3.

Preliminary considerations. A point z_0 in Ω will be called a *singular point of the measure* μ if $\mu(z_0) \geqq 1$. (The symbol z_0 is used here to denote the set consisting of the point z_0.) In every subcompact of Ω there are at most a finite number of such points.

Lemma 1. *Let* α *be an analytic arc which contains no singular point of* μ. *Then* $\int_\alpha e^u |dz| < +\infty$.

[3]) The proof of Theorem 1 will later be given.

Remark. It is easy to construct examples which show that in this lemma the word "analytic" cannot be replaced by "rectifiable".

Proof. Let $\tilde{\alpha}$ denote the segment $0 \leqq \xi \leqq 1$ on the real axis of a complex ζ-plane ($\zeta = \xi + i\eta$). There exists a conformal mapping $z = \varphi(\zeta)$ of a neighborhood $\tilde{V}$ of $\tilde{\alpha}$ onto a neighborhood V of α such that $\tilde{\alpha}$ corresponds to α. We now consider in $\tilde{V}$ the subharmonic functions

$$\begin{aligned} \tilde{u}_1(\zeta) &\equiv u_1(\varphi(\zeta)) + \log|\varphi'(\zeta)| \\ \tilde{u}_2(\zeta) &\equiv u_2(\varphi(\zeta)) \end{aligned} \tag{2.8}$$

and define $\tilde{u} = \tilde{u}_1 - \tilde{u}_2$, so that $e^u|dz| = e^{\tilde{u}}|d\zeta|$. One proves, without difficulty, that $\mu_1(e) = \tilde{\mu}_1(\tilde{e})$ and $\mu_2(e) = \tilde{\mu}_2(\tilde{e})$ for corresponding sets e and $\tilde{e}$, $\tilde{\mu}_1$ and $\tilde{\mu}_2$ denoting the measures associated with $\tilde{u}_1$ and $\tilde{u}_2$, respectively. Because of the existence of such a transformation we may, without loss of generality, assume α to be the segment $0 \leqq x \leqq 1$ on the real axis.

Given an arbitrary point x_0 on α, there always exists a radius $r_0(x_0) > 0$ such that $\mu_2(|z - x_0| < 2r_0) = p < 1$. By a well known theorem of F. RIESZ [27, II, p. 350] we have the representations

$$u_1(z) = h_1(z) + \int_{|\zeta - x_0| < 2r_0} \log|z - \zeta| d\mu_1(e_\zeta) \tag{2.9}$$

and

$$u_2(z) = h_2(z) + \int_{|\zeta - x_0| < 2r_0} \log|z - \zeta| d\mu_2(e_\zeta) \tag{2.10}$$

in $|z - x_0| < 2r_0$, the functions h_1 and h_2 being harmonic. An obvious covering argument yields the existence of $\int_0^1 e^u dx$ if we can show that

$$\int_{x_0 - r_0}^{x_0 + r_0} e^u dx < +\infty .$$

But, by (2.9) and (2.10), this will be achieved if we can prove that

$$I = \int_{x_0 - r_0}^{x_0 + r_0} \exp\{- \int_{|\zeta - x_0| < 2r_0} \log|x - \zeta| d\mu_2(e_\zeta)\} dx < +\infty . \tag{2.11}$$

We first consider the special case where the mass distribution μ_2 is concentrated in one point ζ_0. Then

$$I = \int_{x_0 - r_0}^{x_0 + r_0} |x - \zeta_0|^{-p} dx \leqq \int_{x_0 - r_0}^{x_0 + r_0} |x - x_0|^{-p} dx = \frac{2r_0^{1-p}}{1 - p} . \tag{2.12}$$

Let us now proceed to a measure μ_2 which consists of a finite number of concentrated masses, $\alpha_1 p$ in $\zeta_1, \ldots, \alpha_n p$ in ζ_n, $\Sigma\alpha_i = 1$, $\alpha_i > 0$ $(i = 1, 2, \ldots, n)$. We introduce the notation I_N for the integral I in which $\log|z - \zeta|$

has been replaced by $\log^N|z - \zeta| = \max[\log|z - \zeta|, -N]$, N denoting an arbitrarily large constant. By an application of HÖLDER's inequality [15, p. 140] and of (2.12) we obtain

$$I_N = \int_{x_0-r_0}^{x_0+r_0} \exp\{-\sum_{i=1}^{n} \alpha_i p \log^N|x - \zeta_i|\}dx$$

$$\leqq \prod_{i=1}^{n} [\int_{x_0-r_0}^{x_0+r_0} \exp\{-p \log^N|x - \zeta_i|\}dx]^{\alpha_i}$$

$$\leqq \prod_{i=1}^{n} [\int_{x_0-r_0}^{x_0+r_0} |x-\zeta_i|^{-p}dx]^{\alpha_i} \leqq \frac{2r_0^{1-p}}{1-p} .$$

Let us now drop every special assumption about μ_2. In the general case I_N can always be approximated arbitrarily close for fixed N by substituting for μ_2 a suitable measure of the special type considered above. Hence

$$I_N \leqq 2r_0^{1-p}/(1-p)$$

holds without restriction. Letting $N \to +\infty$ we obtain (2.11) in the limit. Q. E. D.

Lemma 2. *Suppose there exists a sequence* $\{\sigma_n\}$, $n = 1, 2, 3, \ldots$, *of rectifiable curves in* Ω, *a subcompact* K *of* Ω_0 *and a number* C *such that the following is true:*

(a) *each* σ_n *has a non-empty intersection with* K,

(b) $\{\sigma_n\}$ *comes arbitrarily near to* Γ,

(c) $\int_{\sigma_n} e^u|dz| < C$ *for all* n.

Then there exists a locally rectifiable path σ *in* Ω, *tending to* Γ, *such that* (2.7) *is satisfied.*

Proof. We may assume that Ω is a circular ring

$$R_1 < |z| < R_2 \qquad (0 \leqq R_1 < R_2 \leqq +\infty) .$$

For, if this lemma has been proved for a particular region Ω, then it is immediately seen to be valid for the whole class of conformally equivalent domains. This is easily proved by transplanting the metric $e^{u(z)}|dz|$ under conformal representation.

We introduce a sequence of circles $\gamma_m = [|z| = r_m]$, $m = 1, 2, 3, \ldots$, in Ω, supposing that γ_1 encloses K and that $r_m \nearrow R_2$ for $m \nearrow +\infty$. We further assume that no singular point of μ lies on these circles. If one of the curves σ_n tends to Γ, then we have nothing to prove. If this is not the case, then there exists a subsequence $\{\sigma_n'\}$ of $\{\sigma_n\}$ such that σ_n' intersects γ_n for all n. From this and hypothesis (a) we conclude that σ'_n contains an arc which leads from γ_1 to γ_n. We subdivide it into $\sigma'_n(1,2)$ (leading from γ_1 to γ_2), ..., $\sigma'_n(n-1,n)$ (leading from γ_{n-1} to γ_n). By making use of CANTOR's diagonal process we select a subsequence $\{\sigma''_n\}$ of $\{\sigma'_n\}$ such that the common endpoints of $\sigma''_n(m-1,m)$ and $\sigma''_n(m,m+1)$ on γ_m converge to a limit point z_m for all m. Let now m be fixed. By Lemma 1, there exists an open arc α_m of γ_m, containing z_m, such that $\int_{\alpha_m} e^u|dz|<2^{-m}$. Furthermore, there exists an index $N(m)$ such that the following conditions are satisfied:

(a) The inner and outer endpoints of $\sigma''_N(m,m+1)$ lie on α_m and α_{m+1}, respectively,

(b) $\int_{\sigma''_N(m,m+1)} e^u|dz|<\inf_{n>N} \int_{\sigma''_n(m,m+1)} e^u|dz|+2^{-m-1}$.

We define σ to consist of the curves $\sigma''_{N(1)}(1,2)$, $\sigma''_{N(2)}(2,3)$, ..., joined by subarcs of $\alpha_2, \alpha_3, \ldots$. The reader will easily convince himself that

$$\int_\sigma e^u|dz|<C+1 .$$

This proves Lemma 2.

Proof of Theorem 1. There is some expository advantage in assuming that Ω is a circular ring $R_1<|z|<R_2$ $(0 \leqq R_1<R_2 \leqq +\infty)$. This can be done without loss of generality .Indeed, an arbitrary Ω can always be mapped conformally onto a circular annulus $R_1<|\zeta|<R_2$ in such a way that Γ corresponds to the outer circle $|\zeta|=R_2$. Under such a representation both length element ds and flux Φ are invariant if u is transformed according to (2.8).

It follows from the hypotheses that there exists a number η $(0<\eta<1)$ and a radius r_1 $(R_1<r_1<R_2)$ such that

$$\Phi(u_2,|z|=r_1;\Gamma)>1+2\eta , \tag{2.13}$$

$$\Phi_1(\Gamma)-\Phi(u_1,|z|=r_1;\Gamma)<\eta . \tag{2.14}$$

Lemma 3. *Let the radius ϱ_1 $(r_1<\varrho_1<R_2)$ be chosen arbitrarily. Then there is a number C with the following property: Given any ϱ_2 $(\varrho_1<\varrho_2<R_2)$, there exists a rectifiable curve α, leading from $|z|=\varrho_1$ to $|z|=\varrho_2$, such that*

$$\int_\alpha e^u|dz|<C . \tag{2.15}$$

Proof. We introduce three radii s_1, s_2 and r_2 satisfying the inequalities

$$R_1<r_1<s_1<\varrho_1<\varrho_2<s_2<r_2<R_2 . \tag{2.16}$$

In addition we require that there should be no singular point of μ on $|z|=s_1$.

In the following all radii with index 1 have to be considered fixed. On the other hand, ϱ_2, s_2 and r_2 are variable within the above limits and our estimation ultimately does not depend on their choice. By a theorem of F. RIESZ [27, II, p. 357] the representations

$$u_1(z) = h_1(z) - \int_\omega g(z, \zeta)\, d\mu_1(e_\zeta) \tag{2.17}$$

and

$$u_2(z) = h_2(z) - \int_\omega g(z, \zeta)\, d\mu_2(e_\zeta) \tag{2.18}$$

hold in $\omega = [r_1 < |z| < r_2]$, where h_1 and h_2 are the best harmonic majorants of u_1 and u_2, respectively, in ω and g denotes GREEN's function for this domain. We define

$$v(z) = h_1(z) - u_2(z) = u(z) + \int_\omega g(z, \zeta)\, d\mu_1(e_\zeta)\ . \tag{2.19}$$

v is superharmonic in ω and admits the representation (F. RIESZ [27, II, p. 350])

$$v(z) = H(z) - \int_\omega \log|z - \zeta|\, d\mu_2(e_\zeta)\ , \tag{2.20}$$

where H is harmonic in ω. It follows from (2.13) and (2.14) that

$$\frac{1}{2\pi}\int_\gamma \frac{\partial H}{\partial n}|dz| < -1 - \eta\ , \tag{2.21}$$

γ being an arbitrary smooth JORDAN curve in ω which encloses $|z| = r_1$, n denoting the outer normal.

Let $\vartheta = (\varrho_1 - s_1)/2$. We define

$$\omega_0 = [r_1 < |z| \leqq s_1 + \vartheta]\ , \qquad \omega_1 = [s_1 + \vartheta < |z| < r_2]\ ,$$

$\mu_{20}(e) = \mu_2(e \frown \omega_0)$, $\mu_{21}(e) = \mu_2(e \frown \omega_1)$, $m_0 = \mu_2(\omega_0)$, $m_1 = \mu_2(\omega_1)$ and

$$v_1(z) = H(z) - \int_{\omega_1} \log|z - \zeta|\, d\mu_{21}(e_\zeta)\ . \tag{2.22}$$

Further, let

$$V(z) = u(z) + \int_{|z|>r_1} G(z, \zeta)\, d\mu_1(e_\zeta)\ , \tag{2.23}$$

where G denotes GREEN's function for $|z| > r_1$. For $|z| = s_1$, $v_1(z) \leqq v(z) + m_0 \log(2\varrho_1)$. Furthermore, V is superharmonic in ω, the associated measure being $\mu_2(e)$, and $v \leqq V$. Since the circle $|z| = s_1$ contains no singular point of μ, we have, by Lemma 1

$$\int_{|z|=s_1} e^{v_1}|dz| \leqq (2\varrho_1)^{m_0} \int_{|z|=s_1} e^{v}|dz| \leqq (2\varrho_1)^{m_0} \int_{|z|=s_1} e^{V}|dz| = C_1 < +\infty\ , \tag{2.24}$$

where C_1 is a constant not depending on the choice of ϱ_2, s_2 and r_2.

For $\varrho_1 \leqq |z| \leqq \varrho_2$, $u \leqq v \leqq v_1 - m_0 \log \vartheta$. Hence, for any curve α contained in this annulus

$$\int_\alpha e^u |dz| \leqq \vartheta^{-m_0} \int_\alpha e^{v_1} |dz| \ . \tag{2.25}$$

We are now going to demonstrate that there exists a rectifiable curva α, leading from $|z| = \varrho_1$ to $|z| = \varrho_2$, and such that

$$\int_\alpha e^{v_1} |dz| \leqq C_2 \int_{|z|=s_1} e^{v_1} |dz| + C_3 \ , \tag{2.26}$$

where C_2 and C_3 are constants not depending on the choice of ϱ_2, r_2 and s_2. Lemma 3 will be an immediate consequence of (2.24), (2.25) and (2.26).

In order to establish (2.26) we first approximate the measure $\mu_{21}(e)$ by a finite number of concentrated masses. This is done by the following construction: Choose an integer $N \geqq 2$ such that

$$m_1 \log \frac{N + 1 + 2\sqrt{2}}{N + 1 - 2\sqrt{2}} < \log 2 \ . \tag{2.27}$$

Let $\zeta_1, \zeta_2, \ldots, \zeta_m$ designate those points (necessarily finite in number) which support a concentrated mass of weight $\geqq \dfrac{1}{4(2N+1)^2}$ in the measure μ_{21}. Denoting the corresponding masses by $p_1, p_2, \ldots, p_m$, we define

$$w(z) = H(z) - \sum_{k=1}^{m} p_k \log |z - \zeta_k| \ . \tag{2.28}$$

We introduce the compact set K which is obtained by subtracting from the annulus $s_1 \leqq |z| \leqq s_2$ the open disks $|z - \zeta_k| < \delta_k$ $(k = 1, 2, \ldots, m)$, where the radii δ_k are chosen small enough so that the following conditions are satisfied:

(a) the sets $|z| = s_1$, $|z - \zeta_1| \leqq 2\delta_1, \ldots, |z - \zeta_m| \leqq 2\delta_m$ are disjoint,

(b) whenever ζ_k is a singular point of μ (i. e. $p_k \geqq 1$), then we choose δ_k so small that

$$\int_\varkappa e^w |dz| > \frac{2^{m_1+2} r_2^{m_1}}{1 - \cos(\pi\eta)} \int_{|z|=s_1} e^{v_1} |dz| \tag{2.29}$$

for any rectifiable curve $\varkappa$ leading from $|z| = s_1$ to $|z - \zeta_k| = 2\delta_k$,[4])

(c) we require the remaining δ_k's to be so small that

$$\sum_{p_k<1} \int_{|z-\zeta_k|=2\delta_k} e^{v_1} |dz| < 1 \ . \tag{2.30}$$

[4]) It is easy to prove that such a choice is always possible.

This condition can always be satisfied since

$$\lim_{\delta \to 0} \int_{|z-\zeta_k|=\delta} e^{v_1}|dz| = 0 \tag{2.31}$$

at any non-singular point ζ_k. (2.31) can be proved by a reasoning which is quite similar to the one used in the proof of Lemma 1 and which we do not reproduce here.

Let $\nu(e)$ denote the measure which is obtained from $\mu_{21}(e)$ after subtracting the concentrated masses p_1 in $\zeta_1, \ldots, p_m$ in ζ_m. We have

$$v_1(z) = w(z) - \int_{\omega_1} \log|z - \zeta|\, d\nu(e_\zeta) \; . \tag{2.32}$$

There exists a number $d > 0$ such that $\nu(e) \leqq \frac{1}{4(2N+1)^2}$ for all BOREL sets e of diameter $\leqq d$. Let such a d be chosen, requiring in addition that $d < 1$.

The function $w(z)$ is uniformly continuous on K.

We now choose a number $L > 0$, so small that the conditions (2.33) to (2.37) are satisfied:

$$\begin{aligned} &|z_1 - z_2| \leqq 3\sqrt{2}\,L \quad \text{implies} \\ &|w(z_1) - w(z_2)| < \log 2 \quad \text{for all} \quad z_1, z_2 \varepsilon K \; , \end{aligned} \tag{2.33}$$

$$m_1 \log \frac{\vartheta}{\vartheta - L} < \log 2 \; , \tag{2.34}$$

$$\sqrt{2}L < d \; , \qquad \sqrt{2}(2N+1)L < 1 \; , \tag{2.35}$$

$$2\sqrt{2}L < \delta_k \qquad (k = 1, 2, \ldots, m) \; , \tag{2.36}$$

$$m_1 d^{-m_1} e^M (N+1)^2 N^{-\frac{1}{4}} \lambda^{\frac{1}{2}} |\log \lambda| < 1 \tag{2.37}$$

for $0 < \lambda \leqq \sqrt{2}L$, M denoting the maximum of $w(z)$ on K.

Now we cover the (x, y)-plane by a net of squares Σ with sides of length L ($x = iL, y = jL; i, j = 0, \pm 1, \pm 2, \ldots$) and replace the measure $\nu(e)$ by a finite number of concentrated masses, assigning to the points $((i + \frac{1}{2})L, (j + \frac{1}{2})L)$ the weights $\nu([iL \leqq x < (i+1)L] \frown [jL \leqq y < (j+1)L])$. Let these be the masses p_{m+1} in $\zeta_{m+1}, \ldots, p_n$ in ζ_n. We define

$$w_1(z) = w(z) - \sum_{k=m+1}^{n} p_k \log|z - \zeta_k| = H(z) - \sum_{k=1}^{n} p_k \log|z - \zeta_k| \; . \tag{2.38}$$

We now state: There exists a polygon β, leading from $|z| = s_1$ to $|z| = \varrho_2$, such that

$$\int_\beta e^{w_1}|dz| < \frac{2}{1 - \cos(\pi\eta)} \int_{|z|=s_1} e^{w_1}|dz| \; . \tag{2.39}$$

We should like to point out that the proof of this statement constitutes the kernel of our demonstration of Lemma 3. In fact, if u were assumed to be superharmonic and of the type w_1 (i. e. harmonic with isolated logarithmic singularities) throughout Ω, then the following reasoning would represent the complete proof of this lemma.

