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1. Introduction

A fundamental result relating the topology of a smooth (i. e., C^∞) manifold X and its global differential geometry is the Theorem of de Rham [5], which we state in the following form: Let $\mathfrak{C}(X, \boldsymbol{R})$ denote the exterior differential algebra (over the real numbers $\boldsymbol{R}$) of smooth differential forms on X, and let $\mathfrak{H}(X, \boldsymbol{R})$ be its derived cohomology algebra. For each $\theta \epsilon \mathfrak{C}(X, \boldsymbol{R})$ we define the smooth singular cochain hθ by the formula

$$h\theta \cdot c = \int_c \theta \tag{1}$$

for smooth real chains c. It follows from Stokes' Formula that h induces a homomorphism h^* on cohomology classes, and de Rham's Theorem asserts that h^* *is an algebra isomorphism of* $\mathfrak{H}(X, \boldsymbol{R})$ *onto the singular cohomology algebra (cup product)* $H(X, \boldsymbol{R})$ *of* X.

Now let A be an integral subdomain of $\boldsymbol{R}$. We will consider certain smooth differential forms with singularities (i.e., forms ω defined on X except perhaps for a closed rare set $e(\omega)$). For such forms the hypotheses of Stokes' Formula are not satisfied; however, the deviation (*called the residue relative to the integral chain c*)

$$\int_c d\omega - \int_{\partial c} \omega \tag{2}$$

plays a fundamental role in the theory. We will construct a differential graded A-module $\mathfrak{C}(X, A)$ of forms with singularities (more precisely, of certain equivalence classes defined by these forms), requiring that all residues lie in A. A product is defined on the cohomology classes of $\mathfrak{C}(X, A)$ (but not the elements of $\mathfrak{C}(X, A)$ themselves). Our basic result (Theorem 4A) is an analogue of de Rham's Theorem, asserting that *the derived cohomology A-algebra* $\mathfrak{H}(X, A)$ *of* $\mathfrak{C}(X, A)$ *is canonically isomorphic to the singular cohomology* $H(X, A)$ *of* X *with coefficients in* A; the isomorphism is given essentially by means of the residues (2); see Section 5. Included in $\mathfrak{H}(X, A)$ are those cohomology classes of X represented by closed forms (without singularities) which correspond under (1) to cocycles with coefficients in A; see Theorem 5C.

The motivating idea of constructing suitable forms with singularities came from a study of the Kronecker index (see Section 2 for examples) and more generally from DE RHAM's Intersection Formula (Section 6). Our proof of Theorem 4A uses the CARTAN-LERAY sheaf theory. In Section 5 we use a simplicial subdivision of X to show that *in every cohomology class of* $\mathfrak{H}^r(X, A)$ *we can find a representative whose singularity is an* $(n-r)$*-cycle* $(dim\ X = n)$. This is a simplification of the construction made in ALLENDOERFER [2] of integer residue forms relative to a particular subdivision of X, and the present work should be considered as an outgrowth of that paper. We end by indicating briefly some applications of the methods of harmonic integrals to the theory of forms with singularities.

Notations

X: a paracompact, connected, differentiable manifold of dimension n and class C^∞ (= smooth).

A: an integral subdomain of the real number field $\boldsymbol{R}$.

$\boldsymbol{Z}$ $(\boldsymbol{Z}_m)$: the integers (integers modulo m).

$\overline{S}_r(X, A)$: the A-module of locally finite smooth singular r-chains of X with coefficients in A.

$S_r(X, A)$: the submodule of finite chains.

$\mathfrak{T}(A)$: the twisted coefficient domain of A (see CARTAN [4, XX] or DE RHAM-KODAIRA [8]; a chain (or form) with twisted coefficients is sometimes said to be of even (or odd) kind); $\mathfrak{T}(A) = A$ if X is orientable.

2. On singular forms and their residues

(A) Suppose we choose an orthonormal coordinate system in EUCLIDEAN space E_n $(n \geq 2)$ and write the coordinates of a point x as $(x_1, \ldots, x_n)$. We will let

$$\omega(x) = k(n) \sum_{i=1}^{n} (-1)^{i+1} x_i (x_1^2 + \ldots + x_n^2)^{-n/2} dx_1 \vee \ldots \vee \widehat{dx_i} \vee \ldots \vee dx_n \quad (1)$$

denote *KRONECKER's index form*, where $k(n)$ is the reciprocal of the area of the unit $(n-1)$-sphere in E_n and $\vee$ denotes the exterior product. Then ω is a harmonic (and in particular a closed, analytic) $(n-1)$-form in $E_n - O$; furthermore, see HADAMARD [9, p. 453], for any oriented n-simplex σ whose boundary $\partial\sigma$ does not contain O,

$$\int_{\partial\sigma} \omega = \begin{cases} 0 \text{ if } O \text{ is not in } \sigma \\ \pm 1 \text{ otherwise,} \end{cases} \quad (2)$$

the sign depending on whether the orientation of σ agrees with or is opposite to that of E_n. If ω is the index form in $E_n - O$, then there is an analytic

$(n-2)$-form ξ in $E_n - E_1^+$, where $E_1^+ = \{x = (x_1, 0, \ldots, 0)\colon\ x_1 \geq 0\}$, such that if σ is an oriented $(n-1)$ simplex not containing O and whose boundary does not intersect E_1^+, then

$$\int_\sigma \omega - \int_{\partial\sigma} \xi = \begin{cases} 0 \text{ if } \sigma \text{ does not intersect } E_1^+ \\ \pm 1 \text{ otherwise.} \end{cases}$$

An explicit construction for ξ has been given by HADAMARD (loc. cit.; see also our Proposition 4B); e. g., for $n = 2$, $\xi(x) = k(2) \arctan(x_2/x_1)$ and for $n = 3$,

$$\xi(x) = k(3) \frac{x_1 x_3 dx_2 - x_1 x_2 dx_3}{(x_1^2 + x_2^2 + x_3^2)^{1/2} (x_2^2 + x_3^2)} .$$

That ξ might be called *the (generalized) angle form;* in this connection see ALLENDOERFER [3, p. 256].

