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Cohomology Operations derived from Cyclic Groups*)

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§1. Introduction

In a previous paper [2], STEENROD defined a family of cohomology operations, called reduced powers, each being associated with some permutation group. It was also shown that these operations have a basis, in the sense of composition, consisting of, firstly, four primitive types of operations (which are: addition, cup-product, homomorphisms induced by coefficient homomorphisms, and BOCKSTEIN-WHITNEY coboundary operators) and, secondly, those reduced powers associated with cyclic permutation groups having degree p and order p where p ranges over primes.

In this paper, we shall improve the result by showing that there is a smaller basis consisting of the same primitive operations and only particular operations associated with cyclic groups: namely, for each prime p the cyclic reduced powers

$$\mathcal{P}^i : H^q(K;Z_p) \to H^{q+2i(p-1)}(K;Z_p) \ , \qquad i = 0,1,\dots \ ,$$

and the PONTRJAGIN pth powers

$$\mathfrak{P}_p : H^{2q}(K;Z_{p^k}) \to H^{2pq}(K;Z_{p^{k+1}}) \ .$$

The latter were defined for $p = 2$ by PONTRJAGIN [1], and generalized for $p > 2$ by THOMAS [5]. When $p = 2$, $\mathcal{P}^i$ is usually written Sq^{2i}.

Throughout the paper an *elementary* cohomology operation will mean one which is a composition of operations of the four primitive types.

§2. Recapitulation

Let p be a prime, and π the cyclic permutation group of order p and degree p. The reduced power operations based on π are obtained as elements of cohomology groups

$$H^r(W \otimes_\pi M^p \otimes G) \ . \tag{2.1}$$

In this expression W denotes a π-free acyclic chain complex, G is any coefficient group, and M is a cochain complex having two free generators u,v of dimensions $q, q+1$, respectively, and the coboundary relation

$$\delta u = \theta v \tag{2.2}$$

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where θ is an integer >1. Finally, π acts on $M^p = M \otimes \cdots \otimes M$ (p factors) by cyclic permutations of the factors. Then an element ξ of the group 2.1 determines a cohomology operation

$$\xi : H^q(K;Z_\theta) \to H^r(K;G) \tag{2.3}$$

defined for all complexes K.

We refer to Z_θ, G as the *initial* and *terminal* coefficient groups respectively. In a paper by STEENROD [3], it is shown that a basis for cohomology operations is provided by the four primitive types and those operations whose initial and terminal coefficient groups are cyclic of infinite or prime power order. Thus we have only to consider the cases $\theta = 0$, $\theta =$ a prime power, and $G = Z_m$ where $m = 0$ or $m =$ a prime power. In case $\theta = 0$, the cochain complex M is simplified by setting $v = 0$; then M^p is a cyclic group generated by the cocycle u^p.

Since the group 2.1 is independent of the choice of W, we shall choose the simplest known π-free acyclic complex W. Let T denote the generator of π which moves each factor of M^p one step to the right and moves the last factor to the first position. In the group ring $Z(\pi)$ set

$$\Delta = T - 1 \quad \text{and} \quad \Sigma = \sum_{j=0}^{p-1} T^j . \tag{2.4}$$

The group of r-chains ($r = 0, 1, \ldots$) of W is the π-free module having one generator e_r (i. e. as a complex W has a single r-cell and its distinct transforms in each dimension r). The boundary operator in W is defined by

$$\partial e_{2i+1} = \Delta e_{2i} , \quad \partial e_{2i+2} = \Sigma e_{2i+1} , \quad i = 0, 1, \ldots . \tag{2.5}$$

Since each of Δ, Σ generates the annihilator of the other in $Z(\pi)$, it follows that W is acyclic.

We have therefore, the problem of computing the cohomology of the specific cochain complex $W \otimes_\pi M^p$. Speaking roughly our method consists in reducing the complex to normal form and reading off the results. We must distinguish special cases depending on the integers p, q, θ, m, r, principally, $p = 2$ and $p > 2$, q odd and q even, $\theta = 0$, $\theta = p^k$ and θ prime to p. The case $\theta = 0$ will be obtained as a subcase of $\theta = p^k$ by the device of setting $v = 0$.

§ 3. The case θ prime to p

In case θ is prime to p, we will show that each $\xi \in H^r(W \otimes_\pi M^p \otimes G)$ gives an elementary operation. In all subsequent sections it will be assumed that θ is a power of p.