Let z_0 be an arbitrary point of the annulus $A = [s_1 \leqq |z| \leqq s_2]$. We define $\Lambda(z_0) = \inf_{\gamma} \int_{\gamma} e^{w_1}|dz|$, where γ varies on the set $P(z_0)$ consisting of all rectifiable, closed curves on A which are not nullhomotopic and pass through z_0. If z_0 is not a singular point of μ, then $\Lambda(z_0)$ is finite and there exists at least one *minimal curve* $\bar{\gamma}(z_0) \in P(z_0)$ such that

$$\Lambda(z_0) = \int_{\bar{\gamma}} e^{w_1}|dz| \ . \tag{2.40}$$

Indeed, $\bar{\gamma}$ can be constructed in the usual way as the limit of a suitably chosen minimal sequence. $\Lambda(z)$ is continuous on $A - \bigcup_{p_k \geqq 1} \zeta_k$.

We decompose the set $D = [s_1 \leqq |z| \leqq \varrho_2]$ into three disjoint subsets $D = \bigcup_{i=1}^{3} D_i$, where

(a) $$D_1 = D \frown \bigcup_{p_k \geqq 1} [|z - \zeta_k| < 2\delta_k] \ ,$$

(b) D_2 consists of those points of $D - D_1$ which possess at least one minimal curve contained in D,

(c) $$D_3 = D - (D_1 \smile D_2) \ .$$

Some of these sets may be empty. D_2 is closed.

We now discuss some properties of minimal curves which contain no double point. Assume first, for simplicity, that $\bar{\gamma}(z_0)$ is completely in the interior of A and that none of the ζ_k's lie on it $(k = 1, 2, \ldots, n)$. Then w_1 is harmonic in some (doubly connected) neighborhood V of $\bar{\gamma}$. Hence the function

$$\zeta = \Psi(z) = \int_{z_0}^{z} e^{w_1 + i w_1^*} \, dz \ , \tag{2.41}$$

where w_1^* is conjugate harmonic to w_1, yields a conformal mapping of the universal covering surface $\tilde{V}$ of V onto some simply connected RIEMANN surface extending over the ζ-plane. Since $e^{w_1}|dz| = |d\zeta|$, the image of any subarc of $\bar{\gamma}$ not containing z_0 in its interior (and considered on any sheet of $\tilde{V}$) is a straight line segment in the ζ-plane. Hence

$$\arg d\zeta = w_1^* + \arg dz = \text{const.} \tag{2.42}$$

along such an arc. Furthermore, (2.21) implies

$$\int_{\bar{\gamma}} \frac{\partial w_1^*}{\partial s} ds = \int_{\bar{\gamma}} \frac{\partial w_1}{\partial n} ds = \int_{\bar{\gamma}} \frac{\partial H}{\partial n} ds - 2\pi \sum_{\zeta_k \text{ inside } \bar{\gamma}} p_k < -2\pi(1+\eta) \ , \tag{2.43}$$

where s denotes the arc length, n the exterior normal, and the integration is performed in the positive sense. From (2.42) and (2.43) we conclude that, in the neighborhood of z_0, $\bar{\gamma}$ consists of two analytic arcs which intersect at an exterior angle $\Theta(\bar{\gamma}, z_0)$ satisfying the inequality

$$\Theta(\bar{\gamma}, z_0) < \pi(1 - 2\eta) \ . \tag{2.44}$$

If we now allow $\bar{\gamma}$ to have points in common with $|z| = s_1$, then (2.42) will no more be true in general. However, it follows easily from a consideration of the mapping (2.41) that $w_1^* + \arg dz$ increases monotonically if $\bar{\gamma}$ is followed in the positive sense. From this and (2.42) the inequality (2.44) is again obtained.

Finally, we also admit that $\bar{\gamma}$ may pass through some of the ζ_k's. In such points $\bar{\gamma}$ will not possess corners because these would make shortcuts (in the metric $e^{w_1}|dz|$) possible, contrary to the definition of minimal curves. Near these points we replace $\bar{\gamma}$ temporarily by small circular arcs lying in the interior region. By making use of the mapping (2.41) (integrating in a neighborhood of the modified curve) and by our knowledge of the behavior of w_1^* along the small circles, we verify again the monotonicity of $w_1^* + \arg dz$ and the ensuing relation (2.44). No essential difficulty arises if z_0 itself coincides with one of the ζ_k's.

If $\bar{\gamma}$ has points in common with $|z| = s_2$, then (2.44) will not be fulfilled in general.

We shall now investigate the behavior of the function $\Lambda(z)$ in the neighborhood U of a point z_0 on D_2. To this end we map U, by (2.41), conformally onto a domain $\tilde{U}$ in the ζ-plane. (Assume first, for simplicity, that w_1 is harmonic at z_0.) There exists a minimal curve $\bar{\gamma}(z_0)$ which is contained in D. For the present we make the additional assumption that $\bar{\gamma}$ is a simple closed curve. Then the image of $\bar{\gamma}$ in $\tilde{U}$ consists of two straight line segments, l_1 and l_2, intersecting at $\zeta_0 = \Psi(z_0)$ under the angle $\Theta(\bar{\gamma}, z_0)$. Let ζ_0' be a point on the bisector b of Θ, and let ζ_1 and ζ_2 denote the points of intersection of the normal n to b through ζ_0' with l_1 and l_2, respectively. (Choose $|\zeta_0 - \zeta_0'|$ small enough so that the triangle $\zeta_0\zeta_1\zeta_2$ lies in $\tilde{U}$.) We have

$$|\zeta_0 - \zeta_0'| = \frac{\cos(\Theta/2)}{2(1-\sin(\Theta/2))} [(|\zeta_0 - \zeta_1| + |\zeta_0 - \zeta_2|) - (|\zeta_0' - \zeta_1| + |\zeta_0' - \zeta_2|)] \cdot$$

The (Euclidean) bisector of Θ in the z-plane is transformed into an analytic curve a tangential to b at z_0. Let ζ_0'' denote the point of intersection of a with n. Then

$$\int_{\zeta_0}^{\zeta_0''} |d\zeta| = |\zeta_0 - \zeta_0'| + o(|\zeta_0 - \zeta_0'|) .$$

Hence, for sufficiently small $|\zeta_0 - \zeta_0'|$

$$\int_{\zeta_0}^{\zeta_0''} |d\zeta| < \frac{\cos(\Theta/2)}{1-\sin(\Theta/2)} [(|\zeta_0 - \zeta_1| + |\zeta_0 - \zeta_2|) - (|\zeta_0'' - \zeta_1| + |\zeta_0'' - \zeta_2|)] . \quad (2.45)$$

Let $z_0'' = \Psi^{-1}(\zeta_0'')$. From (2.44), (2.45) and the definition of Λ we conclude that

$$\int_{z_0}^{z_0''} e^{w_1} |dz| < \frac{1}{1-\cos(\pi\eta)} [\Lambda(z_0) - \Lambda(z_0'')] , \quad (2.46)$$

where $z_0 z_0''$ denotes a (sufficiently small) segment on the (Euclidean) bisector of $\Theta(\bar{\gamma}, z_0)$.

Only slight modifications are needed for the case where z_0 coincides with one of the ζ_k's. Again we make use of the mapping (2.41). In order to obtain uniqueness we slit the domain U along a line which leads from z_0 into the interior of $\bar{\gamma}$. The mapping Ψ is no more conformal at z_0. However, the reader will easily convince himself that Θ is decreased. So our estimations hold a fortiori.

Finally, if $\bar{\gamma}$ contains double points, then it can be proved without difficulty that (2.46) is satisfied if $z_0 z_0''$ is defined to be a (sufficiently small) segment on the tangent to either of the two branches of $\bar{\gamma}$ issuing from z_0.

On D_2 we now define a complex-valued function $T(z)$. Let $z \in D_2$ and φ $(0 \leqq \varphi < 2\pi)$ be fixed and let $t(z, \varphi)$ denote the largest number with the following property: For all τ in the interval $0 < \tau < t$ the point $z = \tau e^{i\varphi}$ is contained in A and the inequality

$$\int_0^{\tau} \exp\{w_1(z + re^{i\varphi})\}\, dr < \frac{1}{1-\cos(\pi\eta)} [\Lambda(z) - \Lambda(z + \tau e^{i\varphi})]$$

holds. Suppose $\{\varphi_i(z)\}$ is a sequence of arguments such that $\{t(z, \varphi_i)\}$ tends to the least upper bound $t_m(z)$ of $t(z,\varphi)$. By (2.46), $t_m(z)$ is always positive. We choose an arbitrary limit point φ_m of $\{\varphi_i\}$ and define $T(z) = t_m e^{i\varphi_m}$. It follows from the definition of $T(z)$ and the continuity of $\Lambda(z)$ that, for all $z \in D_2$

$$\int_0^{t_m} \exp\{w_1(z + re^{i\varphi_m})\}\, dr \leqq \frac{1}{1-\cos(\pi\eta)} [\Lambda(z) - \Lambda(z + T(z))] . \quad (2.47)$$

Furthermore, by making use of the definitions it can be verified in a straightforward way that $t_m(z)$ is lower semicontinuous. Let t_{min} denote the (positive) minimum of $t_m(z)$ on the (compact) set D_2.

Let us now construct β. We choose an arbitrary point z_0 on $|z| = s_1$. Obviously

$$A(z_0) \leqq \int_{|z|=s_1} e^{w_1}|dz| \ . \tag{2.48}$$

If $z_0 \in D_2$, then we define $z_1 = z_0 + T(z_0)$, $z_2 = z_1 + T(z_1)$, etc., until we arrive at a point z_n not contained in D_2. This will always happen after a finite number of steps. This follows from $A(z) > 0$ and the inequalities (implied by (2.47))

$$A(z_k) - A(z_{k+1}) \geqq t_{min} e^{w_{1min}}[1 - \cos(\pi\eta)]$$

$(k = 0, 1, 2, \ldots)$, $w_{1\,min}$ denoting the (finite) minimum of w_1 on A.

Let β_1 denote the polygon $(z_0, z_1, \ldots, z_n)$. From (2.47) (formulated for $z = z_0, z_1, \ldots, z_{n-1}$) and (2.48) we conclude that

$$\int_{\beta_1} e^{w_1}|dz| \leqq \frac{1}{1 - \cos(\pi\eta)} \int_{|z|=s_1} e^{w_1}|dz| \ . \tag{2.49}$$

Assume that $z_n \in D_3$. Then every minimal curve $\bar{\gamma}(z_n)$ intersects $|z| = \varrho_2$. Hence there exists a rectifiable curve, joining z_n with some point on $|z| = \varrho_2$, and of length $\leqq A(z_n)/2 \leqq A(z_0)/2$ in the metric $e^{w_1}|dz|$. Combined with (2.48) this implies the existence of a polygon β_2 with the same endpoints and such that

$$\int_{\beta_2} e^{w_1}|dz| < \int_{|z|=s_1} e^{w_1}|dz| \ . \tag{2.50}$$

We define $\beta = \beta_1 + \beta_2$. (2.39) follows from (2.49) and (2.50).

If $|z_n| > \varrho_2$, then β reduces to a portion of β_1. If $z_0 \in D_3$, then β consists of β_2 only. In these cases (2.39) is true a fortiori.

We are left with the possibility $z_n \in D_1$. This never occurs. Indeed, by (2.38), we have

$$w \leqq w_1 + m_1 \log(2r_2) \ . \tag{2.51}$$

Let S_k be an arbitrary closed square of the net Σ which intersects with ω_1. Then, for any $z \in [|z| = s_1]$ and arbitrary $\zeta \in S_k$

$$\log\left|\frac{z - \zeta}{z - \zeta_k}\right| < \log\frac{\vartheta}{\vartheta - L} \ ,$$

ζ_k denoting the center of S_k. Hence (2.32), (2.34) and (2.38) imply

$$w_1 \leqq v_1 + \log 2 \quad \text{on} \quad |z| = s_1 \ .$$

Therefore

$$\int_{|z|=s_1} e^{w_1}|dz| \leqq 2 \int_{|z|=s_1} e^{v_1}|dz| \ . \tag{2.52}$$

It follows from (2.51), (2.49) and (2.52) that

$$\int_{\beta_1} e^{w}|dz| \leqq \frac{2^{m_1+1}\, r_2^{m_1}}{1-\cos(\pi\eta)} \int_{|z|=s_1} e^{v_1}|dz| \ .$$

Now, if $z_n \in D_1$, this would contradict (2.29).

We thus have proved the existence of β. By applying (2.39) instead of (2.49) in the above reasoning one finds that β does not intersect D_1.

With the polygon β we now associate a curve α which leads from $|z| = \varrho_1$ to $|z| = \varrho_2$. This is done by the following construction: Let $E(z)$ denote the set consisting of those (1, 2 or 4) closed squares of the net Σ which contain the point z. We connect the endpoint z_0 of β on $|z| = s_1$ with the last point of intersection z_1' of β with $E(z_0)$, z_1' with the last point of intersection z_2' of β with $E(z_1')$, and so forth, until we arrive at the endpoint of β on $|z| = \varrho_2$. We obtain a polygon β'. Now, if β' should penetrate into some of the disks

$$|z - \zeta_k| < 2\,\delta_k \qquad (p_k < 1\,;\, k = 1, 2, \ldots, m) \ ,$$

then we replace in each case the subpolygon between the first entry and the last exit by an arc on $|z - \zeta_k| = 2\,\delta_k$. If the endpoint of β on $|z| = \varrho_2$ should lie in $|z - \zeta_k| < 2\,\delta_k$ $(p_k < 1\,;\, k = 1, 2, \ldots, m)$, then we follow the circle $|z - \zeta_k| = 2\,\delta_k$ from the first entry until we reach $|z| = \varrho_2$. The resulting curve contains a portion α which leads from $|z| = \varrho_1$ to $|z| = \varrho_2$ and is contained in $\varrho_1 \leqq |z| \leqq \varrho_2$. We are going to prove that

$$\int_{\alpha} e^{v_1}|dz| \leqq \frac{32}{3}\int_{\beta} e^{w_1}|dz| + \frac{259}{3} \ . \tag{2.53}$$

The construction clearly implies that α does not enter any of the disks $|z - \zeta_k| < \delta_k$ for which $p_k < 1$ $(k = 1, 2, \ldots, m)$. From (2.36) and the fact that β does not intersect D_1 it follows that the same is true for $p_k \geqq 1$. So α lies in K.

We decompose $\alpha = \alpha_P + \alpha_C$, α_P consisting of a finite number of polygons, α_C of a finite number of circular arcs. We have, by (2.30)

$$\int_{\alpha_C} e^{v_1}|dz| < 1 \ . \tag{2.54}$$

α_P is composed of straight line segments $\alpha_{P1}, \alpha_{P2}, \ldots, \alpha_{Pr}$ which are contained in some closed squares $S_1, S_2, \ldots, S_r$ of the net Σ. It follows from the construction of α that these squares are different from each other. We complete the finite sequence $S_1, S_2, \ldots, S_r$ to an infinite sequence $\{S_i\}$ which enumerates all squares of Σ. Let Z_i denote the centers of S_i $(i = 1, 2, 3, \ldots)$. Some of the Z_i's are identical with $\zeta_{m+1}, \zeta_{m+2}, \ldots, \zeta_n$ (introduced for the definition of w_1), say $Z_{i_1} = \zeta_{m+1}, Z_{i_2} = \zeta_{m+2}, \ldots, Z_{i_{n-m}} = \zeta_n$. For the corresponding masses we introduce the change of notation $P_{i_1} = p_{m+1}, P_{i_2} = p_{m+2}, \ldots, P_{i_{n-m}} = p_n$, defining $P_i = 0$ for all other indices.

Let S_1' and S_i'' denote the closed squares of center Z_i with sides of lengths $3L$ and $(2N+1)L$, respectively. We define

$$A_i = \max_{z\in(S_i'\cap K)} w(z) \quad \text{and} \quad a_i = \min_{z\in(S_i'\cap K)} w(z)$$

(2.33) implies

$$A_i - a_i < \log 2 \ . \tag{2.55}$$

Further, let b_{ik} and B_{ik} denote the maximum and minimum, respectively, of the function $\log|z-\zeta|$, z varying on S_i', ζ on S_k. If S_k is not contained in S_i'', then we have, by (2.27)

$$m_1(b_{ik} - B_{ik}) < \log 2 \ , \tag{2.56}$$

Let $Q_i = \nu(S_i'')$. It follows from (2.35) and the definition of d that

$$Q_i \leqq \tfrac{1}{4} \quad (i = 1, 2, 3, \ldots) \ . \tag{2.57}$$

An arbitrary point z of the plane is contained in at most $(2N+2)^2$ of the sets S_i''. From this we infer that

$$\sum_{i=1}^{\infty} Q_i \leqq 4(N+1)^2 m_1 \ . \tag{2.58}$$

We have

$$\begin{aligned}\int_{\alpha_{Pi}} e^{v_1}|dz| &\leqq \int_{\alpha_{Pi}} \exp\{A_i - \sum_{k=1}^{\infty}{}'' P_k B_{ik} - \int_{S_i''} \log|z-\zeta|\,d\nu(e_\zeta)\}|dz| \\ &\leqq 2\exp\{A_i - \sum_{k=1}^{\infty}{}'' P_k B_{ik}\} \int_0^{\lambda_i/2} x^{-Q_i}dx \leqq \tfrac{32}{3}\exp\{a_i - \sum_{k=1}^{\infty}{}'' P_k b_{ik}\}\,\lambda_i^{1-Q_i} \ .\end{aligned} \tag{2.59}$$

Here, as indicated by the double prime, we do not extend the summation over those indices k for which S_k is contained in S_i''. This estimation is based on the representation (2.32). In the first step we replace $w(z)$ as well as the potential of the masses ν outside S_i'' by a constant. Then we observe that the integral attains its maximum when the total mass Q_i is concentrated in the center of α_{Pi}. This follows from an application of HÖLDER's inequality. (The reader is referred to the proof of Lemma 1 where exactly the same reasoning has been reproduced in full detail.) Finally, in the third step we evaluate the integral and make use of (2.55), (2.56), (2.57) and the inequality $\Sigma'' P_k \leqq m_1$.

The segment α_{Pi} has been introduced as a shortcut of some subpolygon β_i of β which, at least for a portion of length $\geqq \lambda_i$, is contained in S_i'. By (2.36) and the construction of α, $S_i' \subset K$ $(i = 1, 2, \ldots, r)$. Hence

$$\int_{\beta_1} e^{w_1}|dz| \geqq \lambda_i \exp\{a_i - \sum_{k=1}^{\infty}{}'' P_k b_{ik}\} \ . \tag{2.60}$$

Here $w(z)$ as well as the potential of the masses ν outside S_i'' have been replaced by a constant. The potential of the masses on S_i'' has been neglected. (We are allowed to do this because of the second relation of (2.35).)