(B) In the following proposition we will use ω to generalize these statements. Let $B_n = \{x \in E_n\colon x \leq 1\}$, $E^{n-r} = \{x \in E_n\colon x = (0, \ldots, 0, x_{r+1}, \ldots, x_n)\}$

$$B^{n-r} = B_n \cap E^{n-r} .$$

Proposition. *There is a smooth $(r-1)$-form ω in $E_n - B^{n-r}$ $(2 \leq r < n)$ such that* 1) *its exterior differential is uniquely extendable to a smooth r-form θ on $E_n - \partial B^{n-r}$, and* 2) *for any oriented r-simplex σ in E_n not intersecting ∂B^{n-r} and whose boundary does not intersect B^{n-r},*

$$\int_\sigma \theta - \int_{\partial\sigma} \omega = \begin{cases} 0 \textit{ if } \sigma \textit{ does not intersect } B^{n-r} , \\ \pm 1 \textit{ otherwise,} \end{cases} \tag{3}$$

the sign again depending on the orientation of σ.

Proof. Let $C = \{x \in E_n\colon\ 4x_1^2 + \ldots + 4x_r^2 + x_{r+1}^2 + \ldots + x_n^2 \leq 1\}$ and $D = \{x \in E_n\colon\ |x| \geq 1\}$. Then $C' = C - \partial B^{n-r}$ and $D' = D - \partial B^{n-r}$ are disjoint closed subsets of the manifold $E_n - \partial B^{n-r}$. By a well known construction (see WHITNEY [12, App. 3]) there is a smooth real function ψ on $E_n - \partial B^{n-r}$ such that $0 \leq \psi \leq 1$, $\psi(x) = 0$ if $x \in D'$, $\psi(x) = 1$ if $x \in C'$.

If $\pi\colon E_n \to E_r$ is the projection map $\pi(x_1, \ldots, x_n) = (x_1, \ldots, x_r)$ and ω_0 is the index form in $E_r - O$, then the induced form $\pi^* \omega_0$ is a closed $(r-1)$-form on $E_n - E^{n-r}$. Set

$$\omega(x) = \begin{cases} \psi(x)\, \pi^* \omega_0(x) & \text{in } E_n - E^{n-r} \\ 0 & \text{if } x \in E^{n-r} \text{ and } |x| > 1; \end{cases}$$

$$\theta(x) = \begin{cases} d\omega(x) & \text{in } E_n - B^{n-r} \\ 0 & \text{if } x \in E^{n-r} \text{ and } |x| < 1 . \end{cases}$$

Clearly ω and θ have the properties described in 1); the uniqueness of the extension of $d\omega$ follows from its continuity (and the fact that B^{n-r} is rare in E_n). The equation (3) follows from (2) by a simple computation (or by appeal to Proposition $2D$ below).

Remark. Another method of constructing such pairs (θ, ω) and valid on any closed RIEMANN manifold can be given by means of GREEN's form; see Section 6.

(C) Let X be a smooth manifold, and let ω be a smooth $(r-1)$-form $(r > 0)$ defined on X except perhaps for a closed rare (= nowhere dense) set $e(\omega)$; we do not require that $X - e(\omega)$ is the maximal domain of definition for ω. Let us agree that if $r = 0$ then ω is the function identically zero on X, whence $e(\omega) = \emptyset$. Suppose that θ is an extension of $d\omega$ to $X - e(\theta)$, where $e(\theta)$ is a closed subset of $e(\omega)$; the extension of θ is necessarily unique, although of course it depends on $e(\theta)$. We will call (θ, ω) a *pair on* X.

Definitions. Let c be a smooth finite singular r-chain on X with real coefficients, and let $|c|$ denote its *support* i.e., $|c|$ is the union of point set images $|s_i|$ in X of the simplexes s_i in the unique expression $c = \Sigma a_i s_i$, where $a_i \neq 0$ and the s_i are distinct. Then

$$|c_1 + c_2| \subset |c_1| \cup |c_2|, \quad |ac| \subset |c|, \quad |\partial c| \subset |c|. \tag{4}$$

We say that c *is admissible for the pair* (θ, ω) if $|c| \cap e(\theta) = \emptyset$, $|\partial c| \cap e(\omega) = \emptyset$.

Given such a chain, we define the *residue of* (θ, ω) *with respect to* c as the number

$$R[(\theta, \omega), c] = \int_c \theta - \int_{\partial c} \omega. \tag{5}$$

It follows from STOKES' Formula that *if* $e(\omega) = \emptyset$, *then every smooth r-chain is admissible and all residues are zero.*

Properties (4) show that the residue is bilinear in the arguments (θ, ω) and c when all terms are defined; however, it can happen for example that $c_1 + c_2$ is admissible for (θ, ω) and yet neither c_i is admissible.

(D) **Proposition.** *Let* c_t $(0 \leq t \leq 1)$ *be a smooth deformation of the chain* c_0 *on* X *which is admissible for the pair* (θ, ω); i. e., *every chain* c_t *is admissible for* (θ, ω). *Then*

$$R[(\theta, \omega), c_0] = R[(\theta, \omega), c_1].$$

Proof. We have the usual smooth chain homotopy formula $c_1 - c_0 = D\partial c_0 + \partial D c_0$, where Db denotes the deformation chain of b. Then by hypothesis $|D\partial c_0| \cap e(\omega) = \emptyset$, whence $D\partial c_0$ is admissible for (θ, ω) and has zero residue; similarly $|\partial D c_0| \cap e(\theta) = \emptyset$, and we have

$$R[(\theta, \omega), c_1 - c_0] = R[(\theta, \omega), D\partial c_0] + \int_{\partial D c_0} \theta = 0.$$

3. The complex $\mathfrak{C}(X, A)$ of forms

(A) **Definition.** An *(A, r)-pair (θ, ω) of forms on* $X (r \geq 0)$ is a pair as in Section 2 C such that 1) the singular sets $e(\theta)$ and $e(\omega)$ lie on smooth locally finite polyhedra of dimensions not exceeding $n - r - 1$ and $n - r$, respectively, and 2) for every smooth r-chain $c \in S_r(X, A)$ which is admissible for (θ, ω) the residue $R[(\theta, \omega), c]$ is an element of A.

Thus if $(\theta, 0)$ is an $(A, 0)$-pair, then θ is a function with constant A-values on the components of $X - e(\theta)$, and $R[(\theta, 0), c_0] = \Sigma_{i=1}^{m} a_i \theta(x_i)$ if $c_0 = \Sigma_{i=1}^{m} a_i x_i$. If (θ, ω) is an (A, n)-pair, then $e(\theta) = \emptyset$.