Let σ be the subgroup of π consisting of the unit element. Then W is σ-free, and

$$W \otimes_\sigma M^p = W \otimes M^p \xrightarrow{g} W \otimes_\pi M^p$$

where g is the natural factorization. In [2; 10.4] it is shown that

3.1. *Each element of* $H^r(W \otimes M^p \otimes G)$ *defines an elementary operation.*

If we now apply the result [2; 3.4], we have

3.2. *Each element of the image of*

$$g^*: \ H^r(W \otimes M^p \otimes G) \to H^r(W \otimes_\pi M^p \otimes G)$$

defines an elementary operation.

As a corollary we have

3.3. *Each cocycle of* $W \otimes_\pi M^p \otimes G$ *of the form* $e_0 \otimes_\pi w$, *where* w *is a cocycle of* $M^p \otimes G$, *defines an elementary operation.*

Now let τ denote the *transfer* chain transformation defined relative to the subgroup σ of π (see [2; 11.1]

$$\tau: \ W \otimes_\pi M^p \to W \otimes M^p \ .$$

Since p is the index of σ in π, [2; 11.2] gives $g\tau = p$ where p means multiplication by p. Passing to cohomology with coefficients in G, it follows that $g^*\tau^* = p$; and therefore

3.4. *Each element of* $H^r(W \otimes_\pi M^p \otimes G)$ *which is divisible by* p *defines an elementary operation.*

Assume now that θ is prime to p. Then there are integers α, β such that

$$\alpha\theta + \beta p = 1 \ .$$

Consider now the cochain mappings $M \to M$ which are multiplications by $\alpha\theta$, βp and 1. We construct a cochain homotopy D of βp into 1 by setting

$$Du = 0 \ , \quad Dv = \alpha u$$

The relation $\delta D + D\delta = \alpha\theta = 1 - \beta p$ follows directly. By [2; 5.2], we have that

$$1 \otimes_\pi (\beta p)^p: \ W \otimes_\pi M^p \to W \otimes_\pi M^p$$

is cochain homotopic to the identity. If we tensor with G and pass to cohomology, it follows that each element of $H^r(W \otimes_\pi M^p \otimes G)$ is divisible by $(\beta p)^p$; and so by 3.4 it defines an elementary operation.

§ 4. The case $p = 2$ and q even

We shall give a normal form for the complex $W \otimes_\pi M^2$, i. e. we express it as a direct sum of *elementary* subcomplexes each with two generators, say x and y, and a coboundary relation of the form $\delta x = ky$.

In M^2 we shall abbreviate $u \otimes u$ by u^2, $u \otimes v$ by uv, etc. Then M^2 has the four generators u^2, uv, vu and v^2. Since $(Te_j) \otimes_\pi w = e_j \otimes_\pi T^{-1} w$, it follows that $W \otimes_\pi M^2$ has the generators $e_j \otimes_\pi u^2$, $e_j \otimes_\pi uv$, $e_j \otimes_\pi vu$ and $e_j \otimes_\pi v^2$ for all $j \geqq 0$. Recall that the definition of the grading of a tensor product of a chain and a cochain complex [2; 2.2] gives $\dim(e_j \otimes_\pi w) = \dim w - j$. Hence, in the highest non-zero dimension $2(q+1)$, there is just one generator, and we set

$$\alpha_0 = e_0 \otimes_\pi v^2 \ . \tag{4.1}$$

In the dimension $2q+1$ there are three generators, and we define a unimodular transformation to a new basis a_0, β_0, γ_0 by the matrix

$$\begin{array}{c|ccc} & e_0 \otimes_\pi uv & e_0 \otimes_\pi vu & e_1 \otimes_\pi v^2 \\ \hline \beta_0 & 1 & 0 & -\frac{1}{2}\theta \\ \gamma_0 & -1 & 1 & -\theta \\ a_0 & 0 & 0 & 1 \end{array} \tag{4.2}$$

By virtue of § 3, we are working under the assumption that θ is a power of $p = 2$, so $-\frac{1}{2}\theta$ is an integer. The determinant is 1, hence a_0, β_0, γ_0 form a basis in this dimension.