From (2.59) and (2.60) we conclude that

$$\int_{\alpha_{P_i}} e^{v_1}|dz| \leqq \frac{32}{3}\int_{\beta_i} e^{w_1}|dz| + \frac{32}{3}\exp\{a_i - \sum_{k=1}^{\infty}{}'' P_k b_{ik}\} Q_i \frac{\lambda_i^{1-Q_i} - \lambda_i}{Q_i} \tag{2.61}$$
$$\leqq \frac{32}{3}\int_{\beta_i} e^{w_1}|dz| + \frac{32}{3}\exp\{a_i - \sum_{k=1}^{\infty}{}'' P_k b_{ik}\} Q_i \lambda_i^{3/4}|\log \lambda_i| \qquad (i = 1, 2, \ldots, r).$$

The first step is obvious. In the second inequality we make use of the mean value theorem of differential calculus, (2.57) and the inequality $\lambda_i < 1$ which is implied by the second relation of (2.35). We observe that the final estimation in (2.61) is also valid if $Q_i = 0$, although in this case the intermediate step has no meaning.

Obviously

$$a_i \leqq M \qquad (i = 1, 2, \ldots, r), \tag{2.62}$$

M denoting the maximum of $w(z)$ on K.

It is easy to verify that

$$\sum_{k=1}^{\infty}{}'' P_k b_{ik} \geqq \tfrac{1}{4}\log(NL) + m_1 \log d . \tag{2.63}$$

We have, by (2.62), (2.63), (2.37) and (2.58)

$$\sum_{i=1}^{r}\exp\{a_i - \sum_{k=1}^{\infty}{}'' P_k b_{ik}\} Q_i \lambda_i^{3/4}|\log \lambda_i| \leqq \sum_{i=1}^{r} e^M (NL)^{-1/4} d^{-m_1} Q_i \lambda_i^{3/4}|\log \lambda_i| \tag{2.64}$$
$$\leqq 2\sum_{i=1}^{r} Q_i e^M N^{-1/4} d^{-m_1} \lambda_i^{1/2}|\log \lambda_i| \leqq \frac{2}{m_1(N+1)^2}\sum_{i=1}^{r} Q_i \leqq 8 .$$

We add the inequalities (2.61) and obtain, by (2.64)

$$\int_{\alpha_P} e^{v_1}|dz| \leqq \frac{32}{3}\int_{\beta} e^{w_1}|dz| + \frac{256}{3} . \tag{2.65}$$

(2.53) follows from (2.54) and (2.65). Since (2.26) is implied by (2.53), (2.39) and (2.52), this completes the proof of Lemma 3. Theorem 1 is a consequence of Lemmas 2 and 3.

Proof of Theorem 3. If there exists a subcompact of Ω_0 which intersects infinitely many γ_n, then Theorem 3 follows immediately from an application of Lemma 2. Otherwise we may suppose that (eventually after extraction of a subsequence) the interior regions of $\{\gamma_n\}$ tend monotonically to Ω_0.

We distinguish between two cases depending on whether Ω_0 is parabolic (i. e. identical with the entire plane) or hyperbolic.

(I) *Let Ω_0 be parabolic.* If $\Phi(\Gamma) < -1$, then Theorem 3 is superseded by Theorem 1. We complete the proof of Theorem 3 for parabolic Ω_0 by showing that in the remaining case the hypotheses are incompatible:

Lemma 4. *Suppose that Ω_0 is the entire plane and that $\Phi(\Gamma) > -1$. Let $\{\gamma_n\}$, $n = 1, 2, 2, \ldots$, be a sequence of rectifiable* Jordan *curves whose interior regions tend increasingly to Ω_0. Then*

$$\lim_{n\to\infty} \int_{\gamma_n} e^{u(z)} |dz| = +\infty . \tag{2.66}$$

Proof. There exists an index N and a number $\eta > 0$ such that

$$\Phi(u_1, \gamma_n; \Gamma) - \Phi_2(\Gamma) < -1 + \eta \tag{2.67}$$

for all $n \geqq N$. We choose a circle $\gamma = [|z| = R]$ which encloses γ_N. Let ω_n denote the annular region bounded by γ and γ_n, n being arbitrary but sufficiently large so that $|z| \leqq R$ lies within γ_n. The theory of F. Riesz [27] implies

$$\frac{1}{2\pi} \int_\delta \frac{\partial h_1}{\partial n} |dz| \geqq \Phi(u_1, \gamma; \Gamma) \tag{2.68}$$

and

$$\frac{1}{2\pi} \int_\delta \frac{\partial h_2}{\partial n} |dz| \leqq \Phi(u_2, \gamma_n; \Gamma) \tag{2.69}$$

for every smooth Jordan curve δ in ω_n which encloses γ. Here h_1 and h_2 denote the best harmonic majorants of u_1 and u_2, respectively, in ω_n, and n designates the outer normal. We define

$$h(z) = h_1(z) - h_2(z) \tag{2.70}$$

and conclude from relations (2.67) to (2.70) that

$$\frac{1}{2\pi} \int_\delta \frac{\partial h}{\partial n} |dz| > -1 + \eta . \tag{2.71}$$

There exists a (uniquely determined) conformal representation $z = \varphi_n(\zeta)$ of a (suitable) circular ring $R < |\zeta| < R_n$ onto ω_n such that $\varphi_n(+R) = +R$. We introduce the function

$$\tilde{h}_n(\zeta) = h(\varphi_n(\zeta)) + \log |\varphi_n'(\zeta)| .$$

$\tilde{h}_n$ is harmonic in $R < |\zeta| < R_n$ and we have

$$\int_\gamma \frac{\partial \tilde{h}_n}{\partial n} |d\zeta| = \int_\delta \frac{\partial h}{\partial n} |dz| , \tag{2.72}$$

γ and δ denoting arbitrary smooth JORDAN curves, not null-homotopic and lying in $R<|\zeta|<R_n$ and ω_n, respectively. We state that

$$\begin{aligned}\int_{-\pi}^{+\pi}\tilde{h}_n(\varrho_2 e^{i\varphi})d\varphi &= \int_{-\pi}^{+\pi}\tilde{h}_n(\varrho_1 e^{i\varphi})d\varphi + 2\pi\int_{\varrho_1}^{\varrho_2}\left[\frac{1}{2\pi}\int_{-\pi}^{+\pi}\frac{\partial\tilde{h}_n(\varrho e^{i\varphi})}{\partial\varrho}\varrho\, d\varphi\right]\frac{d\varrho}{\varrho}\\ &\geqq \int_{-\pi}^{+\pi}\tilde{h}_n(\varrho_1 e^{i\varphi})d\varphi + 2\pi(-1+\eta)(\log\varrho_2-\log\varrho_1)\\ &> A + 2\pi(-1+\eta)\log\varrho_2 \qquad (R<\varrho_1<\varrho_2<R_n)\ ,\end{aligned} \tag{2.73}$$

where A is a constant not depending on n, ϱ_1 and ϱ_2. In this inequality we first apply (2.71) and (2.72). Then we let $\varrho_1\rightarrow R$. We complete the proof of (2.73) by showing that the limits

$$L_n = \lim_{\varrho_1\rightarrow R}\int_{-\pi}^{+\pi}\tilde{h}_n(\varrho_1 e^{i\varphi})d\varphi \qquad (n=1,2,3,\ldots)$$

exist and are uniformly bounded. To this end we consider the function

$$U_1(z) = \begin{cases} h_1(z) & \text{in} \quad \omega_n \\ u_1(z) & \text{on} \quad \gamma\cup(\Delta,\gamma)\ , \end{cases}$$

which is defined and subharmonic throughout the annulus (Δ,γ_n). The conformally transplanted functions, $\tilde{U}_{1n}(\zeta)=U_1(\varphi_n(\zeta))$, $n=1,2,3,\ldots$, are subharmonic in circular rings $R_n'<|\zeta|<R_n$ $(R_n'<R)$. It is well known (see e. g. [25, p. 8]) that the arithmetic mean of a subharmonic function on concentric circles is a continuous function of the radius. Consequently

$$\begin{aligned}\lim_{\varrho_1\rightarrow R}\frac{1}{2\pi}\int_{-\pi}^{+\pi}h_1(\varphi_n(\varrho_1 e^{i\varphi}))\,d\varphi &= \lim_{\varrho_1\rightarrow R}\frac{1}{2\pi}\int_{-\pi}^{+\pi}\tilde{U}_{1n}(\varrho_1 e^{i\varphi})d\varphi\\ &= \frac{1}{2\pi}\int_{-\pi}^{+\pi}\tilde{U}_{1n}(Re^{i\varphi})d\varphi = \frac{1}{2\pi}\int_{-\pi}^{+\pi}u_1(\varphi_n(Re^{i\varphi}))\,d\varphi\ .\end{aligned}$$

An analogous relation holds with index 2. By subtraction we obtain

$$\lim_{\varrho_1\rightarrow R}\int_{-\pi}^{+\pi}h(\varphi_n(\varrho_1 e^{i\varphi}))\,d\varphi = \int_{-\pi}^{+\pi}u(\varphi_n(\mathrm{Re}^{i\varphi}))\,d\varphi\ . \tag{2.74}$$

Furthermore

$$\lim_{\varrho_1\rightarrow R}\int_{-\pi}^{+\pi}\log|\varphi_n'(\varrho_1 e^{i\varphi})|\,d\varphi = \int_{-\pi}^{+\pi}\log|\varphi_n'(Re^{i\varphi})|\,d\varphi\ , \tag{2.75}$$

since φ_n is analytic on $|\zeta|=R$. From (2.74), (2.75) and the definition of $\tilde{h}_n$ we infer that

$$L_n = \int_{-\pi}^{+\pi}u(\varphi_n(Re^{i\varphi}))\,d\varphi + \int_{-\pi}^{+\pi}\log|\varphi_n'(Re^{i\varphi})|\,d\varphi \qquad (n=1,2,3,\ldots)\ . \tag{2.76}$$

We have yet to verify that the sequence $\{L_n\}$ is bounded. Actually we shall prove more, namely

$$\lim_{n\to\infty} L_n = \int_{-\pi}^{+\pi} u(Re^{i\varphi})d\varphi \ . \tag{2.77}$$

we first observe that (2.77) will be an immediate consequence of (2.76) after it has been shown that

$$\lim_{n\to\infty} \varphi_n(\zeta) = \zeta \tag{2.78}$$

(and, consequently, $\lim_{n\to\infty} \varphi_n'(\zeta) \equiv 1$), uniformly on $|\zeta| = R$. Indeed, then

$$\lim_{n\to\infty} \int_{-\pi}^{+\pi} u(\varphi_n(Re^{i\varphi}))\, d\varphi = \lim_{n\to\infty} \int_{-\pi}^{+\pi} u(Re^{i\psi}) \left| \frac{d\varphi_n^{-1}(Re^{i\psi})}{dz} \right| d\psi = \int_{-\pi}^{+\pi} u(Re^{i\psi})d\psi\,,$$

whereas the second term on the right-hand side of (2.76) tends to 0 as $n\to\infty$.

In order to prove (2.78) we first extend the definition of $\varphi_n(\zeta)$ by reflection at $|\zeta| = R$. The resulting function is schlicht in $R^2/R_n<|\zeta|<R_n$, and it maps $|\zeta| = R$ onto $|z| = R$. We mention that $R_n\to\infty$ as $n\to\infty$, due to the fact that ω_n tends to $R<|z|<+\infty$.

By making use of Cantor's diagonal process we select a subsequence $\{\varphi_n^*\}$ of $\{\varphi_n\}$ which converges at an enumerable set of points possessing limit points in both $0<|\zeta|<R$ and $R<|\zeta|<+\infty$. $\{\varphi_n^*\}$ is normal [7, pp. 176, 179] in these regions. Hence, by a theorem of Vitali [7, p. 186], it converges there, uniformly on every subcompact. From this we conclude that $\{\varphi_n^*\}$ converges even in $0<|\zeta|<+\infty$, uniformly on every subcompact. The limit function φ is either schlicht meromorphic or a constant [7, p. 193]. The latter possibility can immediately be excluded, since convergence to a constant, uniformly on $|\zeta| = R$, is incompatible with the fact that φ_n maps $|\zeta| = R$ *onto* $|z| = R$ for all n.

The image region has to be of the same conformal type as $0<|\zeta|<+\infty$. Furthermore, it follows from results of A. Hurwitz [7, pp. 191–192] that φ does not assume the values 0 and ∞. Hence $0<|\zeta|<+\infty$ is mapped onto $0<|z|<+\infty$ and φ is necessarily a linear transformation. The points 0, ∞ and $+R$ are fixed. Consequently, $\varphi(\zeta) \equiv \zeta$.

Finally, if $\{\varphi_n(\zeta)\}$ does not converge to ζ, then $\{\varphi_n^*\}$ can be chosen such that, for some ζ_0 $(0<|\zeta_0|<+\infty)$, $\lim_{n\to\infty} \varphi_n^*(\zeta_0) = \zeta_1 \neq \zeta_0$. We thus obtain a contradiction, since the above reasoning yields $\lim_{n\to\infty} \varphi_n^*(\zeta_0) = \zeta_0$. This completes the proof of (2.78) and, with it, of (2.73).

By the theorem of the arithmetic and geometric means [15, p. 137]

$$\int_{-\pi}^{+\pi} \exp\, \{\tilde{h}_n(\varrho_2 e^{i\varphi})\}\, \varrho_2 d\varphi \geqq 2\pi\varrho_2 \exp\left\{\frac{1}{2\pi} \int_{-\pi}^{+\pi} \tilde{h}_n(\varrho_2 e^{i\varphi}) d\varphi\right\} . \tag{2.79}$$

Furthermore

$$\limsup_{\varrho_2 \to R_n} \int_{-\pi}^{+\pi} \exp\{\tilde{h}_n(\varrho_2 e^{i\varphi})\}\varrho_2 d\varphi \leqq \int_{\gamma_n} e^u |dz| . \tag{2.80}$$

This relation will be verified in the proof of Lemma 5 (see formula (2.130)). By (2.73), (2.79) and (2.80)

$$\int_{\gamma_n} e^u |dz| \geqq 2\pi e^{A/2\pi} R_n^\eta . \tag{2.81}$$

(2.66) is implied by (2.81), since $R_n \to \infty$ as $n \to \infty$.

(II) *Let Ω_0 be hyperbolic.* Again we may suppose that the interior regions of $\{\gamma_n\}$ tend increasingly to Ω_0. Furthermore, we assume that $\Phi(\Gamma) = +\infty$, since otherwise Theorem 3 is superseded by Theorem 2. Then there exists an index N such that

$$\Phi(u_1, \gamma_N; \Gamma) - \Phi_2(\Gamma) > 0 \tag{2.82}$$

and

$$\Phi_2(\Gamma) - \Phi(u_2, \gamma_N; \Gamma) < \frac{1}{16} . \tag{2.83}$$

Let $n > N$ be arbitrary. Theorem 3 (for the case remaining to be treated) is an obvious consequence of Lemma 2 and the following result:

Lemma 5. *There exists a rectifiable curve σ_n, leading from γ_N to γ_n, such that*

$$\int_{\sigma_n} e^u |dz| \leqq \frac{M}{\sin(\pi/8)\cos(\pi/32)} . \tag{2.84}$$

Proof. By F. RIESZ [27, II, p. 357] the representations

$$u_1(z) = h_1(z) - \int_\omega g(z, \zeta) d\mu_1(e_\zeta) \tag{2.85}$$

and

$$u_2(z) = h_2(z) - \int_\omega g(z, \zeta) d\mu_2(e_\zeta) \tag{2.86}$$

hold in the annulus $\omega = (\gamma_N, \gamma_n)$. Here h_1 and h_2 denote the best harmonic majorants of u_1 and u_2, respectively, in ω and g is GREEN's function for this region. The theory of F. RIESZ implies further that

$$\Phi(u_1, \gamma_N; \Gamma) \leqq \frac{1}{2\pi} \int_\gamma \frac{\partial h_1}{\partial n} |dz| \leqq \Phi(u_1, \gamma_n); \Gamma) \tag{2.87}$$

and

$$\Phi(u_2, \gamma_N; \Gamma) \leqq \frac{1}{2\pi} \int_\gamma \frac{\partial h_2}{\partial n} |dz| \leqq \Phi(u_2, \gamma_n; \Gamma) \tag{2.88}$$

for every smooth JORDAN curve γ in ω which encloses $\gamma_N \cdot n$ denotes the outer normal. We define

$$h(z) = h_1(z) - h_2(z) \tag{2.89}$$

and introduce the abbreviation

$$\alpha = \frac{1}{2\pi}\int_{\gamma} \frac{\partial h}{\partial n}|dz| \; . \tag{2.90}$$

α does not depend on the choice of γ. (2.85), (2.86) and (2.89) imply

$$u(z) \leqq h(z) + \int_{\omega} g(z, \zeta) d\mu_2(e_\zeta) \; . \tag{2.91}$$

From (2.83) follows

$$\mu_2(\omega) < \frac{1}{16} \; . \tag{2.92}$$

We begin with the special case where γ_N and γ_n are circles $|z| = R_N$ and $|z| = R_n$ $(0 < R_N < R_n < +\infty)$. We state that

$$\int_{R_N}^{R_n} e^{u(x)} dx \leqq \frac{M}{\sin(\pi/8)\cos(\pi/32)} \; . \tag{2.93}$$

More generally, (2.93) holds if the integration is extended over any radial segment $(R_N e^{i\Theta}, R_n e^{i\Theta})$, $0 \leqq \Theta < 2\pi$. First we prove (2.93) under the additional assumption that the measure $\mu_2(e_\zeta)$ is concentrated in one point ζ_0. We have, for positive real x

$$g(x, \zeta_0) \leqq g(x, |\zeta_0|) \; . \tag{2.94}$$

This estimation can be verified by considering, for fixed x, the Fourier expansion of $g(x, \zeta_0)$ on the circle $|\zeta_0| = \text{const}$. We conclude from (2.91), (2.92) and (2.94) that, for positive real x

$$u(x) \leqq h(x) + \frac{1}{16} g(x, |\zeta_0|) \; . \tag{2.95}$$

If we slit ω along the negative real axis we obtain a simply connected region ω'. We define

$$f(z) = \exp\{(h(z) + ih^*(z))/2\} \; , \tag{2.96}$$

where h^* is conjugate harmonic to h and, for example, $h^*((R_N + R_n)/2) = 0$. f is regular analytic in ω'. It can be continued into ω as a multiple-valued function with (multiplicative) period $e^{\pi\alpha i}$. We consider

$$\varphi(z) = z^{-\alpha/2} f(z) \; , \tag{2.97}$$

defining $z^{-\alpha/2}$ to be positive real on the interval (R_N, R_n). φ is analytic in ω'

and can be continued into ω in a unique way. There exists a decomposition of the form

$$\varphi(z) = \varphi_1(z) + i\varphi_2(z) \ , \tag{2.98}$$

where φ_1 and φ_2 are analytic and one-valued in ω, real for real z. Indeed, just put $\varphi_1(z) = \Sigma a_k z^k$ and $\varphi_2(z) = \Sigma b_k z^k$, where $\Sigma(a_k + i b_k) z^k$ is the LAURENT expansion of φ in ω (a_k, b_k real; $k = 0, \pm 1, \pm 2, \ldots$). From (2.97) and (2.98) we conclude that f admits the decomposition

$$f(z) = f_1(z) + i f_2(z) \ , \tag{2.99}$$

where f_1 and f_2 are analytic in ω' and

$$\arg f_1 \equiv \arg f_2 \equiv 0 \qquad (\mathrm{mod}\ \pi) \tag{2.100}$$

on (R_N, R_n). Furthermore, on $(-R_N, -R_n)$ we have

$$\arg f_1 \equiv \arg f_2 \equiv \pi\alpha/2 \qquad (\mathrm{mod}\ \pi) \tag{2.101}$$

or

$$\arg f_1 \equiv \arg f_2 \equiv -\pi\alpha/2 \qquad (\mathrm{mod}\ \pi) \tag{2.102}$$

depending on whether the real axis is approached from above or below. For real z

$$|f(z)|^2 = |f_1(z)|^2 + |f_2(z)|^2 \ . \tag{2.103}$$

We further have, for $-\pi < t < +\pi$ and $R_N < r < R_n$

$$|f(re^{it})|^2 = |f_1(re^{it})|^2 + |f_2(re^{it})|^2 + i[f_1(re^{-it})f_2(re^{it}) - f_1(re^{it})f_2(re^{-it})] \ .$$