Remark. By (1) of Section 5 below each (A, r)-pair (θ, ω) on X determines a *geometric $r - A$-cochain* of X in the sense of Whitney [11] with *nucleus* $e(\omega)$ and *nuclear boundary* $e(\theta)$. With this identification Whitney's theory can be considered as an abstract form (for polyhedra) of the calculus of (A, r)-pairs just as the theory of flat cochains (for polyhedra; see Whitney [12; Part II]) abstracts the exterior differential calculus.

(B) We will need the following well known result: *Any two smooth chains c_p and $c_q (p + q \leq n)$ on X can be brought into general position* $(|\partial c_p| \cap |c_q| = \emptyset$, $|c_p| \cap |\partial c_q| = \emptyset)$ *by an arbitrarily small smooth deformation.*

The sum $(\theta_1, \omega_1) + (\theta_2, \omega_2)$ of two (A, r)-pairs is defined as the (A, r)-pair $(\theta_1 + \theta_2, \omega_1 + \omega_2)$, where $e(\theta_1 + \theta_2) = e(\theta_1) \cup e(\theta_2)$, $e(\omega_1 + \omega_2) = e(\omega_1) \cup e(\omega_2)$; similarly for $a(\theta, \omega) = (a\theta, a\omega)$ for $a \in A$. The (A, r)-pairs on X do not form an A-module, for the inverse of (θ, ω) is not defined if $e(\omega) \neq \emptyset$. We are thus led to take equivalence classes of (A, r)-pairs: Let us say that $(\theta, \omega) \equiv (\theta', \omega')$ if $R[(\theta, \omega), c] = R[(\theta', \omega'), c]$ for all chains $c \in S_r(X, A)$ which are admissible for both pairs. It is clear that $\equiv$ is reflexive and symmetric; to show that it is transitive let $(\theta, \omega) \equiv (\theta', \omega') \equiv (\theta'', \omega'')$, and take any c admissible for both (θ, ω) and (θ'', ω''). We now make an admissible smooth deformation of c to a chain c_1 which is admissible for all three pairs. By Proposition 2D we have $R[(\theta, \omega), c] = R[(\theta, \omega), c_1] = R[(\theta'', \omega''), c]$; i.e., $(\theta, \omega) \equiv (\theta'', \omega'')$.

Let $[\theta, \omega]$ denote the equivalence class of (θ, ω); it is easily checked that *these classes form an A-module, denoted by* $\mathfrak{C}^r(X, A)$ $(r \geq 0)$.

(C) If (θ, ω) is an (A, r)-pair, then $(0, \theta)$ is an $(A, r + 1)$-pair; it follows easily that we can properly define the *exterior differential d*:

$$\mathfrak{C}^r(X, A) \to \mathfrak{C}^{r+1}(X, A) \quad \text{by} \quad d[\theta, \omega] = [0, \theta].$$

Then d is an A-homomorphism such that $d \cdot d = 0$; its kernel $\mathfrak{Z}^r(X, A)$ consists of those classes $[\theta, \omega]$ such that for any representative (θ, ω) we have

$$\int_{\partial c_{r+1}} \theta = 0 \quad \text{if} \quad |\partial c_{r+1}| \cap e(\theta) = \emptyset. \tag{1}$$

Thus if the class of (θ, ω) is in $\mathfrak{Z}^r(X, A)$ and if $e(\theta) = \emptyset$, then θ is a closed form on X.

Two $(A, 0)$-pairs $(\theta, 0)$, $(\theta', 0)$ are equivalent if and only if $\theta(x) = \theta'(x)$ for all x in their common domain; (1) shows that if they represent an element of $\mathfrak{Z}^0(X, A)$ then θ and θ' have constant A-value in their domains (X is connected), whence the

Proposition. *The natural map* $\mathfrak{Z}^0(X, A) \to A$ *is an isomorphism.*

Definition. The direct sum $\mathfrak{C}(X, A) = \Sigma_{r\geq 0}\mathfrak{C}^r(X, A)$ is a cochain complex, called *the complex of A-pairs of forms on X*. We let $\mathfrak{H}(X, A) = \Sigma_{r\geq 0}\mathfrak{H}^r(X, A)$ denote the derived cohomology module.

(D) We consider briefly now the problem of introducing products in $\mathfrak{H}(X, A)$.

Example. Let (θ^p, ω^{p-1}) be an (A, p)-pair in oriented EUCLIDEAN n-space E_n with $e(\omega^{p-1}) =$ an $(n - p)$-simplex and $e(\theta^p)$ its frontier. We observe that if (θ^p, ω^{p-1}) has any non-zero residue, then the orientation of E_n induces an orientation of $e(\omega^{p-1})$; namely, we take an admissible oriented p-simplex σ_p such that $R[(\theta^p, \omega^{p-1}), \sigma_p] > 0$ and then define the orientation of $e(\omega^{p-1})$ such that the ordered pair σ_p, $e(\omega^{p-1})$ has orientation compatible with that in E_n. If (θ^q, ω^{q-1}) is an (A, q)-pair of the same type whose singularities are in general position with respect to those of (θ^p, ω^{p-1}) (see (B) above), then we define the *exterior product* $[\theta^p, \omega^{p-1}] \vee [\theta^q, \omega^{q-1}]$ of their classes as follows: If $p + q > n$ or if either pair has only zero residues, then we define the product to be zero. Otherwise, the $(n - p - q)$-simplex $e = e(\omega^{p-1}) \cap e(\omega^{q-1})$ has orientation induced from that in $e(\omega^{p-1})$, $e(\omega^{q-1})$; we use Proposition 2B to construct an $(A, p + q)$-pair (θ, ω) with e, ∂e as singularities and with $R[(\theta^p, \omega^{p-1}), \sigma_p]\, R[(\theta^q, \omega^{q-1}), \sigma_q]$ as residue on the product cell $\sigma_p \times \sigma_q$. Set $[\theta^p, \omega^{p-1}] \vee [\theta^q, \omega^{q-1}] = [\theta, \omega]$.