Since the bracket contains an odd function of t, it follows that

$$\int_{-\pi}^{+\pi} |f(re^{it})|^2 dt = \int_{-\pi}^{+\pi} (|f_1(re^{it})|^2 + |f_2(re^{it})|^2) dt \tag{2.104}$$

for $R_N < r < R_n$. Let $\omega'' = \omega \cap [\mathrm{Im}\, z > 0]$. Let $g^*(z, |\zeta_0|)$ be conjugate harmonic to $g(z, |\zeta_0|)$ in ω'', satisfying

$$g^*(x, |\zeta_0|) = 0 \quad \text{for} \quad R_N < x < |\zeta_0| \ . \tag{2.105}$$

We then have

$$g^*(x, |\zeta_0|) = \pi \quad \text{for} \quad |\zeta_0| < x < R_n \tag{2.106}$$

and

$$0 < g^*(x, |\zeta_0|) < \pi \tag{2.107}$$

throughout ω''. We define

$$F_1(z) = f_1^2(z) \exp\left\{\frac{1}{16}[g(z, |\zeta_0|) + i g^*(z, |\zeta_0|)]\right\} \ . \tag{2.108}$$

We state that

$$\sin(\pi/8)\cos(\pi/32)\int_{r_N}^{r_n}|f_1(x)|^2\exp\left\{\frac{1}{16}g(x,|\zeta_0|)\right\}dx$$
$$\leqq r_N\exp\left\{\frac{1}{16}g(r_N,|\zeta_0|)\right\}\int_0^\pi|f_1(r_Ne^{it})|^2dt+r_n\exp\left\{\frac{1}{16}g(r_n,|\zeta_0|)\right\}\int_0^\pi|f_1(r_ne^{it})|^2dt \quad (2.109)$$

for any two radii r_N and r_n $(R_N<r_N<r_n<R_n)$. For the proof of this relation we distinguish between two cases:

(A) $|\alpha-q|\geqq 1/4$, *where q denotes the or one of the odd integers nearest to* α.

(B) *There exists an odd integer q such that* $|\alpha-q|<1/4$.

Case (A): We apply CAUCHY's integral theorem to F_1. We integrate along the boundary of ω'' (described in the positive sense), replacing the circular boundary by the approximating circles $|z|=r_N$ and $|z|=r_n$. At first we bypass ζ_0 on a small semicircle of radius ε. But we observe that the integral along this semicircle tends to 0 with ε. So we have

$$I_1+I_2+I_3+I_4=0\ , \quad (2.110)$$

where I_1, I_2, I_3 and I_4 denote the respective integrals along $(r_N,|\zeta_0|)$, $(|\zeta_0|, r_n)$, $(-r_n,-r_N)$ and the two semicircles. (2.100), (2.105) and (2.108) imply

$$\arg I_1=0\ . \quad (2.111)$$

By (2.100), (2.106) and (2.108) we have

$$\arg I_2=\pi/16\ . \quad (2.112)$$

From (2.111) and (2.112) we infer

$$|I_1+I_2|\geqq\cos(\pi/32)(|I_1|+|I_2|)\ . \quad (2.113)$$

We further have

$$0\leqq\arg(I_1+I_2)\leqq\pi/16\ . \quad (2.114)$$

From the assumption that $|\alpha-q|\geqq 1/4$ (q denoting the or one of the odd integers nearest to α), (2.101), (2.107) and (2.108) we conclude that $\arg I_3$ distinguishes itself from the nearest odd multiple of π by at least $3\pi/16$. Consider the triangle with sides I_1+I_2, I_3 and I_4 in the complex number plane. Since the angle between I_1+I_2 and I_3 is $\geqq\pi/8$ we obtain the estimation

$$|I_4|\geqq\sin(\pi/8)|I_1+I_2|\ . \quad (2.115)$$

From (2.113), (2.115) and the property (2.94) of Green's function it follows indeed that

$$\sin(\pi/8)\cos(\pi/32)\int_{r_N}^{r_n}|f_1(x)|^2\exp\left\{\frac{1}{16}g(x,|\zeta_0|)\right\}dx$$
$$= \sin(\pi/8)\cos(\pi/32)(|I_1|+|I_2|) \leqq |I_4|$$
$$\leqq r_N \exp\left\{\frac{1}{16}g(r_N,|\zeta_0|)\right\}\int_0^\pi |f_1(r_N e^{it})|^2 dt$$
$$+ r_n \exp\left\{\frac{1}{16}g(r_n,|\zeta_0|)\right\}\int_0^\pi |f_1(r_n e^{it})|^2 dt \ .$$

Case (*B*): From (2.82), (2.87), (2.88), (2.90) and the inequality $|\alpha - q| < 1/4$ we conclude that $q > 0$.

We first make the additional hypothesis that $f_1 \neq 0$ *on*

$$D = [r_N \leqq |z| \leqq r_n] \frown [\operatorname{Im} z \geqq 0] \ .$$

Our method is again based on an application of CAUCHY's integral theorem to the function F_1, but this time we do not integrate along the boundary of D. (The previous argument breaks down because it does not yield any more an inequality of the type (2.115)).

We state that there exists an *analytic curve* τ with the following properties: (a) τ *is contained in D and leads from* $|z| = r_n$ *to* $|z| = r_N$, (b) $\arg[f_1^2(z)dz] \equiv 0$ *along* τ.

In order to verify the existence of τ we decompose the function $\log f_1^2$ into its real and imaginary parts, $\log f_1^2 = H_1(z) + iH_1^*(z)$. With every point z on D we associate the unit vector $\exp\{-iH_1^*(z)\}$. The thus defined vector field has no singular points, since $f_1 \neq 0$ on D. Through every point z on D there passes exactly one streamline (i. e. solution of the differential equation

$$dz/dt = \exp\{-iH_1^*(z)\}) \ ,$$

which begins and ends on the boundary of D. Obviously, $\arg[f_1^2(z)dz] \equiv 0$ along these lines. They are analytic since the conformal mapping $w = \int_{z_0}^{z} f_1^2(z)dz$ transforms them into straight lines. So the existence of τ will be asserted if we can verify that at least one of these streamlines leads from $|z| = r_n$ to $|z| = r_N$.

On the intervals (r_N, r_n) and $(-r_n, -r_N)$, $H_1^* = 0$ and $H_1^* = \pi\alpha$, respectively. Since $q \geqq 1$, $\alpha > \frac{3}{4}$. The continuity of H_1^* and the relations $H_1^*(r_N) = 0$, $H_1^*(-r_N) = \pi\alpha > \frac{3}{4}\pi$ imply: There exists at least one open subarc $S = [\Theta_1 > \Theta = \arg z > \Theta_2]$ of the semicircle

$$C_N = [|z| = r_N] \frown [\operatorname{Im} z \geqq 0]$$

with the following properties:

(a) $H_1^*(r_N e^{i\Theta_1}) \equiv -\Theta_1 + \frac{3\pi}{2} \qquad (\mathrm{mod}\ 2\pi)$,

(b) $H_1^*(r_N e^{i\Theta_2}) \equiv -\Theta_2 + \frac{\pi}{2} \qquad (\mathrm{mod}\ 2\pi)$,

(c) $-\Theta + \frac{3\pi}{2} > H_1^*(r_N e^{i\Theta}) > -\Theta + \frac{\pi}{2} \qquad (\mathrm{mod}\ 2\pi)$

for all Θ in S.

Since $H_1^*(r_N e^{i\Theta})$ is of bounded variation in Θ, there exists but a finite number of such arcs. We denote them by $S_1, S_2, \ldots, S_m$ and suppose that they have been arranged in such a way that $\Theta_{k-1,2} > \Theta_{k1}$ $(k = 2, 3, \ldots, m)$, where $S_k = [\Theta_{k1} > \Theta > \Theta_{k2}]$. In each S_k there is (at least) one point Θ_{k0} at which $H_1^*(r_N e^{i\Theta_{k0}}) = -\Theta_{k0} + \pi\ (\mathrm{mod}\ 2\pi)$.

The geometrical meaning of these conditions is the following: At $r_N e^{i\Theta_{k1}}$ and at $r_N e^{i\Theta_{k2}}$ the field vector is tangential to C_N, directed away from S_k. It points into $|z| < r_N$ at all points in S_k and is, in particular, normal to C_N at $r_N e^{i\Theta_{k0}}$.

We are now going to prove that at least one of the m streamlines ending at $r_N e^{i\Theta_{10}}, r_N e^{i\Theta_{20}}, \ldots, r_N e^{i\Theta_{m0}}$, must begin on $|z| = r_n$. This will complete the existence proof for τ.

We state that $H_1^*(r_N e^{i\Theta_{10}}) \geqq \pi q - \Theta_{10}$. Indeed, otherwise

$$H_1^*(r_N e^{i\Theta_{10}}) \leqq \pi(q-2) - \Theta_{10}.$$

From this would follow

$$H_1^*(r_N e^{i\Theta_{11}}) \leqq \pi\left(q - \frac{3}{2}\right) - \Theta_{11} ,$$

which, in turn, would imply the existence of at least one S_k between $-r_N$ and $r_N e^{i\Theta_{11}}$, contrary to hypothesis.

Let z_0 be the point at which the streamline γ ending at $r_N e^{i\Theta_{10}}$ begins. We consider all possibilities:

(a) $|z_0| = r_n$. Then there is nothing left to prove.

(b) z_0 positive real. This never occurs because the positive real axis is itself a streamline.

(c) $z_0 = r_N e^{i\Theta_0}$ $(\Theta_{10} > \Theta_0 > 0)$. It is easy to verify that $\arg dz$ increases by at least $\frac{\pi}{2} + \Theta_{10} - \Theta_0$ if we follow γ from z_0 to $r_N e^{i\Theta_{10}}$. From this, the inequality $H_1^*(r_N e^{i\Theta_{10}}) \geqq \pi q - \Theta_{10}$ and the fact that $\arg[f_1^2(z)dz] \equiv 0$ along γ we conclude that $H_1^*(z_0) \geqq \pi(q + \frac{1}{2}) - \Theta_0$. Hence, there exists at least one more S_k between z_0 and r_N. Furthermore, we have again

$$H_1^*(r_N e^{i\Theta_{k0}}) \geqq \pi q - \Theta_{k0} \ .$$

(d) $z_0 = r_N e^{i\Theta_0}$ $(\pi > \Theta_0 > \Theta_{10})$. It is easy to see that $\arg dz$ decreases by at least $\frac{\pi}{2} + \Theta_0 - \Theta_{10}$ if we follow γ from z_0 to $r_N e^{i\Theta_{10}}$. Hence

$$H_1^*(r_N e^{i\Theta_{10}}) - H_1^*(z_0) \geqq \frac{\pi}{2} + \Theta_0 - \Theta_{10} \ .$$

Furthermore, $H_1^*(z_0) \geqq \pi(q + \frac{1}{2}) - \Theta_0$. (Otherwise, since the vector $\exp\{-iH_1^*(z_0)\}$ points into D, we would have $H_1^*(z_0) \leqq \pi(q - \frac{1}{2}) - \Theta_0$. But this would imply the existence of an S_k between $-r_N$ and z_0, contrary to hypothesis). We conclude that $H_1^*(r_N e^{i\Theta_{10}}) \geqq \pi(q+1) - \Theta_{10}$. But, knowing that $H_1^*(r_N e^{i\Theta_{10}}) + \Theta_{10}$ is an odd multiple of π, we infer that even $H_1^*(r_N e^{i\Theta_{10}}) \geqq \pi(q+2) - \Theta_{10}$ is satisfied. Consequently,

$$H_1^*(r_N e^{i\Theta_{12}}) \geqq \pi\left(q + \frac{3}{2}\right) - \Theta_{12} \ .$$

So there must exist a second arc S_2 and we have $H_1^*(r_N e^{i\Theta_{20}}) \geqq \pi q - \Theta_{20}$.

(e) z_0 negative real. The previous argument applies also to this case and yields the same conclusion.

As a result of this discussion we have now the following alternative: Either (a) occurs, and then the proof is completed, or the above constructions lead to another S_k such that $H_1^*(r_N e^{i\Theta_{k0}}) \geqq \pi q - \Theta_{k0}$. In the second case we repeat the above reasoning. (The reader will convince himself that this can be done without difficulty. The following trivial observation is useful in this connection: Streamlines do not intersect. Hence, for example, the one ending at $r_N e^{i\Theta_{k0}}$ does not begin on $|z| = r_N$ between the points z_0 and $r_N e^{i\Theta_{10}}$.) We arrive at the same alternative again. But there are only a finite number of S_k's. Hence, if we iterate this argument we must meet with case (a) after a finite number of steps. So τ exists.

We now apply CAUCHY's integral theorem to F_1, integrating in the positive sense along the closed curve consisting of τ, (r_N, r_n) and subarcs of $|z| = r_N$ and $|z| = r_n$. We have

$$I_1 + I_2 + I_3' + I_4' = 0 \ , \tag{2.116}$$

where I_1, I_2, I_3' and I_4' denote the respective integrals along $(r_N, |\zeta_0|)$, $(|\zeta_0|, r_n)$, τ and the two connecting circular arcs. From (2.107) and the fact that $\arg [f_1^2(z)dz] \equiv 0$ along τ we infer

$$0 \leqq \arg I_3' \leqq \pi/16 \ . \tag{2.117}$$

Consider the triangle with sides $I_1 + I_2$, I_3' and I_4' in the complex number plane. By (2.114) and (2.117), the angle between $I_1 + I_2$ and I_3' is $\geqq 15\pi/16$. Hence

$$|I_4'| \geqq |I_1 + I_2| \ . \tag{2.118}$$

It follows from (2.113), (2.118) and the property (2.94) of GREEN's function that

$$\cos(\pi/32)\int_{r_N}^{r_n} |f_1(x)|^2 \exp\left\{\frac{1}{16} g(x, |\zeta_0|)\right\} dx = \cos(\pi/32)(|I_1| + |I_2|) \leqq |I_4'|$$

$$\leqq r_N \exp\left\{\frac{1}{16} g(r_N, |\zeta_0|)\right\} \int_0^\pi |f_1(r_N e^{it})|^2 dt + r_n \exp\left\{\frac{1}{16} g(r_n, |\zeta_0|)\right\} \int_0^\pi |f_1(r_n e^{it})|^2 dt \ .$$

This inequality implies (2.109). It has been deduced under the assumption that $f_1 \neq 0$ on D. *We now admit zeros, but still assume that* $f_1 \neq 0$ *on the circles* $|z| = r_N$ *and* $|z| = r_n$. Let $a_1, a_2, \ldots, a_s$ denote the zeros of f_1 on D. We introduce the function

$$f_{10}(z) = f_1(z) \prod_{i=1}^{s} \exp\{[g(z, a_i) + g(z, \bar{a}_i)] + i[g^*(z, a_i) + g^*(z, \bar{a}_i)]\} \ . \tag{2.119}$$

Here g denotes GREEN's function for the annulus $r_N < |z| < r_n$ and g^* is conjugate harmonic to g, defined to vanish at $z = r_N$. f_{10} is analytic and $\neq 0$ on D. Obviously

$$|f_{10}(z)| = |f_1(z)| \tag{2.120}$$

on $|z| = r_N$ and on $|z| = r_n$. Furthermore

$$|f_{10}(z)| \geqq |f_1(z)| \ , \tag{2.121}$$

everywhere on D. We conclude from (2.100) and (2.119) that

$$\arg f_{10} \equiv 0 \quad (\mathrm{mod}\ \pi) \tag{2.122}$$

(2.101) and (2.119) imply

$$\arg f_{10} \equiv \pi\alpha_0/2 \quad (\mathrm{mod}\ \pi) \ , \tag{2.123}$$

where $\alpha_0 > \alpha$. We can write $\alpha_0 = q_0 + \vartheta_0$, where q_0 is a positive odd integer and $|\vartheta_0| \leqq 1$. We distinguish between two cases:

(B_1) $|\vartheta_0| < 1/4$. The above reasoning can be applied to $f_{10}(z)$ since this function has no zeros on D. We obtain relation (2.109), but with f_{10} taking the place of f_1. It follows from (2.120) and (2.121) that the unmodified inequality (2.109) is true a fortiori.

(B_2) $|\vartheta_0| \geqq 1/4$. By the method used in case (A) we prove (2.109), f_1 being replaced by f_{10}. The unmodified inequality (2.109) follows as mentioned.

We can easily free ourselves from the assumption that $f_1 \neq 0$ *on the circles* $|z| = r_N$ *and* $|z| = r_n$. For, if this hypothesis should not be fulfilled, then we first prove (2.109) for neighboring circles and afterwards pass to the limit.

(2.109) is also satisfied by f_2. Analogous estimations hold for the lower half-annulus. By adding these four inequalities and making use of (2.95), (2.96), (2.103) and (2.104) we obtain

$$2 \sin(\pi/8) \cos(\pi/32) \int_{r_N}^{r_n} e^{u(x)} dx \leqq r_N \exp\left\{\frac{1}{16} g(r_N, |\zeta_0|)\right\} \int_{-\pi}^{+\pi} |f(r_N e^{it})|^2 dt$$
$$+ r_n \exp\left\{\frac{1}{16} g(r_n, |\zeta_0|)\right\} \int_{-\pi}^{+\pi} |f(r_n e^{it})|^2 dt \ . \tag{2.124}$$

The best harmonic majorants h_1 and h_2 are limits of decreasing sequences, $\{h_{1k}\}$ and $\{h_{2l}\}$ $(k, l = 1, 2, 3, \ldots)$, consisting of functions which are harmonic in the region ω, continuous on its closure. Furthermore, on the boundary $h_{1k} \searrow u_1$ and $h_{2l} \searrow u_2$ for $k, l \nearrow \infty$.