Clearly we cannot hope to define the exterior product of elements of $\mathfrak{C}(X, A)$ in general, for we cannot always alter the position of the singularities of a pair without changing its equivalence class. (The same sort of problem is faced in defining the intersection product of cycles of X.) On the other hand we can use the above construction to define the exterior product of cohomology classes of pairs. In fact, given two classes $z^p \in \mathfrak{H}^p(X, A)$, $z^q \in \mathfrak{H}^q(X, A)$, we can choose (using (B)) representations $[\theta^p, \omega^{p-1}]$, $[\theta^q, \omega^{q-1}]$ whose singularities are in general position, and then define their exterior product. It follows from standard reasoning that *this induces an associative, distributive, anti-commutative* ($z^p \vee z^q = (-1)^{pq} z^q \vee z^p$) *product pairing* $\mathfrak{H}^p(X, A)$, $\mathfrak{H}^q(X, A)$ *to* $\mathfrak{H}^{p+q}(X, A)$.

(E) A smooth (or even regular) map $f: X \to Y$ of one manifold into another may not carry (A, r)-pairs into (A, r)-pairs; however, if f is a bi-regular homeomorphism then any (A, r)-pair (θ, ω) on Y is transformed into an (A, r)-pair $(\theta^* = f^*\theta, \omega^* = f^*\omega)$ on X. Furthermore, if c is a smooth admissible r-chain for (θ^*, ω^*), then fc is admissible for (θ, ω), and by the transformation of integral formula $R[(\theta^*, \omega^*), c] = R[(\theta, \omega), fc]$.

4. The isomorphism theorem

(A) We will now make the appropriate modifications in the sheaf proof of DE RHAM's Theorem (see CARTAN [4] or HIRZEBRUCH [6] for properties of sheaves) to obtain our basic result.

Theorem. *Let X be a paracompact smooth manifold of dimension n, and let A be an integral subdomain of $\boldsymbol{R}$. Then there is a canonical (algebra) isomorphism of the cohomology algebra $\mathfrak{H}(X, A)$ of the A-pairs of forms onto the cohomology algebra $H(X, A)$ of X.*

Example 1. Suppose X is compact and $A = \boldsymbol{Z}$; then $\mathfrak{H}^n(X, \boldsymbol{Z})$ is isomorphic to $\boldsymbol{Z}$ or $\boldsymbol{Z}_2$, depending on whether X is orientable or not. In either case a generator $[\theta, \omega]$ for $\mathfrak{H}^n(X, \boldsymbol{Z})$ can be given as follows ($n \geq 2$): Take any $x_0 \in X$ and let $\psi\colon U \to B_n$ (the unit ball in $\boldsymbol{R}_n$) be a coordinate system (U is a coordinate ball) such that $\psi(x_0) = 0$. Let ω_0 be the index form in $\boldsymbol{R}_n$ with singularity at 0, and let φ be a smooth real function in $\boldsymbol{R}_n$ such that $\varphi(x) = 1$ if $|x| \leq \frac{1}{2}$ and $\varphi(x) = 0$ if $|x| \geq \frac{3}{4}$; then $\omega_1 = \varphi\omega_0$ and $\theta_1 = d\omega_1$ (extended over all $\boldsymbol{R}_n$) is a $(\boldsymbol{Z}, n)$-pair on $\boldsymbol{R}_n$. It follows easily that $\theta = \psi^*\theta_1$, $\omega = \psi^*\omega_1$ (both defined to be zero outside U) determines the desired class $[\theta, \omega]$ on X. If X is not compact, then the same construction gives a generator of $H^n_K(X, \boldsymbol{Z})$, the n^{th} singular cohomology module with compact supports.

Example 2. We will see in Section 5C that any closed r-form on X with integral periods determines a closed $(\boldsymbol{Z}, r)$-pair. It follows that such pairs can be used to generate the integral cohomology algebras of manifolds without torsion (e. g., the complex STIEFEL and GRASSMANN manifolds, the LIE groups $SU(n)$, $Sp(n)$).

Example 3. Let P_n denote the real projective n-space ($n \geq 2$). Then generators for $\mathfrak{H}^r(P_n, \boldsymbol{Z})$ are given by the previous examples if $r = 0, n$ ($n \geq 2$). Supposing $n \geq 2$, the other generators can be constructed as follows: Using the notations of Section 2B, let us represent P_n by identifying the antipodal points of the boundary of B_n. Then the pair (θ, ω) constructed in Proposition 2B determines a closed $(\boldsymbol{Z}, r)$-pair $(2 \leq r < n)$ on P_n with $e(\omega) = |P^{n-r}|$ (the antipodal identification of B^{n-r}) $e(\theta) = |P^{n-r-1}|$, which is noncobounding if r is even and cobounding if r is odd. We will see in Section 5B that (θ, ω) is cohomologous to a pair (θ_1, ω_1) such that $e(\omega_1) = |P^{n-r}|$ and $e(\theta_1) = \emptyset$.

(B) First of all, let us prove the theorem in case X is an open ball U in E_n. If $r = 0$, the result follows from Proposition 3C; if $r > 0$ we use a standard homotopy construction (see e. g., WHITNEY [12, Chapter 4]).

Proposition. *Suppose* $r > 0$ *and* $[\theta, \omega] \in \mathfrak{Z}^r(U, A)$. *Then there is an* $[\eta, \xi] \in \mathfrak{C}^{r-1}(U, A)$ *such that* $d[\eta, \xi] = [\theta, \omega]$.

Proof. By Section 3E we reduce the problem to the case that the singularities of a representative $(\theta, \omega) \in [\theta, \omega]$ lie along rectilinear polyhedra of the appropriate dimensions. Choose $x_0 \in U - e(\omega)$, and let V be the maximal region in $U - e(\omega)$ which is star-shaped with respect to x_0; take a neighborhood W of $V \times I$ in $V \times \boldsymbol{R}$ such that $g\colon W \to V$, where

$$g(x, t) = (1 - t)\, x + t x_0.$$

If $r > 1$ set $ke(\omega) = U - V$, a (rectilinear) locally finite polyhedron of dimension $\leq n - r + 1$. Then for all $x \in V$ we define

$$(k\omega)\, x = \int_0^1 (g^* \omega)(x, t)\, dt\,, \tag{1}$$

a smooth $(r - 2)$-form in $U - ke(\omega)$; if ω is a 0-form set $ke(\omega) = \emptyset$ and $(k\omega)\, x = 0$. Similarly for the definition of $ke(\theta)$ and $k\theta$ in an appropriate maximal star-shaped region $V' \supset V$. Then $\omega(x) = dk(\omega)\, x + k\theta(x)$ in V.