Let k, l be fixed. The function $|z| \exp\{h_{1k}(z) - h_{2l}(z)\}$ is subharmonic in ω. Hence the integral

$$L(r \exp\{h_{1k} - h_{2l}\}; r) = \int_{-\pi}^{+\pi} r \exp\{h_{1k}(re^{it}) - h_{2l}(re^{it})\} dt$$

is a convex function of $\log r$. Consequently, for $R_N < r < R_n$

$$L(r \exp\{h_{1k} - h_{2l}\}; r) \leqq \max [L(r \exp\{h_{1k} - h_{2l}\}; R_N),$$
$$L(r \exp\{h_{1k} - h_{2l}\}; R_n)] \ .$$

Letting first $k \to \infty$, then $l \to \infty$, we obtain, by hypothesis (b) of Theorem 3

$$L(r|f|^2; r) = L(re^h; r) \leqq \max [L(re^u; R_N), L(re^u; R_n)] < M \ .$$

Since, furthermore, $g(r_N, |\zeta_0|) \to 0$ for $r_N \to R_N$ and $g(r_n, |\zeta_0|) \to 0$ for $r_n \to R_n$, (2.93) is a consequence of (2.124).

Let us now proceed to general measures μ_2. We define

$$S_K = \int_{R_N}^{R_n} \exp\{h^K(x) + \int_{\omega} g^K(x, \zeta) d\mu_2(e_\zeta)\} dx \ , \tag{2.125}$$

where $g^K(x, \zeta) = \min[K, g(x, \zeta)]$ and $h^K(x) = \min[K, h(x)]$, K denoting an arbitrary positive constant. We first examine the case where $\mu_2(e_\zeta)$ consists of a finite number of concentrated masses: $\alpha_1 p$ in ζ_1, $\alpha_2 p$ in $\zeta_2, \ldots, \alpha_m p$ in ζ_m, $\Sigma_1^m \alpha_i = 1$, $\alpha_i > 0$ $(i = 1, 2, \ldots, m)$ and $p < 1/16$. An application of HÖLDER's inequality [15, p. 140] and the above result yields

$$\int_{R_N}^{R_n} \exp\{h^K(x) + p\sum_{i=1}^{m}\alpha_i g^K(x,\zeta_i)\}\,dx = \int_{R_N}^{R_n} \prod_{i=1}^{m} [\exp\{h^K(x) + p g^K(x,\zeta_i)\}]^{\alpha_i}\,dx$$

$$\leqq \prod_{i=1}^{m} [\int_{R_N}^{R_n} \exp\{h^K(x) + p g^K(x,\zeta_i)\}dx]^{\alpha_i} \leqq \prod_{i=1}^{m} [\int_{R_N}^{R_n} \exp\{h(x) + p g(x,\zeta_i)\}dx]^{\alpha_i}$$

$$\leqq \frac{M}{\sin(\pi/8)\cos(\pi/32)} \,. \tag{2.126}$$

One proves without difficulty that it is always possible to approximate S_K arbitrarily close by substituting for $\mu_2(e_\zeta)$ a finite number of concentrated masses of total weight $p = \mu_2(\omega) < 1/16$. Therefore, we infer from (2.126) that

$$S_K \leqq \frac{M}{\sin(\pi/8)\cos(\pi/32)} \,. \tag{2.127}$$

Letting $K \to \infty$ we obtain (2.93) as a consequence of (2.91), (2.125) and (2.127).

We now admit arbitrary rectifiable Jordan curves γ_N and γ_n. Then there exists a conformal representation $z = \varphi(\zeta)$ of some suitable circular ring $R_N < |\zeta| < R_n$ $(0 < R_N < R_n < +\infty)$ onto the annular region ω, bounded by γ_N and γ_n, such that the boundary components γ_N and γ_n correspond to $|\zeta| = R_N$ and $|\zeta| = R_n$, respectively. Then the flux $\Phi(u,\gamma;\Gamma)$ is invariant if u is transformed according to (2.8). We are going to prove that (2.84) is satisfied if σ_n is the image of the interval (R_N, R_n) on the ξ-axis $(\zeta = \xi + i\eta)$. Let $\tilde{u}(\zeta)$ be defined by (2.8). Obviously, (2.84) is equivalent to

$$\int_{R_N}^{R_n} e^{\tilde{u}(\xi)}d\xi \leqq \frac{M}{\sin(\pi/8)\cos(\pi/32)} \,. \tag{2.128}$$

All relevant quantities $(e^u|dz|$, mass, flux) are invariant under the transformation (2.8). For this reason the proof of (2.128) is essentially an application of the already treated special case to the function $\tilde{u}$. The only difficulty which arises stems from the boundary behavior of φ.

First we assume that μ_2 is concentrated in one point. Let $h_1(z)$ and $h_2(z)$ denote the best harmonic majorants in ω of $u_1(z)$ and $u_2(z)$, respectively. We define $h(z) = h_1(z) - h_2(z), f(z) = \exp\{(h(z) + ih^*(z))/2\}$ (h^* being conjugate harmonic to h) and introduce the transplanted functions $\tilde{h}(\zeta) = h(\varphi(\zeta))$, $\tilde{f}(\zeta) = f(\varphi(\zeta))$. Assume for a moment that the inequalities

$$\limsup_{r\to R_N} \int_{-\pi}^{+\pi} |\tilde{f}^2(re^{it})\varphi'(re^{it})|\,r\,dt \leqq \int_{\gamma_N} e^{u(z)}|dz| \tag{2.129}$$

and

$$\limsup_{r\to R_n} \int_{-\pi}^{+\pi} |\tilde{f}^2(re^{it})\varphi'(re^{it})|\,r\,dt \leqq \int_{\gamma_n} e^{u(z)}|dz| \tag{2.130}$$

have been demonstrated. Then we would obtain (2.128) as an immediate consequence of (2.129), (2.130), Hypothesis (b) of Theorem 3 and (2.124) (u, f^2 being replaced by $\tilde{u}, \tilde{f}^2\varphi'$). Since an application of HÖLDER's inequality would enable us again to get rid of the special hypothesis about μ_2, the proof of Lemma 5 would thus be complete.

We are left to verify (2.129) and (2.130). Let $\{h_{1k}(z)\}$ and $\{h_{2l}(z)\}$ be defined as above, i. e. sequences of functions which are harmonic in ω, continuous on the closure and which tend decreasingly to $h_1(z)$ and $h_2(z)$, respectively. Let $h_{kl} = h_{1k} - h_{2l}$. We introduce the functions

$$f_{kl}(z) = \exp\{(h_{kl}(z) + i h_{kl}^*(z))/2\} \qquad (k, l = 1, 2, 3, \ldots) , \qquad (2.131)$$

where h_{kl}^* is conjugate harmonic to h_{kl}, and define

$$\tilde{f}_{kl}(\zeta) = f_{kl}(\varphi(\zeta)) \qquad (k, l = 1, 2, 3, \ldots) . \qquad (2.132)$$

We state that

$$\limsup_{r \to R_N} \int_{-\pi}^{+\pi} |\tilde{f}_{kl}^2(re^{it})\varphi'(re^{it})| r\,dt \leqq \int_{\gamma_N} |f_{kl}(z)|^2 |dz| \qquad (2.133)$$

and

$$\limsup_{r \to R_n} \int_{-\pi}^{+\pi} |\tilde{f}_{kl}^2(re^{it})\varphi'(re^{it})| r\,dt \leqq \int_{\gamma_n} |f_{kl}(z)|^2 |dz| . \qquad (2.134)$$

We briefly indicate a proof of these two inequalities [5]). (For more details the reader is referred to the quoted articles.) Certain statements concerning the boundary behavior of φ will have to be verified. The analogous properties of the conformal mapping of simply connected domains are well known and we shall make extensive use of them.

Let us first verify that φ is continuous and of bounded variation on the boundary. We consider the outer boundary, $|\zeta| = R_n$. Let $z = \varphi_1(w)$ denote a conformal mapping of a suitable circular disk $|w| < R$ onto the interior of γ_n. We define $w = \varphi_2(\zeta) = \varphi_1^{-1}(\varphi(\zeta))$. This function represents the domain $R_N < |\zeta| < R_n$ conformally onto an annulus Ω with outer boundary $|w| = R$. φ has thus been decomposed into two steps, $\varphi = \varphi_1(\varphi_2(\zeta))$. φ_1 is known to be continuous on $|w| = R$ (for references see C. GATTEGNO and A. OSTROWSKI [14, p. 27]). Since γ_n is rectifiable, φ_1 is also of bounded variation. Furthermore, $w = \varphi_2(\zeta)$ is analytic on $|\zeta| = R_n$ because this part of the boundary is mapped onto a circle. Consequently, $z = \varphi(\zeta)$ is continuous and of bounded variation on $|\zeta| = R_n$. The same is true for the immer circle $|\zeta| = R_N$.

Next we prove that φ is absolutely continuous on the boundary. From the LAURENT expansion we conclude that φ admits the representation $\varphi(\zeta) = \varphi_{\rm I}(\zeta) + \varphi_{\rm II}(\zeta)$, $\varphi_{\rm I}$ and $\varphi_{\rm II}$ being analytic in $|\zeta| < R_n$ and in $|\zeta| > R_N$, respec-

[5]) Professor M. RIESZ kindly suggested to us the following demonstration.

tively. Since φ is continuous and of bounded variation on $|\zeta| = R_n$, the same is true for φ_{II}. Hence, by a well known theorem of F. and M. RIESZ [28], φ_{II} is absolutely continuous on $|\zeta| = R_n$. Consequently, φ is absolutely continuous on $|\zeta| = R_n$. An analogous reasoning proves the same for $|\zeta| = R_N$.

φ can be written as the POISSON integral of its boundary values

$$\varphi(\zeta) = \int K(\zeta, Z)\, \varphi(Z)\, |dZ| \ ,$$

where K denotes the POISSON kernel, ζ an interior and Z a boundary point of $R_N < |\zeta| < R_n$. Because of the circular symmetry of the domain the kernel K depends only on $|\zeta|$, $|Z|$ and $\arg(\zeta - Z)$. Consequently, if we put $\zeta = \varrho e^{i\vartheta}$ and $Z = R_N e^{it}$ (or $Z = R_n e^{it}$), we have $\partial K/\partial\vartheta = -\partial K/\partial t$. This relation allows us to convert the differentiation of $\int K(\zeta, Z)\varphi(Z)|dZ|$ with respect to ζ into a differentiation along the boundary (see M. RIESZ [29, p. 55]). A partial integration then permits us to conclude that $\varphi'(\zeta)$ is given by the POISSON integral of its (almost everywhere existing) values

$$\varphi'(R_N e^{it}) = \frac{1}{i R_N e^{it}} \frac{\partial \varphi(R_N e^{it})}{\partial t} \quad \text{and} \quad \varphi'(R_n e^{it}) = \frac{1}{i R_n e^{it}} \frac{\partial \varphi(R_n e^{it})}{\partial t}$$

on the bounding circles. It follows that

$$|\varphi'(\zeta)| \leqq A_1(\zeta) + A_2(\zeta) \ , \tag{2.135}$$

where

$$A_1(\zeta) = \int_{-\pi}^{+\pi} K(\zeta, R_N e^{it})\, |\varphi'(R_N e^{it})|\, R_N\, dt$$

and

$$A_2(\zeta) = \int_{-\pi}^{+\pi} K(\zeta, R_n e^{it})\, |\varphi'(R_n e^{it})|\, R_n\, dt \ .$$

For $|Z| = R_N$, $|\zeta| \to R_n$ implies $K(\zeta, Z) \to 0$, the convergence being uniform in Z and $\arg\zeta$. Consequently

$$\lim_{r \to R_n} \int_{-\pi}^{+\pi} A_1(re^{it})\, r\, dt = 0 \ . \tag{2.136}$$

Let K_0 denote the POISSON kernel for the domain $|\zeta| < R_n$. For $|Z| = R_n$, $K(\zeta, Z) \leqq K_0(\zeta, Z)$. Hence

$$\begin{aligned} \int_{-\pi}^{+\pi} A_2(re^{it})\, r\, dt &\leqq \int_{|\zeta|=r} \int_{|Z|=R_n} K_0(\zeta, Z)\, |\varphi'(Z)|\, |d\zeta|\, |dZ| \\ &\leqq \int_{-\pi}^{+\pi} |\varphi'(R_n e^{it})|\, R_n\, dt = \int_{\gamma_n} |dz| \ . \end{aligned} \tag{2.137}$$

From (2.135), (2.136) and (2.137) we infer

$$\limsup_{r \to R_n} \int_{-\pi}^{+\pi} |\varphi'(re^{it})|\, r\, dt \leqq \int_{-\pi}^{+\pi} |\varphi'(R_n e^{it})|\, R_n\, dt = \int_{\gamma_n} |dz| \ . \tag{2.138}$$

We state that, more generally

$$\limsup_{r\to R_n} \int_{t_1}^{t_2} |\varphi'(re^{it})|\, r\,dt \leqq \int_{t_1}^{t_2} |\varphi'(R_n e^{it})|\, R_n\,dt = \int_{\gamma_n'} |dz| \tag{2.139}$$

holds for any pair of values t_1 and t_2 $(-\pi \leqq t_1 < t_2 \leqq +\pi)$, γ_n' denoting the portion of γ_n which corresponds to the arc$(R_n e^{it_1}, R_n e^{it_2})$. Indeed, suppose that (2.139) is not fulfilled for some such arc. Then there exists a sequence of radii $\{r_k\} \to R_n$ such that

$$\lim_{k\to\infty} \int_{t_1}^{t_2} |\varphi'(r_k e^{it})|\, r_k\,dt > \int_{t_1}^{t_2} |\varphi'(R_n e^{it})|\, R_n\,dt$$

and

$$\lim_{k\to\infty} \int_{-\pi}^{+\pi} |\varphi'(r_k e^{it})|\, r_k\,dt \leqq \int_{-\pi}^{+\pi} |\varphi'(R_n e^{it})|\, R_n\,dt \ .$$

These two inequalities imply

$$\lim_{k\to\infty} \int_{t_2}^{t_1+2\pi} |\varphi'(r_k e^{it})|\, r_k\,dt < \int_{t_2}^{t_1+2\pi} |\varphi'(R_n e^{it})|\, R_n\,dt \ .$$

But since $|\varphi'(r_k e^{it})| \to |\varphi'(R_n e^{it})|$ for almost all t, we have thus obtained a contradiction to FATOU's lemma [30, p. 29]. This proves (2.139). (2.134) is an immediate consequence of (2.139). (2.133) can be proved by the same method.

Now, let $k \to \infty$. We conclude from (2.133) and (2.134) that

$$\begin{aligned} &\limsup_{r\to R_N} \int_{-\pi}^{+\pi} \exp\,\{h_1(\varphi(re^{it})) - h_{2l}(\varphi(re^{it}))\}\, |\varphi'(re^{it})|\, r\,dt \\ &\leqq \int_{\gamma_N} \exp\,\{u_1(z) - h_{2l}(z)\}\, |dz| \end{aligned} \tag{2.140}$$

and

$$\begin{aligned} &\limsup_{r\to R_n} \int_{-\pi}^{+\pi} \exp\,\{h_1(\varphi(re^{it})) - h_{2l}(\varphi(re^{it}))\}\, |\varphi'(re^{it})|\, r\,dt \\ &\leqq \int_{\gamma_n} \exp\,\{u_1(z) - h_{2l}(z)\}\, |dz| \qquad (l = 1,2,3,\ldots)\ . \end{aligned} \tag{2.141}$$

Indeed, the symbol „$>$" in either of these inequalities leads to a contradiction: An easy argument then yields the conclusion that, for any fixed l and sufficiently large $k(l)$, the integral on the left-hand sides of (2.133) and (2.134) is *not* a convex function of $\log r$. But, on the other hand, it should possess this convexity property, since it represents the mean value of a subharmonic function.

Finally we let $l \to \infty$. By making again use of the convexity we obtain (2.129) and (2.130) as consequences of (2.140) and (2.141), respectively. This completes the proof of Lemma 5.

Theorem 1, 2 and 3 are concerned with the behavior of u near Γ. Corresponding results can be obtained for the neighborhood of Δ. We state them without giving detailed proofs. Let $\Phi(\Delta)$ be defined analogously to $\Phi(\Gamma)$. (Interchange Γ with Δ and assume that n designates the inner normal in all definitions.) Suppose that $\Phi(\Delta)$ exists.

Theorem 4. *If $\Phi(\Delta) < +1$, then there exists a locally rectifiable path σ in Ω, tending to Δ, such that* (2.7) *is fulfilled.*

Theorem 5. *If Δ contains more than one point, and if $\Phi(\Delta) < +\infty$, then there exists a locally rectifiable path σ, tending to Δ, such that* (2.7) *is satisfied.*

Theorem 6. *Suppose there exists a sequence $\{\gamma_n\}$, $n = 1,2,3\ldots$, of rectifiable* Jordan *curves, enclosing Δ, in Ω and a number M such that*

(a) $\{\gamma_n\}$ *comes arbitrarily near to Δ,*

(b) $\int_{\gamma_n} e^u |dz| < M$ *for all n.*

Then, if $\Phi(\Delta) \neq +1$, there exists a locally rectifiable path σ, tending to Δ, such that (2.7) *is fulfilled.*

These results follow from the previous ones by an inversion. We mention that Theorems 4 and 6 can also be obtained as corollaries of Theorems 8 and 9 (section 4), respectively.

We conclude this section with a remark concerning a special case. Let $w = f(z)$ be a (not necessarily single-valued) complex analytic function which is defined and $\neq 0$ throughout Ω except possibly at a (finite or infinite) number of isolated points $a_1, a_2, a_3, \ldots$. Suppose that in the neighborhood of these f admits the representation

$$f(z) = (z - a_k)^{p_k} g(z) \qquad (k = 1,2,3,\ldots)\ ,$$

where g is regular and $\neq 0$ at a_k, and p_k denotes an arbitrary real number. (In particular, f may be of the form $[F(z)]^\lambda$, where F denotes a meromorphic function and λ is a real number.) Then the (single-valued) function

$$u(z) = \log |f(z)|$$

is harmonic throughout Ω except at $a_1, a_2, a_3, \ldots$, where it possesses isolated logarithmic singularities. Hence u admits the representation (2.1). Let γ denote a Jordan curve in Ω, enclosing Δ, which does not pass through any of the a_k's. Then

$$\Phi(u, \gamma; \Gamma) = \frac{1}{2\pi} \int_\gamma d[\arg f(z)]\ ,$$

γ being described in the positive sense. Since $e^{u(z)}|dz| = |f(z)||dz|$, Theorems 1 to 6 thus imply certain results on integrals of moduli of analytic functions in the case where something is known about the variation of the argument along closed curves.