Set $\xi(x) = (k\omega)\, x$ in $U - e(\xi)$ and $\eta(x) = \omega(x) - (k\theta)\, x$ in $U - e(\eta)$, where $e(\xi) = ke(\omega) = U - V$ and $e(\eta) = ke(\theta) \cup e(\omega)$. If c is a (rectilinear) chain in $S_{r-1}(U, A)$ which is admissible for (η, ξ) then the join $J(x_0, c)$ is an r-chain admissible for (θ, ω), and $R[(\eta, \xi), c] = -R[(\theta, \omega), J(x_0, c)]$ since

$$\int_c k\theta = \int_{J(x_0, c)} \theta \text{ and } \int_{\partial c} k\omega = \int_{J(x_0, \partial c)} \omega\,;$$

it follows in particular that (η, ξ) is an $(A, r - 1)$-pair in U. Because $[\theta, \omega]$ is closed we have $(0, \eta) \equiv (\theta, \omega)$; i.e., $d[\eta, \xi] = [\theta, \omega]$, for if c is admissible for both pairs,

$$R[(0, \eta), c] = R[(\theta, \omega), c] - \int_{\partial J(x_0, c)} \theta\,.$$

The integral in the right member is zero by (1) of Section 3C. This completes the proof of the proposition.

(C) We now proceed to the proof of the theorem. Given any coordinate ball U on X we let $\mathcal{C}_U$ denote the complex of A-pairs on U; since the restriction of any class $[\theta, \omega]$ to an open subball V is in $\mathcal{C}_V$ we have the natural restriction homomorphism $\mathcal{C}_U \to \mathcal{C}_V$, which is clearly reflexive and transitive. The differential graded sheaf associated with this presheaf is denoted by

$$\mathcal{C} = \Sigma_{r \geq 0} \mathcal{C}^r .$$

Letting $d\colon \mathcal{C}^r \to \mathcal{C}^{r+1}$ denote the sheaf homomorphism associated with the exterior differential, we have the sheaf sequence

$$0 \to A \to \mathcal{C}^0 \xrightarrow{d} \mathcal{C}^1 \xrightarrow{d} \mathcal{C}^2 \to \dots , \tag{2}$$

where $A \to \mathcal{C}^0$ is the natural imbedding; it follows as usual from the proposition in (B) and the identity $d \cdot d = 0$ that the sequence (2) is exact.

Lemma. (2) *is a sheaf resolution of* A; *i. e., the* Čech *modules* $H^p(X, \mathcal{C}^r) = 0$ *for* $p > 0, r \geq 0$.

Apparently (2) is not a fine resolution; however, we can use a slight modification of the method used to prove the lemma in the fine case. For simplicity we will make use of A. Weil's construction [10, § 1] of a differentiably simple cover $\mathfrak{U} = (U_i)_{i \in I}$ of X; $\mathfrak{U}$ is a locally finite cover by open coordinate balls such that every intersection $U_{i_0} \cap \dots\dots \cap U_{i_p}$ of elements of $\mathfrak{U}$ can be smoothly retracted to a point. Then (see Weil [10]) the nerve $N(\mathfrak{U})$ of $\mathfrak{U}$ has the same homotopy type as X, and we have a canonical isomorphism $H^p(X, \mathcal{C}^r) \approx H^p(\mathfrak{U}, \mathcal{C}^r)$.

To prove the lemma we take any $f \in Z^p(\mathfrak{U}, \mathcal{C}^r)$; for each simplex

$$(i) = (i_0 \dots i_p) \quad \text{in} \quad N(\mathfrak{U})$$

we choose a representative $(\theta_{(i)}, \omega_{(i)}) \in f(i)$. Let e_j denote the union of all singular points of all the $\omega_{(i)}$ which lie in U_j; then e_j lies on a locally finite (with respect to U_j) polyhedron of dimension $\leq n - r$. Set

$$B_j = \bar{e}_j \cap (\bar{U}_j - U_j);$$

then $B = \cup_{j \in I} B_j$ is a locally finite union of closed sets and therefore is closed. Let $e'_j = e_j \cap (X - B)$, and take a locally finite open covering $\mathfrak{V} = (V_i)_{i \in I}$ of $X - B$ such that $e'_j \subset V_j \subset \bar{V}_j \subset U_j$.

Take a smooth function $\bar{\varphi}_j$ on the manifold $X - B$ such that $\bar{\varphi}_j(x) = 1$ if $x \in V_j$, $\bar{\varphi}_j(x) = 0$ if $x \notin \bar{U}_j$, $\bar{\varphi}_j(x) > 0$ in U_j; of course $\bar{\varphi}_j$ cannot be smoothly extended to X. Setting $\varphi_j = \bar{\varphi}_j / \Sigma_k \bar{\varphi}_k$, we obtain a smooth partition of unity on $X - B$ (depending on the choices of f and $(\theta_{(i)}, \omega_{(i)})$). We extend $\varphi_j \omega_{(i)}$ to be zero outside $\bar{U}_j$ in $U_{i_0} \cap \dots \cap U_{i_p}$; similarly for $\varphi_j \theta_{(i)}$. It follows easily that $(\varphi_j \theta_{(i)}, \varphi_j \omega_{(i)})$ is an (A, r)-pair in $U_{(i)}$, with

$$e(\varphi_j \omega_{(i)}) = U_j \cap e(\omega_{(i)}), \quad e(\varphi_j \theta_{(i)}) = U_j \cap e(\theta_{(i)}) .$$

Define $g_j \in C^{p-1}(\mathfrak{U}, \mathcal{C}^r)$ by

$$g_j(i_0 \ldots i_{p-1}) = \begin{cases} [\varphi_j \theta_{ji_0 \ldots i_{p-1}}, \varphi_j \omega_{ji_0 \ldots i_{p-1}}] & \text{in} \quad U_{ji_0 \ldots i_{p-1}} \\ 0 \text{ in the rest of } U_{i_0 \ldots i_{p-1}} . \end{cases}$$

Then $g = \Sigma_j g_j$ (locally finite sum) is a cochain easily seen to satisfy $dg = f$; the lemma follows.

Using the resolution (2) the theorem is now completed by application of standard methods in sheaf theory. (For the canonical isomorphism of $\mathfrak{H}^r(X, A)$ onto $H^r(X, A)$ see HIRZEBRUCH [6, § 2]; for the product isomorphism, see CARTAN [4, XX].)

Remarks. The theorem and proof are valid for manifolds of class C^{k+1} $(k \geq 0)$ with $\mathfrak{C}^r(X, A)$ based on (A, r)-pairs of forms, requiring both θ and ω to be of class C^k. The theorem can be modified by requiring that the (A, r)-pairs and the cochains have compact supports.

(D) Let m be any positive integer; the quotient $\mathfrak{C}^r(X, \boldsymbol{Z}) / \mathfrak{C}^r(X, m\boldsymbol{Z})$ is (roughly speaking) the module of $(\boldsymbol{Z}, r)$-pairs whose residues are in $\boldsymbol{Z}_m$. We let $\mathfrak{C}(X, \boldsymbol{Z}_m)$ denote the complex defined by these quotient modules. The following statement is an application of the five-lemma and Theorem 4A.

Corollary. *There is a canonical isomorphism of $\mathfrak{H}^r(X, \boldsymbol{Z}_m)$ onto $H^r(X, \boldsymbol{Z}_m)$.*

(E) DE RHAM's Theorem was originally formulated in terms of existence and uniqueness of a class of closed forms having prescribed real periods on a set of linearly independent (with respect to real homology) cycles on a compact manifold; see DE RHAM [5, Chapitre III]. We will now give an analogous formulation of Theorem 4A, restricted (for simplicity of statement) to the compact case and $A = Z$.

Definitions. An integral chain $c \in S_r(X, \boldsymbol{Z})$ is said to be a *cycle mod m* $(m \geq 2)$ if $\partial c = ma$ for some $a \in S_{r-1}(X, \boldsymbol{Z})$; two r-cycles c, c' mod m are *homologous mod m* if there are integral chains a and b such that

$$c' - c = \partial a + mb .$$

We must not confuse the r-cycles mod m with the elements of $Z_r(X, \boldsymbol{Z}_m)$; however, it is easy to see that if $r_m: \boldsymbol{Z} \to \boldsymbol{Z}_m$ is the coset homomorphism, then r_m induces an isomorphism of the module $H_r^{(m)}(X, \boldsymbol{Z})$ of homology classes mod m onto $H_r(X, \boldsymbol{Z}_m)$.

Because X is compact its integral homology modules are finitely generated; let β denote its r^{th} Betti number and $\tau_1 < \ldots < \tau_k$ its $(r-1)^{th}$ torsion numbers (τ_i divides τ_{i+1} for $i = 1, \ldots, k-1$). Let us take a *system of integral r-chains*

$$c_1^{(0)}, \ldots, c_\beta^{(0)}; \quad c_1^{(\tau_i)}, \ldots, c_{\alpha_i}^{(\tau_i)} \quad (1 \leq i \leq k) \tag{3}$$

such that 1) the $c_j^{(0)}$ $(1 \leq j \leq \beta)$ form a base for the free part of $H_r(X, \boldsymbol{Z})$, and 2) the $c_j^{(\tau i)}$ $(1 \leq j \leq \alpha_i)$ are linearly independent with respect to homology mod τ_i and the classes of the $(r-1)$-cycles $\partial c_1^{(\tau i)}/\tau_i, \ldots, \partial c_{\alpha_i}^{(\tau i)}/\tau_i$ generate the part of $H_{r-1}(X, \boldsymbol{Z})$ of order τ_i. It is well known that such a system (3) exists (and in fact for simplicial homology (3) is part of a canonical base for the integral r-chains).

Remark. A system (3) for X (for every r) is adequate for a description of the integral homology of X; in fact, it is known (ALEXANDROFF-HOPF [1, p. 228]) that $H(X, \boldsymbol{Z})$ is determined by the collection $H_r(X, \boldsymbol{Z}_m)$.

Given a system (3) and an integral r-cocycle f, the *periods* $\pi_j^{(0)} \in \boldsymbol{Z}(1 \leq j \leq \beta)$ and *modular periods* $\pi_j^{(\tau i)} \in \boldsymbol{Z}_{\tau_i}$ $(1 \leq j \leq \alpha_i)$ *of* f are defined by

$$\pi_j^{(0)} = f \cdot c_j^{(0)} \quad \text{and} \quad \pi_j^{(\tau i)} = r_{\tau_i}[f \cdot c_j^{(\tau i)}]. \tag{4}$$

Clearly these π's depend only on the homology classes (integral or mod m) of the c's and on the cohomology class of f. Conversely, for any set of periods and modular periods relative to (3) there is a cohomology class $\bar{f} \in H^r(X, \boldsymbol{Z})$, such that (4) is satisfied for any $f \in \bar{f}$; the proof is elementary.

We can now reformulate a special case of Theorem 4A as follows:

Theorem. *Let X be a compact smooth manifold, and let* (3) *be a system of integral r-chains on X.*

1) *If (θ, ω) is a closed $(\boldsymbol{Z}, r)$ pair on X for which the chains* (3) *are admissible and if all periods and modular periods are zero, then (θ, ω) is derived.*

2) *For any set of periods and modular periods there is a closed $(\boldsymbol{Z}, r)$-pair (θ, ω) on X for which the chains in* (3) *are admissible, and*

$$\begin{aligned} \int_{c_j^{(m)}} \theta - \int_{\partial c_j^{(m)}} \omega &= \pi_j^{(m)} \quad \textit{if} \quad m = 0 \quad \textit{and} \quad 1 \leq j \leq \beta; \\ &\equiv \pi_j^{(m)} \ \operatorname{mod} m \quad \textit{if} \quad m = \tau_i \quad \textit{and} \quad 1 \leq j \leq \alpha_i. \end{aligned}$$

5. Pairs relative to a subdivision of X

(A) Let X be simplicially subdivided into a locally finite combinatorial manifold K, and let K_* denote its dual cell complex; we will suppose that the star of every vertex of K is contained in a coordinate ball of X. It is known (see WHITNEY [12, Chapter 4]) that any smooth manifold admits such a subdivision.