If the region of definition of f can be extended to Ω_0, then

$$\Phi(u, \gamma; \Gamma) = \sum_{a_k \in \omega} p_k ,$$

where ω designates the interior region of γ.

3. A characteristic property of polynomials

Theorem 7. *An entire analytic function* $w = f(z)$ *is a polynomial if and only if there exists a positive number* λ *such that*

$$\int_\sigma |f(z)|^{-\lambda} |dz| = +\infty \tag{3.1}$$

for every locally rectifiable path σ *tending to infinity.*

Remarks. It is natural to ask whether this theorem remains valid if the class of admissible curves σ is more restricted. We have no results in this direction. For example, the following question is still open: *Let* $w = f(z)$ *be an entire analytic function. Suppose there exists a positive number* λ *such that*

$$\int_1^{+\infty} |f(\varrho e^{i\Theta})|^{-\lambda} d\varrho = +\infty$$

for all Θ $(0 \leqq \Theta < 2\pi)$. *Does this imply that* f *is a polynomial?*

One might also consider other regions of definition of $f(z)$ instead of the entire plane. In this respect Theorem 2 immediately yields the following statement: *Suppose the function* $w = f(z) \not\equiv 0$ *is defined and analytic in a simply-connected, proper subregion* Ω_0 *of the z-plane. Then, given any* $\lambda > 0$, *there exists a locally rectifiable path* σ, *tending to the boundary* Γ *of* Ω_0, *such that*

$$\int_\sigma |f(z)|^{-\lambda} |dz| < +\infty .$$

Proof of Theorem 7. Suppose the function $w = f(z)$ satisfies condition (3.1) for some $\lambda > 0$. Let N denote the number of zeros of f $(N \leqq +\infty)$. We exclude the trivial function $f \equiv 0$ and define $u(z) = u_1(z) - u_2(z)$, where $u_1(z) \equiv 0$ and $u_2(z) = \lambda \log |f(z)|$. Let Ω_0 denote the entire z-plane. Using the notations of section 2 we then have $\Phi(\Gamma) = -\lambda N$. Since u does not satisfy the statement of Theorem 1, $\Phi(\Gamma) \geqq -1$. Hence $N \leqq 1/\lambda < +\infty$. So f admits the representation

$$f(z) = e^{\Psi(z)} \prod_{k=1}^{N} (z - a_k) , \tag{3.2}$$

where Ψ is an entire function and $a_1, a_2, \ldots, a_N$ are the zeros of f. The function

$$\zeta = \varphi(z) = \int_0^z e^{-\lambda \psi(z)} dz \tag{3.3}$$

yields a conformal mapping of the finite z-plane onto a RIEMANN surface R without branch points, extending over the ζ-plane. We have to distinguish between two cases

(a) R coincides with the entire finite plane. Then $\zeta = \varphi(z)$ is necessarily an entire linear function. So $\psi(z)$ is a constant and, therefore, $f(z)$ is a polynomial.

(b) There exists a finite point ζ_0 with the following property: On some sheet of R the half-open segment $\zeta(t) = \zeta_0 t$ $(0 \leqq t < 1)$ belongs to R, whereas ζ_0 lies on the boundary. This segment is the image of an analytic curve σ_0 in the z-plane which tends to infinity. By (3.3) we have

$$\int_{\sigma_0} |e^{\psi(z)}|^{-\lambda} |dz| = \int_{\sigma_0} |\varphi'(z)||dz| = |\zeta_0| < +\infty \ . \tag{3.4}$$

From (3.2) and (3.4) we conclude that

$$\int_{\sigma_0} |f(z)|^{-\lambda} |dz| < +\infty \ , \tag{3.5}$$

contrary to hypothesis. (It is understood that σ_0 has to be slightly modified if zeros of f should lie on it.)

So f must be a polynomial. The converse is obvious.

4. On complete conformal metrics defined on finitely connected, open RIEMANN surfaces

We first give a conformally invariant formulation of Theorems 1 and 3. Let Ω be a doubly connected, open RIEMANN surface on which a conformal metric (1.7) is defined. We assume that u admits the local representation (1.9). Let Γ and Δ denote the ideal boundary components of Ω and let γ be a JORDAN curve [6]) in Ω which is not nullhomotopic.

Assume for a moment that u is of class C^2 and that γ is analytic. Then we define

$$I(e^u|dz|, \gamma; \Gamma) = \int_\gamma \left(k_e + \frac{\partial u}{\partial n}\right) |dz| \ , \tag{4.1}$$

[6]) The definitions of the concepts JORDAN *curve*, *homotopic*, *locally rectifiable*, *analytic* are to be understood with respect to the underlying RIEMANN surface.

where k_e and n are to be determined as in (1.5), the orientation of γ being chosen such that Γ lies on the right. The integral (4.1) (in fact, even the differential $\left(k_e + \frac{\partial u}{\partial n}\right)|dz|$) is a conformal invariant. This is immediately clear from the geometrical interpretation (1.6), but it can also be verified by direct calculation.

In order to define $I(e^u|dz|,\gamma;\Gamma)$ for general u and γ we consider a sequence $\{\delta_k\}$, $k = 1,2,3,\ldots$, of JORDAN curves in Ω, chosen such that the annuli (Δ,δ_k) tend increasingly to (Δ,γ) as $k \to \infty$. In (δ_k,γ) we introduce a conformal metric $e^{h_k(z)}|dz|$, defining

$$h_k(z) = u(z) + \int_{(\delta_k,\gamma)} g_k(z,\zeta)d\mu(e_\zeta) \ . \tag{4,2}$$

Here g_k denotes GREEN's function for (δ_k,γ) and $\mu = \mu_1 - \mu_2$, where μ_1 and μ_2 are the measures associated with u_1 and u_2, respectively. (It should be noticed that μ does not depend on the choice of decomposition or uniformizer. Hence the integral on the right-hand side of (4.2) is a scalar. Consequently, $e^{h_k(z)}|dz|$ is indeed a conformal invariant.) Let δ_k' be an arbitrary analytic JORDAN curve in (δ_k,γ) which is not nullhomotopic. Then $I(e^{h_k}|dz|,\delta_k';\gamma)$ is well defined. If Ω is a schlicht region in the finite z-plane, then $h_k(z)$, defined by (4.2), is identical with the function designated in the same way in (2.4). This is implied by the decomposition theorem of F. RIESZ [27, II, p. 357]). If, furthermore, Γ denotes the outer boundary of Ω, then, obviously

$$I(e^{h_k}|dz|,\delta_k';\gamma) = 2\pi[\Phi(h_k,\delta_k';\gamma) + 1] \ . \tag{4.3}$$

We observe that $I(e^{h_k}|dz|,\delta_k';\gamma)$ does not depend on the choice of δ_k', this being true for the right-hand side of (4.3). We further conclude from (4.3) and the existence of the limit (2.4) that

$$I(e^u|dz|,\gamma;\Gamma) = \lim_{k\to\infty} I(e^{h_k}|dz|,\delta_k';\gamma) \tag{4.4}$$

always exists, being finite and independent of the choice of $\{\delta_k\}$. Clearly

$$I(e^u|dz|,\gamma;\Gamma) = 2\pi[\Phi(u,\gamma;\Gamma) + 1] \ . \tag{4.5}$$

These relations hold under the above mentioned special assumptions. But Ω can always be mapped conformally onto a schlicht annulus such that Γ corresponds to the outer boundary. Furthermore, since $I(e^{h_k}|dz|,\delta_k';\gamma)$ is a conformal invariant, the existence of $I(e^u|dz|,\gamma;\Gamma)$ is thus assured in any case.

Now, let $\{\gamma_l\}$, $l = 1,2,3,\ldots$, be an arbitrary sequence of JORDAN curves which are not nullhomotopic and such that the regions (Δ,γ_l) tend increasingly to Ω as $l \to \infty$. Assume that the limit

$$I(\Gamma) = \lim_{l \to \infty} I(e^u |dz|, \gamma_l; \Gamma) \tag{4.6}$$

exists for any such sequence, admitting the values $+\infty$ and $-\infty$. Of course, $I(\Gamma)$ is necessarily independent of the choice of $\{\gamma_l\}$.

From (4.5) we infer that

$$I(\Gamma) = 2\pi[\Phi(\Gamma) + 1] \tag{4.7}$$

in the case where Ω is a schlicht region and Γ denotes its outer boundary. Since $I(\Gamma)$ is a conformal invariant, this relation gives rise immediately to the following extensions of Theorems 1 and 3[7]):

Theorem 8. *If* $I(\Gamma) < 0$, *then there exists a locally rectifiable path* σ, *tending to* Γ, *such that* $\int_\sigma e^{u(z)} |dz| < +\infty$.

Theorem 9. *Suppose there exists a sequence* $\{\gamma_n\}$, $n = 1, 2, 3, \ldots$, *of locally rectifiable* Jordan *curves which are not nullhomotopic and a number* M *such that*

(a) $\{\gamma_n\}$ *comes arbitrarily near to* Γ,

(b) $\int_{\gamma_n} e^{u(z)} |dz| < M$ *for all* n.

Then, if $I(\Gamma) \neq 0$, *there exists a locally rectifiable path* σ, *tending to* Γ, *such that* $\int_\sigma e^{u(z)} |dz| < +\infty$.

After this preparation we take up the concepts developed in the introduction. Consider an open Riemann surface S on which a conformal metric (1.7) is defined. Assume that u admits the local representation (1.9). We define: The metric $e^{u(z)} |dz|$ is said to be *complete* if $\int_\sigma e^{u(z)} |dz| = +\infty$ for every locally rectifiable path σ which tends to the ideal boundary of S.

Theorem 10. *Let* S *be a finitely connected, open* Riemann *surface on which a complete conformal metric* $e^{u(z)} |dz|$ *is defined. Suppose that the curvatura integra* C *exists. Then* $C \leqq 2\pi\chi$, *where* χ *denotes the* Euler-Poincaré *characteristic of* S.

Remark. This is a result of S. Cohn-Vossen (Satz 6, p. 79 in [10]) in extended form. (For further comments see introduction.)

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[7]) The meaning of "*σ tends to Γ*" and of "*$\{\gamma_n\}$ comes arbitrarily near to Γ*" has been defined in section 2 for schlicht annular regions. It is clear how these definitions have to be reformulated for arbitrary Riemann surfaces.

Proof. S is homeomorphic to a closed surface from which a finite number (say N) of points have been removed[8]). There exists a subcompact K of S, bounded by N JORDAN curves, $\Delta_1, \Delta_2, \ldots, \Delta_N$, such that the open set $S-K$ consists of N doubly connected components, $\Omega_1, \Omega_2, \ldots, \Omega_N$. Thereby each Ω_r is bounded by Δ_r and a second (ideal) boundary component Γ_r. Let γ_r be an arbitrary JORDAN curve in Ω_r which is not nullhomotopic with respect to Ω_r ($r = 1, 2, \ldots, N$). Let Σ denote the subregion of S whose boundary consists of $\gamma_1, \ldots, \gamma_{N-1}$ and γ_N. We are going to prove the GAUSS-BONNET formula[9])

$$2\pi\mu(\Sigma) + 2\pi\chi = \sum_{r=1}^{N} I(e^u|dz|, \gamma_r; \Gamma_r) \ . \tag{4.8}$$

Assume first that u is of class C_2 and that $\gamma_1, \ldots, \gamma_{N-1}$ and γ_N are analytic. Consider a triangulation of the closure of Σ consisting of analytic arcs. Let T_j denote the interiors, B_j the boundaries, and α_{jl} $(l = 1, 2, 3)$ the exterior angles of the triangles $(j = 1, 2, \ldots, M)$. We may suppose that one and the same local uniformizer can be used in a neighborhood of $T_j \cup B_j$. By GAUSS's theorem and the definition of k_e

$$\iint_{T_j} \Delta u\, dx_j dy_j + 2\pi - \sum_{l=1}^{3} \alpha_{jl} = \int_{B_j} \left(k_e + \frac{\partial u}{\partial n}\right) |dz_j| \qquad (j = 1, 2, \ldots, M) \ ,$$

if we integrate along B_j in the positive sense. We add all these relations. Because of the conformal invariance of $\left(k_e + \dfrac{\partial u}{\partial n}\right) |dz|$, and the coherence of the orientation most integrals on the right-hand side drop out. Furthermore, the EULER-POINCARÉ characteristic χ of S appears in a well known way. We obtain

$$\iint_{\Sigma} \Delta u\, dx dy + 2\pi\chi = \sum_{r=1}^{N} \int_{\gamma_r} \left(k_e + \frac{\partial u}{\partial n}\right) |dz| \ ,$$

i. e. relation (4.8) for the considered special case.

We now proceed to the general case but still assume that the γ_r's are analytic and free of mass. Without losing generality we then may suppose that the entire „skeleton'' of the triangulation, $L = \bigcup_{j=1}^{M} B_j$, is free of mass. Let $\{D_k\}$, $k = 1, 2, 3, \ldots,$ be a sequence of regions, tending decreasingly to L as $k \to \infty$, each of which is bounded by $M + N$ JORDAN curves lying, respectively, in $T_1, T_2, \ldots, T_M$, (γ_1, Γ_1), $(\gamma_2, \Gamma_2), \ldots, (\gamma_N, \Gamma_N)$. Consider the conformal metric

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[8]) cf. B. v. KERÉKJÁRTÓ [19, chapter 5].

[9]) For the functiontheoretical aspects of the GAUSS-BONNET formula see also R. NEVANLINNA [21].

$e^{H_k(z)}|dz|$ in D_k, defining

$$H_k(z) = u(z) + \int_{D_k} G_k(z,\zeta)\,d\mu(e_\zeta) \ , \tag{4.9}$$

where G_k designates GREEN's function for D_k $(k = 1,2,3,\dots)$. Then

$$\int_{B_j} \frac{\partial H_k}{\partial n} |dz_j| + 2\pi - \sum_{l=1}^{3} \alpha_{jl} = \int_{B_j} \left(k_e + \frac{\partial H_k}{\partial n}\right) |dz_j| \qquad (j = 1,2,\dots, M) \ ,$$

and, by addition

$$\sum_{j=1}^{M} \int_{B_j} \frac{\partial H_k}{\partial n} |dz_j| + 2\pi\chi = \sum_{r=1}^{N} \int_{\gamma_r} \left(k_e + \frac{\partial H_k}{\partial n}\right) |dz| \ . \tag{4.10}$$

We state that

$$\lim_{k\to\infty} \int_{B_j} \frac{\partial H_k}{\partial n} |dz_j| = 2\pi\mu(T_j) \qquad (j = 1,2,\dots, M) \tag{4.11}$$

and

$$\lim_{k\to\infty} \int_{\gamma_r} \left(k_e + \frac{\partial H_k}{\partial n}\right) |dz| = I(e^u|dz|, \gamma_r; \Gamma_r) \qquad (r = 1,2,\dots, N) \ . \tag{4.12}$$

In order to verify (4.11) we introduce (for a fixed j) a sequence of doubly connected regions $\{E_k\}$ which are bounded by analytic JORDAN curves and tend decreasingly to B_j. Consider the conformal metric $e^{h_k(z)}|dz|$ in E_k, defining

$$h_k(z) = u(z) + \int_{E_k} g_k(z,\zeta)\,d\mu(e_\zeta) \ , \tag{4.13}$$

where g_k denotes GREEN's function for E_k $(k = 1,2,3,\dots)$. From the results of F. RIESZ [27] one concludes without difficulty that

$$\lim_{k\to\infty} \int_{B_j} \frac{\partial h_k}{\partial n} |dz_j| = 2\pi\mu(T_j) \ . \tag{4.14}$$

(4.9) and (4.13) imply

$$\int_{B_j} \left(\frac{\partial H_k}{\partial n} - \frac{\partial h_k}{\partial n}\right) |dz_j| = \int_{D_k} \left[\int_{B_j} \frac{dG_k(z,\zeta)}{\partial n_z} |dz|\right] d\mu(e_\zeta) - \int_{E_k} \left[\int_{B_j} \frac{\partial g_k(z,\zeta)}{\partial n_z} |dz|\right] d\mu(e_\zeta) \ . \tag{4.15}$$

Obviously we have

$$\left| \frac{1}{2\pi} \int_{B_j} \frac{\partial G_k(z,\zeta)}{\partial n_z} |dz| \right| \leq 1 \tag{4.16}$$

and

$$\left| \frac{1}{2\pi} \int_{B_j} \frac{\partial g_k(z,\zeta)}{\partial n_z} |dz| \right| \leq 1 \ , \tag{4.17}$$

if we interprete these integrals as fluxes. From (4.15), (4.16) and (4.17) we infer that

$$\left|\int_{B_j}\left(\frac{\partial H_k}{\partial n}-\frac{\partial h_k}{\partial n}\right)|dz_j|\right| \leqq 2\pi[\mu(E_k)+\mu(D_k)] \ . \tag{4.18}$$

The right-hand side of this inequality tends to 0 as $k\to\infty$, because $\mu(L) = \mu(B_j) = 0$. Consequently, (4.11) follows from (4.14) and (4.18). (4.12) can be demonstrated in a similar way.

Since $\sum_{j=1}^{M}\mu(T_j) = \mu(\Sigma)$, (4.11) and (4.12) allow us to conclude that (4.8) is the limit of (4.10) as $k\to\infty$.

In order to get rid of the hypotheses that the γ_r's are analytic and free of mass we exhaust an arbitrary Σ by an increasing sequence of regions whose bounding curves satisfy these conditions. Relation (4.8), formulated for Σ, is immediately obtained as the limit of the corresponding equalities already verified for the subregions.

From (4.8) one can conclude that the limits $I(\Gamma_r)$, $r = 1,2,\ldots,N$, exist. (This is a consequence of the existence of $C = -2\pi\mu(S)$ and proved by letting an arbitrary one of the γ_r's move to the boundary while all others are kept fixed.) Now, if Σ tends increasingly to S, then (4.8) yields in the limit

$$C = 2\pi\chi - \sum_{r=1}^{N} I(\Gamma_r) \ . \tag{4.19}$$

From Theorem 8 and the completeness of the metric we infer that $I(\Gamma_r) \geqq 0$ for $r = 1,2,\ldots,N$. Hence, by (4.19), $C \leqq 2\pi\chi$. Q. E. D.