For each $0 \leq r \leq n$ let $\mathfrak{C}^r(K, A) = \{[\theta, \omega] \in \mathfrak{C}^r(X, A)\colon\ e(\omega) \subset K_*^{(n-r)}$ and $e(\theta) \subset K_*^{(n-r-1)}\}$, where $K_*^{(p)}$ denotes the p-skeleton of K_*. Then $\mathfrak{C}(K, A) = \Sigma_{r=0}^n \mathfrak{C}^r(K, A)$ is easily seen to be a cochain subcomplex of

$\mathfrak{C}(X, A)$. Letting $C^r(K, A)$ denote the module of simplicial r-cochains of K (with coefficients in A) we have the

Proposition. *The map* $h\colon \mathfrak{C}^r(K, A) \to C^r(K, A)$ *defined by*

$$h([\theta, \omega])\cdot c = \int_c \theta - \int_{\partial c} \omega \tag{1}$$

for all simplicial chains $c \in C_r(K, \mathbf{Z})$ *is an isomorphism satisfying* $dh = -hd$.

Proof. First of all, $h([\theta, \omega])$ is clearly well defined for all c and satisfies $dh = -hd$. If $h([\theta, \omega]) = 0$, then for any representative (θ, ω) all residues with respect to the chains of K are zero; we apply Proposition 2D to show that $[\theta, \omega] = 0$; i. e., h is one-one. Now take any cochain $f \in C^r(K, A)$ and express it $f = \Sigma a_i f_i$, where $f_i(\sigma_j) = \delta_{ij}$ (KRONECKER delta) with $a_i \in A$. Using Proposition 2B we construct a $(\mathbf{Z}, r)$-pair (θ_i, ω_i) on X with residue δ_{ij} on σ_j and such that $e(\omega)$ is the support of the cell σ_*^i in K_* dual to σ_i and $e(\theta)$ is the support of $\partial\sigma_*^i$. Then setting $\theta = \Sigma a_i \theta_i$, $\omega = \Sigma a_i \omega_i$ (locally finite sums), we have an (A, r)-pair satisfying

$$f(c) = \int_c \theta - \int_{\partial c} \omega \tag{2}$$

for all $c \in C_r(K, \mathbf{Z})$; thus h is onto, and the proof is complete.

Corollary. *h induces an isomorphism h^* of the r-cohomology module* $\mathfrak{H}^r(K, A)$ *derived from* $\mathfrak{C}^r(K, A)$ *onto* $H^r(K, A)$ *(and therefore onto* $H^r(X, A)$*).*

Remark. A proof of Theorem 4A can be given based on the above proposition and the construction of deformations of arbitrary (A, r)-pairs into those relative to K. Such a proof parallels DE RHAM's original, with the elements of $\mathfrak{C}^r(K, A)$ playing the role of the "elementary forms"; see DE RHAM [5,24].

(B) Let $\overline{C}_p(K_*, \mathfrak{T}(A))$ denote the module of locally finite simplicial p-chains of K_* with twisted coefficient domain $\mathfrak{T}(A)$. We construct an isomorphism $k\colon \overline{C}_{n-r}(K, \mathfrak{T}(A)) \to \mathfrak{C}^r(K, A)$ by taking for each oriented $(n-r)$-cell τ in K_* its (unique) orthogonal oriented r-simplex σ_i in K and defining $k(\tau)$ as the class of an (A, r)-pair (θ, ω) constructed as in Proposition 2B, with $e(\omega) = |\tau|$, $e(\theta) = |\partial\tau|$, and $R[(\theta, \omega), \sigma_j] = \delta_{ij}$ for all r-simplexes σ_j in K. Compare the construction in Proposition 6A below. Then

$$\begin{array}{ccc} \mathfrak{C}^r(K, A) & \xrightarrow{h} & C^r(K, A) \\ k \uparrow & & \downarrow \mathfrak{D} \\ \overline{C}_{n-r}(K_*, \mathfrak{T}(A)) & & \end{array} \tag{3}$$

is easily seen to be a commutative diagram of isomorphisms, where $\mathfrak{D}$ is the cap product of an r-cochain with the fundamental n-cycle of X. As a consequence of this and Proposition 5A we have the

Proposition. *The class* $[\theta, \omega] \in \mathfrak{Z}^r(K, A)$ *if and only if the class contains a pair* (θ, ω) *such that the singular set* $e(\omega)$ *corresponds to an* $(n-r)$*-cycle of* K_*, *and* $e(\theta) = \emptyset$.

Corollary. *Every cohomology class of* $\mathfrak{H}^r(X, A)$ *has a representative* $[\theta, \omega]$ *with* θ *defined and closed on all* X.

(C) **Theorem.** *Given a closed smooth r-form* θ *on* X *with periods (relative to a base of integral r-cycles) in* A, *there is a smooth* $(r-1)$*-form* ω *on* $X - e(\omega)$, *where* $e(\omega)$ *lies on an* $(n-r)$*-cycle, such that* (θ, ω) *is an* (A, r)*-pair.*

Proof. Construct a simplicial subdivision K of X as in (A), and let $f \in C^r(K, A)$ be such that

$$f(z) = \int_z \theta \tag{4}$$

for all $z \in Z_r(K, \boldsymbol{Z})$. By the propositions in (A) and (B) there is an (A, r)-pair (α, β) such that 1) $f(c) = R[(\alpha, \beta), c]$ for all $c \in C_r(K, \boldsymbol{Z})$, 2) α is defined and closed on all X, and 3) $e(\beta)$ lies on an $(n-r)$-cycle of K_*. Now the closed form $\theta - \alpha$ has zero periods relative to a base of r-cycles of K, whence by de Rham's Theorem there is a smooth $(r-1)$-form γ on X such that $d\gamma = \theta - \alpha$. Setting $\omega = \beta + \gamma$, we conclude that $R[(\theta, \omega), c] = R[(\alpha, \beta), c]$ for any $c \in C_r(K, A)$; in particular (θ, ω) is an (A, r)-pair on K and $e(\omega) = e(\beta)$.

Taking $A = \boldsymbol{R}$ we obtain the following result of Allendoerfer [2, Theorem 6]:

Corollary. *Any smooth closed r-form on* X *is derivable from a smooth* $(r-1)$*-form with singularities lying on an* $(n-r)$*-cycle.*

(D) Replacing (3) by its induced homology-cohomology diagram, we see the role played by the A-pairs of forms in the Poincaré duality of X (see Cartan [4, XX]). In fact, we have the

Proposition. *The map* k *induces an isomorphism of* $\overline{H}_{n-r}(X, \mathfrak{S}(A))$ *onto* $\mathfrak{H}^r(X, A)$; *furthermore, by this isomorphism the intersection of homology classes corresponds to the product of elements in* $\mathfrak{H}(X, A)$; i. e.,

$$k(\bar{c}_{n-r} \circ \bar{c}_{n-s}) = k(\bar{c}_{n-r}) \vee k(\bar{c}_{n-s}),$$

where $\bar{c}_p$ *denotes the homology class of* c_p.

6. Use of a Riemann metric

In this section we will suppose that X is a closed smooth (or analytic) manifold with smooth (or analytic) Riemann structure. Using the metric properties of X we will construct (A, r)-pairs (θ, ω) whose singularities are the supports of a given $(n-r)$-chain and its boundary; ω *(and therefore* θ*) are currents on* X. This implies that ω and θ have singularities of a special "integrable" type; it follows easily that *in Theorem* 4*A we can replace* $\mathfrak{C}(X, A)$ *by a cochain complex based on pairs* (θ, ω) *which are currents on* X. The construction is merely a reformulation of the development of DE RHAM's Intersection Formula [8, p. 75]; we assume familiarity with the notations and results of that paper. For further properties of harmonic forms with singularities, we refer to KODAIRA [7, Chapter 4].

(A) For each $r\ (0 \leq r \leq n)$ let $g_r(x, y)$ be GREEN's form on X; then $g_r(x, y)$ is a symmetric double form $(x \neq y)$ satisfying

$$d_x g_r(x, y) = \delta_y g_{r+1}(x, y), \tag{1}$$

where δ_y denotes the codifferential taken with respect to y. Recall that the adjoint of a form on X is twisted (is of odd kind, in DE RHAM's terminology) if X is non-orientable. Given any chain $c_{n-r} \in S_{n-r}(X, \mathfrak{G}(A))$ we associate a smooth (or analytic) r-form $G(c_{n-r}) = \alpha$ on $X - |c_{n-r}|$ by the formula

$$G(c_{n-r}) = \alpha(x) = \int_{c_{n-r}(y)} g_r(x, y^*), \tag{2}$$

the integration taken with respect to y (of a twisted form on a twisted chain if X is non-orientable); then α is a current on X. It follows easily from (1) and the properties of the adjoint operator that $d_x g_r(x, y^*) = (-1)^{r+1} d_y g_{r+1}(x, y^*)$, whence α is closed (derived from a form of type (2)) if c_{n-r} is a cycle (boundary). Set $\omega(x) = -\delta_x \alpha(x)$; because of the identity [8, p. 73]

$$d_x \delta_x g_r(x, y^*) + d_y \delta_y g_r(x, y^*) = -h_r(x, y^*) \tag{3}$$

for a suitable harmonic double form h_r on X (in fact, $h_r(x, y)$ is the kernel of the harmonic projection operator), we have

$$d_x \omega(x) = \int_{\partial c_{n-r}(y)} \delta_y g_r(x, y^*) + \int_{c_{n-r}(y)} h_r(x, y^*) \tag{4}$$

on $X - |c_{n-r}|$; the right member of (4) is actually an r-form defined on $X - |\partial c_{n-r}|$, which we denote by θ.

Given $c_r \in S_r(X, A)$ admissible for the pair (θ, ω), we have $R[(\theta, \omega), c_r] =$

$$\int_{c_r(x)} \int_{\partial c_{n-r}(y)} \delta_y g_r(x, y^*) + \int_{c_r(x)} \int_{c_{n-r}(y)} h_r(x, y^*) + \int_{\partial c_r(x)} \int_{c_{n-r}(y)} \delta_x g_r(x, y^*)$$
$$= \int_{c_{n-r}(y)} \int_{c_r(x)} d_x \delta_x g_r(x, y^*) - \int_{c_r(x)} \int_{c_{n-r}(y)} d_x \delta_x g_r(x, y^*),$$

the last identity being DE RHAM's integral expression for the algebraic number $c_r \circ c_{n-r}$ of intersections of c_r and c_{n-r}. Thus we obtain the

Proposition. *Given any chain* $c_{n-r} \in S_{n-r}(X, \mathfrak{D}(A))$ *there is an* (A, r)-*pair* (θ, ω) *on* X *with* $e(\omega) = |c_{n-r}|$, $e(\theta) = |\partial c_{n-r}|$, *and*

$$R[(\theta, \omega), c_r] = c_r \circ c_{n-r}$$

for any admissible chain $c_r \in S_r(X, A)$. *If* c_{n-r} *is a cycle (boundary), then* (θ, ω) *determines a closed (derived) pair, such that* θ *is harmonic on* X *(is zero).*

If in (4) we make the substitution $\delta_y g_r(x, y^*) = (-1)^{r+1} \delta_x g_{r+1}(x, y^*)$ we find that the defining expressions for both currents ω and θ are given in terms of their HODGE decomposition; see [8, p. 65]. In particular we note that although (θ, ω) is a pair, the current differential of ω is not generally equal to θ (considered as a current), for $\theta = 0$ if derived from a current.

(B) Let G: $S_{n-r}(X, \mathfrak{D}(A)) \to \mathfrak{C}^r(X, A)$ be the map constructed in Proposition 6A. If (θ, ω) is a pair defined by an $(n - r)$-cycle c, then the harmonic r-form θ (but not ω) is unique in its cohomology and equivalence classes. For if (θ', ω') is a second such pair which is cohomologous to (θ, ω), then θ' and θ have the same periods on a base of integral r-cycles of X, whence by HODGE's Theorem we have $\theta' = \theta$. In combination with the POINCARÉ Duality Theorem we obtain the

Theorem. *Let* X *be a compact smooth (or analytic)* RIEMANN *manifold, and let* A *be an integral subdomain of* $\mathbf{R}$. *Then every cohomology class of* $\mathfrak{H}^r(X, A)$ *can be represented by an* (A, r)-*pair* (θ, ω) *such that* θ *is harmonic on* X; *furthermore,* θ *is unique in its cohomology class.*

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(Received May 10, 1957.)