Theorem 11. *Let S be a finitely connected, open* RIEMANN *surface on which a complete conformal metric $e^{u(z)}|dz|$ is defined. Suppose there exists a sequence $\{\gamma_n\}$, $n = 1,2,3,\ldots$, of locally rectifiable* JORDAN *curves with the following properties:*

(1) *they are not nullhomotopic,*

(2) *their lengths $\int_{\gamma_n} e^{u(z)}|dz|$ are uniformly bounded,*

(3) *$\{\gamma_n\}$ comes arbitrarily near to every boundary component of S* [10]*.*

Assume further that the curvatura integra C exists. Then $C = 2\pi\chi$, where χ denotes the EULER-POINCARÉ *characteristic of S.*

Remarks. This result implies a theorem of S. COHN-VOSSEN (Satz 7, p. 79 in [10]) which states that $C = 2\pi\chi$ for every finitely connected, open, two-

[10]) i. e. the γ_n's penetrate into each Ω_r, $r = 1,2,\ldots,N$, for any choice of K.

dimensional RIEMANNian manifold whose curvatura integra exists and which does not possess a so-called "eigentlicher Kelch". By going back to the definition of this concept and using the notations introduced for the proof of Theorem 10 one arrives to the following formulation of COHN-VOSSEN's hypothesis: Let

$$m(\Omega_r) = \inf_{\gamma} \left[\int_{\gamma} e^{u(z)} |dz|\right] ,$$

admitting to competition all locally rectifiable JORDAN curves in Ω_r which are not nullhomotopic. A sequence $\{\gamma_n\}$ of such curves is called a *minimal sequence* of Ω_r if

$$\lim_{n \to \infty} \int_{\gamma_n} e^{u(z)} |dz| = m(\Omega_r) .$$

COHN-VOSSEN postulated that, given an arbitrary subcompact K_0 of S, there should always exist a connected subcompact K, containing K_0, such that all components Ω_r, $r = 1, 2, \ldots, N$, of $S - K$ have the following property: Each minimal sequence $\{\gamma_n\}$ of Ω_r comes arbitrarily near to Γ_r.

It is clear that COHN-VOSSEN's hypothesis is stronger than ours. The ordinary circular cylinder imbedded in 3-space is a trivial example of a manifold to which our result applies while COHN-VOSSEN's theorem does not.

The following statement is also a corollary of Theorem 11: *Let M be a finitely connected, complete, open, two-dimensional RIEMANNian manifold whose curvatura integra C exists. Suppose there exists a sequence of subcompacts, tending increasingly to M, the boundaries of which are of uniformly bounded length. Then $C = 2\pi\chi$.* Professor H. HOPF points out to us that if we make the additional assumptions that M is analytic and has everywhere non-negative curvature (and is therefore necessarily simply connected), then this result follows from two theorems of F. FIALA (Theorems A and D, pp. 299–300 in [12]) and the previously mentioned result of COHN-VOSSEN (Satz 6, p. 79 in [10]).

Proof. If infinitely many of the γ_n's would intersect K, then a reasoning quite similar to the one used in the proof of Lemma 2 would yield a contradiction to the hypothesis of completeness.

Hence only a finite number of γ_n's intersect K. From this we conclude that each Ω_r contains a subsequence $\{\gamma_n^{(r)}\}$ of $\{\gamma_n\}$ which comes arbitrarily near to Γ_r. Therefore, by Theorem 9 and the completeness of the metric, $I(\Gamma_r) = 0$ for $r = 1, 2, \ldots, N$. Consequently, by (4.19), $C = 2\pi\chi$. Q. E. D.

Theorem 12. *Let S be a finitely connected, open RIEMANN surface on which a complete conformal metric $e^{u(z)}|dz|$ of finite total area $A = \iint_S e^{2u} dx dy$ is defined. Suppose that the curvatura integra C exists. Then $C = 2\pi\chi$, where χ denotes the EULER-POINCARÉ characteristic of S.*

Proof. Again we make use of the concepts introduced for the proof of Theorem 10. Considering an arbitrary one of the regions Ω_r we shall prove that $I(\Gamma_r) = 0$. This, combined with relation (4.9), will demonstrate the theorem.

The region Ω_r is conformally equivalent to a schlicht circular ring in the z-plane, $R_1 < |z| < R_2$ $(0 < R_1 < R_2 \leqq +\infty)$. We distinguish between two cases depending on whether R_2 is finite or infinite. Let us begin by showing that the first possibility cannot occur.

(I) $R_2 < +\infty$. Let R $(R_1 < R < R_2)$ be chosen arbitrarily. By SCHWARZ's inequality

$$\int_{R_1}^{R} \int_0^{2\pi} e^{2u(\varrho e^{i\varphi})} \varrho d\varrho d\varphi \geqq R_1 \int_0^{2\pi} [\int_{R_1}^{R} e^{2u(\varrho e^{i\varphi})} d\varrho] d\varphi \geqq \frac{R_1}{R - R_1} \int_0^{2\pi} [\int_{R_1}^{R} e^{u(\varrho e^{i\varphi})} d\varrho]^2 d\varphi$$

$$\geqq \frac{2\pi R_1}{R - R_1} [\inf_{0 \leqq \varphi < 2\pi} \int_{R_1}^{R} e^{u(\varrho e^{i\varphi})} d\varrho]^2 . \tag{4.20}$$

Furthermore

$$\lim_{R \to R_2} [\inf_{0 \leqq \varphi < 2\pi} \int_{R_1}^{R} e^{u(\varrho e^{i\varphi})} d\varrho] = +\infty , \tag{4.21}$$

since otherwise an application of Lemma 2 would yield a contradiction to the completeness of the metric. It follows from (4.20) and (4.21) that

$$\iint_{\Omega_r} e^{2u} dx dy = +\infty ,$$

contrary to hypothesis. Hence $R_2 = +\infty$.

(II) $R_2 = +\infty$. Since

$$\iint_{\Omega_r} e^{2u(\varrho e^{i\varphi})} \varrho d\varrho d\varphi < +\infty ,$$

there must exist a sequence of radii $\{\varrho_n\} \to \infty$ for which

$$\int_0^{2\pi} e^{2u(\varrho_n e^{i\varphi})} \varrho_n d\varphi < \frac{1}{\varrho_n} .$$

Hence, by SCHWARZ's inequality

$$[\int_0^{2\pi} e^{u(\varrho_n e^{i\varphi})} \varrho_n d\varphi]^2 \leqq 2\pi \varrho_n \int_0^{2\pi} e^{2u(\varrho_n e^{i\varphi})} \varrho_n d\varphi < 2\pi .$$

Now we apply Theorem 9, letting $\gamma_n = [|z| = \varrho_n]$ and $M = \sqrt{2\pi}$. It follows that $I(\Gamma_r) = 0$. This completes the proof of Theorem 12.

We mention that there exist finitely connected, complete, open, two-dimensional RIEMANNian manifolds which belong to any prescribed topological type.

In order to construct such examples we take a parabolic RIEMANN surface S[11]) which possesses the required topological structure. (Such a RIEMANN surface can always be obtained by removing a finite number of points from a suitable closed surface.) Then the Ω_r's are all conformally equivalent to schlicht circular rings of the type $R_1 < |z| < +\infty$ $(R_1 > 0)$. In these we define the conformal metric

$$ds = \frac{|dz|}{|z| \log (|z| + 1)} .$$

On the remaining portion of S the metric is "filled out" arbitrarily, but such that it is everywhere positive definite and of class C^2. By making use of the fact that

$$\int_1^{+\infty} \frac{d\varrho}{\varrho \log (\varrho + 1)} = +\infty, \quad \text{but} \quad \int_1^{+\infty} \frac{d\varrho}{\varrho \log^2 (\varrho + 1)} < +\infty ,$$

one verifies easily that this metric is complete, but of finite total area.

5. On complete conformal metrics defined on infinitely connected RIEMANN surfaces

Theorem 13. *Suppose that the conformal metric $e^{u(z)}|dz|$, defined on an infinitely connected RIEMANN surface S, is complete. Then $C^- = +\infty$.*

Remark. This result complements Theorem 10. It was suggested to us as a conjecture by Professor H. HOPF.

Proof. Assume that $C^- < +\infty$. We are going to show that $e^{u(z)}|dz|$ cannot be complete under this hypothesis.

We exhaust S by an increasing sequence $\{\Sigma_r\}$ of subcompacts, each being bounded by a finite number of analytic JORDAN curves $(\beta_{r1}, \beta_{r2}, \ldots, \beta_{rm_r})$ which we suppose to be free of mass. We may further request that each β_{rs} constitutes the boundary of exactly one component Ω_{rs} of $S - \Sigma_r$ $(s = 1, 2, \ldots, m_r)$. By the GAUSS-BONNET formula (4.8)

$$2\pi\mu(\Sigma_r) + 2\pi\chi_r = \sum_{s=1}^{m_r} I(e^u |dz|, \beta_{rs}; B) , \tag{5.1}$$

where χ_r denotes the EULER-POINCARÉ characteristic of Σ_r and B designates the (ideal) boundary of S. The left-hand side of (5.1) tends to $-\infty$ as $r \to \infty$, since $2\pi\mu(S) = -C < +\infty$. Hence, for sufficiently large r

$$\sum_{s=1}^{m_r} I(e^u |dz|, \beta_{rs}; B) < -4\pi\mu^+(S) .$$

[11]) i. e. a RIEMANN surface with nullboundary (cf. R. NEVANLINNA [24, p. 319]).

Then, for at least one index s

$$I(e^u|dz|, \beta_{rs}; B) = -4\pi[\mu^+(\Omega_{rs}) + \eta] \ , \tag{5.2}$$

where $\eta > 0$. Let such r and s be chosen. We change the notation by writing β_1, Ω, μ_1 and μ_2 instead of β_{rs}, Ω_{rs}, μ^+ and μ^-, respectively, and introduce an analytic JORDAN curve δ_1 in Ω which is homotopic to β_1, free of mass and so close to β_1 that $\mu_2(\beta_1, \delta_1) < \eta$.

Lemma 6. *There is a number C with the following property: Given an arbitrary index r, there exists a rectifiable curve α, leading from δ_1 to the boundary of Σ_r, such that*

$$\int_\alpha e^u |dz| < C \ . \tag{5.3}$$

Remark. Theorem 13 follows immediately from this result by means of an obvious generalization of Lemma 2.

Proof. Our demonstration is similar to the one of Lemma 3. Let γ_1 and γ_0 be analytic JORDAN curves in (β_1, δ_1), both homotopic to β_1 and free of mass, γ_0 lying in (γ_1, δ_1). We introduce the notations δ_2, γ_2 and β_2 for the respective intersections (non-empty for sufficiently large r) of the boundaries of Σ_r, Σ_{r+1} and Σ_{r+2} with Ω. Further, let $\omega = (\beta_1, \beta_2)$, $\omega_0 = (\beta_1, \gamma_0)$, $\omega_1 = (\gamma_0, \beta_2)$, $\mu_{20}(e) = \mu_2(e \frown \omega_0)$, $\mu_{21}(e) = \mu_2(e \frown \omega_1)$, $m_0 = \mu_2(\omega_0)$ and $m_1 = \mu_2(\omega_1)$. We have

$$m_0 \leqq \mu_2(\beta_1, \delta_1) < \eta \ . \tag{5.4}$$

We define the metrics $e^{h(z)}|dz|$, $e^{v(z)}|dz|$, $e^{v_1(z)}|dz|$ and $e^{V(z)}|dz|$ by putting

$$h(z) = u(z) + \int_\omega g(z,\zeta)\, d\mu(e_\zeta) \ , \tag{5.5}$$

$$v(z) = u(z) + \int_\omega g(z,\zeta)\, d\mu_1(e_\zeta) \tag{5.6}$$

$$= h(z) + \int_\omega g(z,\zeta)\, d\mu_2(e_\zeta) \ ,$$

$$v_1(z) = h(z) + \int_\omega g(z,\zeta)\, d\mu_{21}(e_\zeta) \tag{5.7}$$

and

$$V(z) = u(z) + \int_\omega G(z,\zeta)\, d\mu_1(e_\zeta) \ , \tag{5.8}$$

where G and g denote the GREEN's functions of Ω and ω, respectively[12]. Since $0 \leqq g \leqq G$ throughout ω, $v_1(z) \leqq v(z) \leqq V(z)$ and, consequently

$$\int_{\gamma_1} e^{v_1} |dz| \leqq \int_{\gamma_1} e^{V(z)} |dz| = C_1 < +\infty \ . ^{13)} \tag{5.9}$$

[12]) Cf. R. NEVANLINNA [24].

[13]) $C_1, \ldots, C_5$ are constants not depending on the choice of β_2, γ_2 and δ_2.

Furthermore, throughout (δ_1, δ_2), $u(z) \leq v(z) \leq v_1(z) + C_2$, where C_2 denotes the (finite) upper bound of

$$\int_{\omega_0} G(z,\zeta)\, d\mu_{20}(e_\zeta) \quad \text{for } z \text{ varying on} \quad \Omega - (\beta_1, \delta_1) \; .$$

(In the case where μ_{20} is concentrated in one point the existence of such a bound is an immediate consequence of the properties of G. One proceeds to general measures μ_{20} by an application of HÖLDER's inequality.) Hence, for every curve α in (δ_1, δ_2)

$$\int_\alpha e^u |dz| \leq C_3 \int_\alpha e^{v_1} |dz| \; . \tag{5.10}$$

We are now going to prove that there exists a rectifiable curve α, leading from δ_1 to δ_2, such that

$$\int_\alpha e^{v_1} |dz| \leq C_4 \int_{\gamma_1} e^{v_1} |dz| + C_5 \; . \tag{5.11}$$

Lemma 6 is an immediate consequence of (5.9), (5.10) and (5.11).

In order to establish (5.11) we first approximate the measure μ_{21} by a finite number of concentrated masses. This is the purpose of the following construction.

Consider a triangulation T_0 of the closure $\bar{\omega}$ of ω, consisting of the triangles Δ_k^0 [14] $(k = 1, 2, \ldots, M)$ whose boundaries we suppose to be analytic and free of mass.

We subdivide T_0 in the following way: There exists a conformal representation $t_k = \varphi_k(z)$ of the interior of Δ_k^0 onto the equilateral triangle E $(1/2, -1/2, i\sqrt{3}/2)$ in a t_k-plane such that the vertices of Δ_k^0 correspond to those of E $(k = 1, 2, \ldots, M)$. (In the following the letters z and ζ are used to designate points on S whereas t_k and τ_k denote the corresponding values of the just introduced uniformizers.) We join the mid-points of the sides of E, thus breaking it up into four smaller triangles. To this subdivision of E there corresponds a triangulation T_1 of $\bar{\omega}$, consisting of the triangles

$$\Delta_{kj}^1 \;\; (k = 1, 2, \ldots, M; \, j = 1, 2, 3, 4) \; .$$

By iterating the subdivision of E we obtain a sequence $\{T_n\}$ of triangulations, each T_n being composed of $4^n M$ triangles Δ_{kj}^n $(k = 1, 2, \ldots, M;$ $j = 1, 2, \ldots, 4^n)$. We are going to select an index n large enough for our purpose.

We define a subcompact K_1 of $\bar{\omega}$ by subtracting neighborhoods of all vertices of T_0. We require that these neighborhoods be bounded by analytic

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[14]) By Δ_k^0 and Δ_{kj}^n (below) we understand the *closures* of the respective triangles.

curves (let Γ_1 denote their totality) such that

$$\int_{\Gamma_1 \cap \varpi} e^{v_1} |dz| < 1 \ . \tag{5.12}$$

Further, let K_2 be a second compact of the same type, containing K_1 and satisfying the condition

$$\mu_{21}(\omega - K_2) g(z, \zeta) < \log 2 \tag{5.13}$$

for all $z \epsilon K_1$ and $\zeta \epsilon (\omega - K_2)$. It is always possible to fulfill the conditions (5.12) and (5.13), since, by hypothesis, the vertices of T_0 support no mass.

The region of definition of the conformal representation $t_k = \varphi_k(z)$ can be extended to include an open set O_k containing $\Delta_k^0 \cap K_2$. Let G_k denote a region which also contains $\Delta_k^0 \cap K_2$ and whose closure lies in O_k $(k = 1, 2, \ldots, M)$. G_k's belonging to adjacent triangles intersect. It can be inferred from the construction that there exists a number A $(1 \leqq A < +\infty)$ such that, uniformly in k and l

$$\frac{1}{A} \leqq \left| \frac{dt_k}{dt_l} \right| \leqq A \tag{5.14}$$

throughout $G_k \cap G_l$.

We define the notion of N-neighborhood $(N = 0, 1, 2, \ldots)$ of a triangle Δ_{kj}^n by recursion as follows: The 0-neighborhood is identical with Δ_{kj}^n. The N-neighborhood of Δ_{kj}^n consists of those (closed) triangles $\Delta_{k'j'}^n$ which intersect the $(N-1)$-neighborhood of $\Delta_{kj}^n (N = 1, 2, 3, \ldots)$.

The following property is obvious:

(I) If $\Delta_{k'j'}^n$ lies in the N-neighborhood of Δ_{kj}^n, then Δ_{kj}^n belongs to the N-neighborhood of $\Delta_{k'j'}^n$.

Given an arbitrary positive integer N, there always exists an index $n_0(N)$ such that the following condition is fulfilled for all k, j and any $n > n_0(N)$: If the N-neighborhood of Δ_{kj}^n intersects K_2, then it lies in G_k and overlaps from Δ_k^0 into at most one Δ_l^0 $(l \neq k)$. In this case every such N-neighborhood (considered in the t_k-plane) is contained in a circle of radius $(N+1)A/n$. Since, on the other hand, the area of every composing triangle is $\geqq \sqrt{3}/4A^2n^2$, we conclude:

(II) Each N-neighborhood intersecting K_2 contains less than $8(N+1)^2A^4$ triangles Δ_{kj}^n.

The statements (I) and (II) imply:

(III) Each Δ_{kj}^n intersecting K_2 is contained in less than $8(N+1)^2A^4$ different N-neighborhoods.

We choose an integer $N_1 > A + 1$ and introduce the abbreviation U_{kj}^n for the N_1-neighborhood of Δ_{kj}^n. It is easy to verify that the following is true for all $n > n_0(N_1)$:

(IV) Let ϱ be a rectifiable curve on S, leading from z to ζ, both points lying in Δ^n_{kj}. Suppose that $\Delta^n_{kj} \frown K_2 \neq 0$. Then $\varrho \frown U^n_{kj}$ has at least the length[15] $|t_k - \tau_k|$.

We state that there exists an integer $N_2 > N_1 + 1$ with the following property:

(V) Let V^n_{kj} denote the N_2-neighborhood of Δ^n_{kj}. Suppose that $\Delta^n_{kj} \frown K_2 \neq 0$. Assume further that $n > n_0(N_2)$. Then the inequality

$$m_1 |g(z,\zeta) - g(z',\zeta')| < \log 2 \tag{5.15}$$

holds for arbitrary $z, z' \in \Delta^n_{kj}$ and $\zeta, \zeta' \in \Delta^n_{k'j'}$, admitting any $\Delta^n_{k'j'}$ which is not contained in V^n_{kj}.

It is sufficient to prove the existence of N_2 for a fixed index k. Throughout G_k we have the representation

$$g(z,\zeta) = \log \frac{1}{|t_k - \tau_k|} + r_k(t_k, \tau_k) \ , \tag{5.16}$$

where r_k is a regular function. We first limit ourselves to those triangles $\Delta^n_{k'j'}$ which lie in G_k. If n is large enough, then, by the continuity of r_k

$$m_1 |r_k(t_k, \tau_k) - r_k(t'_k, \tau'_k)| < \frac{\log}{2} \tag{5.17}$$

for $z, z' \in \Delta^n_{kj}$ and $\zeta, \zeta' \in \Delta^n_{k'j'} \subset G_k$. Since $n_0(N_2) \to \infty$ as $N_2 \to \infty$, (5.17) is fulfilled for all $n > n_0(N_2)$ if only N_2 is chosen large enough. Furthermore, the diameter[15] of any $\Delta^n_{k'j'}$ in G_k is at most A/n, whereas the distance[15] between U^n_{kj} and the boundary of V^n_{kj} is at least $(N_2 - N_1 - 1)/An$. Hence, if N_2 is sufficiently large, then

$$m_1 \log \left| \frac{t_k - \tau_k}{t'_k - \tau'_k} \right| \leqq m_1 \log \frac{\dfrac{N_2 - N_1 - 1}{An} + \dfrac{2A}{n}}{\dfrac{N_2 - N_1 - 1}{An} - \dfrac{2A}{n}} \leqq \frac{\log 2}{2} \ (n > n_0(N_2)) \tag{5.18}$$

for $z, z' \in U^n_{kj}$ and $\zeta, \zeta' \in \Delta^n_{k'j'}$, where $\Delta^n_{k'j'}$ is supposed to lie in G_k but not in V^n_{kj}. (5.15) follows from (5.16), (5.17) and (5.18). We have yet to treat the case where $\Delta^n_{k'j'}$ intersects $\omega - G_k$. But then (5.15) (for sufficiently large n) is an obvious consequence of the continuity of g.

From now on N_1 and N_2 are to be considered fixed. n is still variable. Let $\zeta_1, \zeta_2, \ldots, \zeta_m$ designate those points which support a concentrated mass of weight $\geqq \frac{1}{32(N_2+1)^2 A^4}$ in the measure $\mu_{21}(e \frown K_2)$. We denote the corresponding masses by $p_1, p_2, \ldots, p_m$ and define the metric $e^{w(z)}|dz|$ by putting

[15]) with respect to the Euclidean metric in the t_k-plane.

$$w(z) = h(z) + \sum_{l=1}^{m} p_l g(z, \zeta_l) \ . \tag{5.19}$$

We enclose each ζ_l by two analytic JORDAN curves ε_l and ε_l' which satisfy the following conditions:

(a) ε_l' is contained in the interior of ε_l $(l = 1, 2, \ldots, m)$;

(b) $\gamma_1, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m$ do not intersect;

(c) whenever ζ_l is a singular point of μ (i. e. $p_l \geqq 1$), then ε_l shall be so close to ζ_l that

$$\int_{\varkappa} e^{w} |dz| > \frac{4}{1 - \cos(\pi\eta)} \int_{\gamma_1} e^{v_1} |dz| \tag{5.20}$$

for every rectifiable curve $\varkappa$ leading from γ_1 to ε_l;

(d)
$$\sum_{p_l < 1} \int_{\varepsilon_l} e^{v_1} |dz| < 1 \ . \tag{5.21}$$

Let $\nu(e)$ denote the measure which originates from $\mu_{21}(e \frown K_2)$ after removal of the concentrated masses p_1 in $\zeta_1, \ldots, p_m$ in ζ_m. For all $z \in K_1$ we have, by (5.13) and (5.19)

$$v_1(z) = v_2(z) + \int_{\omega - K_2} g(z, \zeta) d\mu_{21}(e_\zeta) \leqq v_2(z) + \log 2 \ , \tag{5.22}$$

where

$$v_2(z) = w(z) + \int_{\omega} g(z, \zeta) d\nu(e_\zeta) \ . \tag{5.23}$$

Let C designate the compact which is obtained by subtracting the interiors of $\varepsilon_1', \varepsilon_2', \ldots, \varepsilon_l'$ from the closure of (γ_1, γ_2).

Now we choose n, large enough so that conditions (5.24) to (5.30) are satisfied for all k, j and l:

$$n > n_0(N_2) \ ; \tag{5.24}$$

$$\Delta_{kj}^n \frown \varepsilon_l \neq 0 \quad \text{implies} \quad U_{kj}^n \frown \varepsilon_l' = 0 \ ; \tag{5.25}$$

$$\nu(\Delta_{kj}^n) \leqq \frac{1}{32(N_2 + 1)^2 A^4} \ ; \tag{5.26}$$

$$m_1 | g(z, \zeta_1) - g(z, \zeta_2) | < \log 2 \tag{5.27}$$

for all $z \in \gamma_1$ and $\zeta_1, \zeta_2 \in \Delta_{kj}^n$, provided that $\Delta_{kj}^n \frown \omega_1 \neq 0$;

$$| w(t_k) - w(\tau_k) | < \log 2 \tag{5.28}$$

and

$$| r_k(t_k, \tau_k) - r_k(t_k', \tau_k') | < \log 2 \ , \tag{5.29}$$

whenever t_k, τ_k, t_k' and τ_k' lie in the same V_{kj}^n.

There exists a positive number L such that, for all $z \in \omega$

$$\nu(E[\zeta \,|\, g(z,\zeta) > L]) < \tfrac{1}{4} \ .$$

Let

$$W = \max_k [\sup_{t_k \in G_k \cap C} w(t_k)] \quad \text{and} \quad R = \max_k [\sup_{t_k, \tau_k \in G_k} r_k(t_k, \tau_k)] \ .$$

We postulate that, for $0 < \lambda \leqq 1/n$

$$\frac{2^9 m_1 A^{\frac{17}{4}} (N_2 + 1)^2}{(N_2 - N_1 - 1)^{\frac{1}{4}}} e^{W+R+m_1 L} \cdot \lambda^{\frac{1}{2}} |\log \lambda| < 1 \ . \tag{5.30}$$

On each Δ^n_{kj} intersecting ω_1 there is a point $\zeta_{\min}(\Delta^n_{kj})$ such that

$$\int_{\gamma_1} \frac{\partial g(z,\zeta_{\min})}{\partial n_z} |dz| \leqq \int_{\gamma_1} \frac{\partial g(z,\zeta)}{\partial n_z} |dz| \tag{5.31}$$

for all $\zeta \in \Delta^n_{kj}$, n_z denoting the normal to γ_1 which points into (γ_1, δ_1).

Now we concentrate in each $\zeta_{\min}(\Delta^n_{kj})$ the mass ν which is associated with the interior of Δ^n_{kj} and part of its boundary, defining the latter such that every point is covered exactly once. Let these be the masses p_{m+1} in $\zeta_{m+1}, \ldots, p_r$ in ζ_r. We introduce the metric $e^{w_1(z)}|dz|$, where

$$w_1(z) = h(z) + \sum_{l=1}^{r} p_l g(z,\zeta_l) = w(z) + \sum_{l=m+1}^{r} p_l g(z,\zeta_l) \ . \tag{5.32}$$

Let γ be a Jordan curve in ω which is homotopic to β_1. We state that

$$I(e^{w_1}|dz|, \gamma; \beta_2) < -2\pi\eta \ . \tag{5.33}$$

In order to prove this inequality we first observe that, by the Gauss-Bonnet formula

$$I(e^{w_1}|dz|, \gamma; \beta_2) \leqq I(e^{w_1}|dz|, \gamma_1; \beta_2) \ . \tag{5.34}$$

From (5.31) and the construction of w_1 we infer that

$$I(e^{w_1}|dz|, \gamma_1; \beta_2) \leqq I(e^{v_2}|dz|, \gamma_1; \beta_2) \ . \tag{5.35}$$

By (4.8), (5.22), (5.6) and (5.7)

$$\begin{aligned} I(e^{v_2}|dz|, \gamma_1; \beta_2) &= I(e^{v_2}|dz|, \gamma_0; \beta_2) \\ &\leqq I(e^{v_1}|dz|, \gamma_0; \beta_2) \leqq I(e^{v}|dz|, \gamma_0; \beta_2) + 2\pi m_0 \\ &\leqq I(e^{u}|dz|, \gamma_0; \beta_2) + 2\pi[m_0 + \mu_1(\Omega)] \ . \end{aligned} \tag{5.36}$$

From (4.8) and (5.2) we infer that

$$I(e^u|dz|,\gamma_0;\beta_2) = I(e^u|dz|,\beta_1;\beta_2) + 2\pi\mu(\beta_1,\gamma_0) < -2\pi[\mu_1(\Omega) + 2\eta] \ . \tag{5.37}$$

(5.33) is implied by relations (5.4) and (5.34) to (5.37).

We state that there exists a rectifiable curve β, leading from γ_1 to δ_2, such that

$$\int_\beta e^{w_1}|dz| < \frac{2}{1-\cos(\pi\eta)} \int_{\gamma_1} e^{w_1}|dz| \ . \tag{5.38}$$

The proof of this inequality is so similar to the one of (2.39) that we do not reproduce it here. We limit ourselves to the following remarks:

(a) In the definition

$$\Lambda(z_0) = \inf_\gamma \int_\gamma e^{w_1}|dz|$$

we admit to competition all rectifiable JORDAN curves γ which lie in the closure of (γ_1,γ_2), pass through z_0 and are homotopic to β_1. Then (2.44) holds again for every minimal curve $\bar{\gamma}(z_0)$ which as neither double points nor points in common with γ_2. This follows from (5.33).

(b) β can be constructed as a "polygon", i. e. a contiguous chain of "straight line segments". Thereby a "straight line segment" σ in ω is defined to be a smooth curve with the property that, for all k, the set $\varphi_k(\sigma \cap \Delta_k^0)$ in the t_k-plane consists of (Euclidean) straight line segments.

(c) β does not intersect any ε_l for which $p_l \geqq 1$. Indeed, by (5.32)

$$\int_\beta e^{w}|dz| \leqq \int_\beta e^{w_1}|dz| \ . \tag{5.39}$$

From (5.27) and (5.22) we infer that

$$\int_{\gamma_1} e^{w_1}|dz| \leqq 2\int_{\gamma_1} e^{v_2}|dz| \leqq 2\int_{\gamma_1} e^{v_1}|dz| \ . \tag{5.40}$$

(5.38), (5.39) and (5.40) imply the inequality

$$\int_\beta e^{w}|dz| < \frac{4}{1-\cos(\pi\eta)} \int_{\gamma_1} e^{v_1}|dz| \ , \tag{5.41}$$

which would contradict (5.20) if β would intersect any ε_l for which $p_l \geqq 1$.

With the "polygon" β we now associate a curve α which leads from δ_1 to δ_2. This is done by the following construction: Let $E(z)$ denote the set consisting of those Δ_{kj}^n's which contain the point z. We join the endpoint z_0 of β on γ_1 with the last point of intersection z_1' of β with $E(z_0)$ by a "straight line segment", z_1' with the last point of intersection z_2' of β with $E(z_1')$, and so forth, until we arrive at the endpoint of β on δ_2. We obtain a "polygon" β'. Now, if β' should penetrate into $\bar{\omega} - K_1$ or into the interiors of some curves ε_l for which $p_l < 1$, then we replace the "subpolygon" between the first entry and the last exit by a boundary arc. The resulting curve contains a portion α which leads from δ_1 to δ_2 and is contained in the closure of (δ_1, δ_2). We state that

$$\int_\alpha e^{v_2}|dz| \leqq \frac{32}{3} \int_\beta e^{w_1}|dz| + 3 \ . \tag{5.42}$$

We decompose $\alpha = \alpha_P + \alpha_C$, α_P consisting of a finite number of "polygons", α_C of the detours introduced above. By (5.12), (5.21) and (5.22)

$$\int_{\alpha_C} e^{v_2}|dz| < 2 \ .$$

We are left to show that

$$\int_{\alpha_P} e^{v_2}|dz| \leqq \frac{32}{3} \int_\beta e^{w_1}|dz| + 1 \ . \tag{5.43}$$

The verification of this inequality is quite analogous to the proof of (2.65). We leave it to the reader and limit ourselves to the following remarks:

(a) From (5.26) and property (II) of N-neighborhoods we infer

$$\nu(V^n_{kj}) \leqq \tfrac{1}{4} \ . \tag{5.44}$$

Every point z in K_2 belongs to at most 6 Δ^n_{kj}'s. From this and property (III) of N-neighborhoods we conclude that

$$\sum_{k,j} \nu(V^n_{kj}) \leqq 48\, m_1 (N_2 + 1)^2 A^4 \ . \tag{5.45}$$

Relations (5.44) and (5.45) correspond to (2.57) and (2.58), respectively.

(b) In the estimations corresponding to (5.59), (5.60) and (5.61) it is convenient to integrate in the plane of the respective uniformizer t_k. One makes use of the decomposition (5.16) of Green's function. The logarithmic term is handled in the same way as in the proof of (2.65). The additional function r_k is taken care of by relation (5.29) and the fact that R occurs in (5.30).

Inequality (5.11) follows from (5.22), (5.42), (5.38) and (5.40). We have thus proved Lemma 6 and, with it, Theorem 13.

6. Further results

Theorem 14. *Let S be an open Riemann surface on which a complete conformal metric $e^{u(z)}|dz|$ is defined. Suppose that the measure μ^+ has a compact support. Then the total area $A = \iint_S e^{2u} dx\, dy$ is infinite.*

Remark. For manifolds which possess a continuous Gaussian curvature K our hypothesis simply means that $K \geqq 0$ outside a compact subdomain. We mention that Theorem 14 has already been demonstrated by F. FIALA (Theorem A, p. 300 in [12]) for (necessarily simply connected) analytic manifolds whose curvature is everywhere non-negative.

Proof. Since $C^- = 2\pi\mu^+(S) < +\infty$, we conclude from Theorem 13 that S is finitely connected. Furthermore, the curvatura integra C exists. Consequently[16], $I(\Gamma_r) \geqq 0$ for $r = 1, 2, \ldots, N$. Consider an arbitrary one of the regions Ω_r. It is conformally equivalent to a schlicht circular ring $R_1 < |z| < R_2$ $(0 < R_1 < R_2 \leqq +\infty)$. Again we distinguish between two cases:

(I) $R_2 < +\infty$. It has been verified in the proof of Theorem 12 that the completeness of the metric yields indeed $A = +\infty$. (Actually this case does not occur at all. For, $R_2 < +\infty$ implies that S is hyperbolic, and it will be shown in Theorem 15 that this is incompatible with $C^- < +\infty$.)

(II) $R_2 = +\infty$. We may assume that Ω_r does not intersect the support of μ^+. (If necessary we increase R_1.) Then u is superharmonic throughout Ω_r. Since $I(\Gamma_r) \geqq 0$, we have, by (4.7)

$$\lim_{\varrho \to \infty} \Phi(u, |z| = \varrho; \Gamma_r) = \Phi(\Gamma_r) \geqq -1 . \tag{6.1}$$

But, u being superharmonic, $\Phi(u, |z| = \varrho; \Gamma_r)$ is a non-increasing function of ϱ. Hence (6.1) implies

$$\Phi(u, |z| = \varrho; \Gamma_r) \geqq -1 \quad (R_1 < \varrho < +\infty) . \tag{6.2}$$

Furthermore

$$\int_0^{2\pi} u(\varrho_2 e^{i\varphi}) d\varphi - \int_0^{2\pi} u(\varrho_1 e^{i\varphi}) d\varphi = 2\pi \int_{\varrho_1}^{\varrho_2} \Phi(u, |z| = \varrho; \Gamma_r) d \log \varrho , \tag{6.3}$$

where $R_1 \leqq \varrho_1 \leqq \varrho_2 < +\infty$. In the case of sufficiently regular u this relation can be verified immediately by a direct calculation. It is more generally true (and essentially known) for all functions u which admit the decomposition (1.9). (The reader looking for a proof will find section 5.14, p. 35 in [25], helpful.) From (6.2) and (6.3) we infer that

$$\frac{1}{2\pi} \int_0^{2\pi} u(\varrho e^{i\varphi}) d\varphi \geqq B_1 - \log \varrho \tag{6.4}$$

for some real constant B_1 and arbitrary ϱ $(R_1 \leqq \varrho < +\infty)$. By making use of the theorem of the arithmetic and geometric means [15, p. 137] we obtain, for arbitrary ϱ

$$\int_0^{2\pi} e^{2u(\varrho e^{i\varphi})} d\varphi \geqq 2\pi e^{\frac{2}{2\pi} \int_0^{2\pi} u(\varrho e^{i\varphi}) d\varphi} \geqq \frac{B_2}{\varrho^2} , \tag{6.5}$$

[16]) See proof of Theorem 10.

B_2 denoting a positive constant. (6.5) yields

$$A > \int_{R_1}^{+\infty} \int_0^{2\pi} e^{2u(\varrho e^{i\varphi})} \varrho \, d\varrho \, d\varphi \geqq B_2 \int_{R_1}^{+\infty} \frac{d\varrho}{\varrho} = +\infty \ . \qquad \text{Q. E. D.}$$

Theorem 15. *If an open RIEMANN surface S admits a complete conformal metric $e^{u(z)}|dz|$ with finite C^-, then it is parabolic.*

Remark. This result is known in the simply connected case, where it has been proved by CH. BLANC and F. FIALA [5].

Proof. We assume that S is hyperbolic and show that this leads to a contradiction.

Under this hypothesis S would possess a GREEN's function (cf. P. J. MYRBERG [21], R. NEVANLINNA [24, chapter 10]). Consider the conformal metric $e^{v(z)}|dz|$, where

$$v(z) = u(z) + 2g(z, \zeta_0) + \int_S g(z, \zeta) d\mu^+(e_\zeta) \ . \qquad (6.6)$$

Here ζ_0 denotes an arbitrary, but fixed point on S. The integral on the right-hand side of (6.6) is not identically infinite, since $\mu^+(S) = C^-/2\pi < +\infty$. One verifies easily that $C^+ \geqq 4\pi$ and $C^- = 0$ for this metric. But since always $\chi \leqq 1$ we conclude from Theorems 10 and 13 that $e^{v(z)}|dz|$ cannot be complete. Hence $e^{u(z)}|dz|$ would not be complete either, contrary to hypothesis.

We observe that the assumption $C^- < +\infty$ has only been used for the purpose of showing that the integral in (6.6) ist not identically infinite. Hence we have actually proved a statement which is slightly stronger than Theorem 15. *Let S be a hyperbolic RIEMANN surface carrying a complete conformal metric $e^{u(z)}|dz|$ whose curvatura integra C may or may not exist. Then*

$$\int_S g(z, \zeta) d\mu^+(e_\zeta) \equiv +\infty \ ,$$

where g denotes GREEN's function for S. In the case of infinitely connected S Theorem 15 is obviously superseded by Theorem 13, whereas the stronger result contains new information.

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