In the dimension $2q$, we have four generators, and we define a new basis $b_0, c_0, \alpha_1, \delta_1$ by the unimodular transformation

$$\begin{array}{c|cccc} & e_0 \otimes_\pi u^2 & e_1 \otimes_\pi uv & e_1 \otimes_\pi vu & e_2 \otimes_\pi v^2 \\ \hline b_0 & 1 & \theta & 0 & 0 \\ c_0 & 0 & 1 & 0 & 0 \\ \delta_1 & 0 & 1 & 1 & 0 \\ \alpha_1 & 0 & 0 & 0 & 1 \end{array} \tag{4.3}$$

In the dimensions $2q - 2i$ for $i \geqq 1$, we define new generators b_i, c_i, δ_{i+1}, α_{i+1} by the unimodular matrix

	$e_{2i}\otimes_\pi u^2$	$e_{2i+1}\otimes_\pi uv$	$e_{2i+1}\otimes_\pi vu$	$e_{2i+2}\otimes_\pi v^2$
b_i	1	0	0	0
c_i	0	1	0	0
δ_{i+1}	0	1	1	0
α_{i+1}	0	0	0	1

(4.4)

Finally, in dimensions $2q-2i+1$, for $i\geqq 1$, we define new generators $\beta_i, \gamma_i, d_i, a_i$ by

	$e_{2i-1}\otimes_\pi u^2$	$e_{2i}\otimes_\pi uv$	$e_{2i}\otimes_\pi vu$	$e_{2i+1}\otimes_\pi v^2$
β_i	1	$\frac{1}{2}\theta$	$\frac{1}{2}\theta$	0
d_i	0	1	0	$\frac{1}{2}\theta$
γ_i	0	-1	1	$-\theta$
a_i	0	0	0	1

(4.5)

With this new basis the coboundary in $W\otimes_\pi M^2$ takes the normal form

$$\delta a_i = -2\alpha_i \,, \qquad i\geqq 0 \,, \tag{4.6}$$

$$\delta b_0 = 2\theta\beta_0 \,, \tag{4.7}$$

$$\delta b_i = 2\beta_i \,, \qquad i\geqq 1 \,, \tag{4.8}$$

$$\delta c_i = \gamma_i \,, \qquad i\geqq 0 \,, \tag{4.9}$$

$$\delta d_i = \delta_i \,, \qquad i\geqq 1 \,. \tag{4.10}$$

As an example, we compute 4.7 in detail.

$$\begin{aligned}\delta(e_0\otimes_\pi u^2) &= (\partial e_0)\otimes_\pi u^2 + e_0\otimes_\pi \delta(u^2)\\ &= \theta e_0\otimes_\pi (vu+uv)\end{aligned}$$

since q is even.

$$\begin{aligned}\delta(e_1\otimes_\pi uv) &= (\partial e_1)\otimes_\pi uv - e_1\otimes_\pi \delta(uv)\\ &= (\Delta e_0)\otimes_\pi uv - \theta e_1\otimes_\pi v^2 \,.\end{aligned}$$

Now use $\Delta = T-1$, and $(Te_0)\otimes_\pi uv = e_0\otimes_\pi T^{-1}uv = e_0\otimes_\pi vu$. Then

$$(\Delta e_0)\otimes_\pi uv = e_0\otimes_\pi (vu-uv) \,.$$

Therefore

$$\begin{aligned}\delta b_0 &= \delta(e_0\otimes_\pi u^2 + \theta e_1\otimes_\pi uv)\\ &= \theta e_0\otimes_\pi(vu+uv) + \theta e_0\otimes_\pi(vu-uv) - \theta^2 e_1\otimes_\pi v^2\\ &= 2\theta e_0\otimes_\pi uv - \theta^2 e_1\otimes_\pi v^2 = 2\theta\beta_0 \,.\end{aligned}$$

The computations of the other coboundaries are similar. It should be pointed out that the assumption that q is even is used in obtaining the relations

$$Tu^2 = u^2 \ , \quad Tv^2 = -v^2 \ , \quad \delta(u^2) = \theta(vu + uv) \ .$$

Using this normal form, we can now read off the cohomology of $W \otimes_\pi M^2$; it is the direct sum of the cohomologies of the subcomplexes 4.6 to 4.10. Obviously 4.9 and 4.10 are acyclic complexes, and their cohomologies are zero. Before treating the other three, we will prove a lemma which greatly reduces the task of showing that the cohomology operations corresponding to the various cocycles are compositions of the operations specified in § 1. In particular the lemma eliminates the need of considering various terminal coefficient groups Z_m.

4.11. Lemma. *Let N be an elementary subcomplex of $W \otimes_\pi M^p$ generated by x, y with $\delta x = ky$. Then the cohomology operation which corresponds to any cocycle of $N \otimes G$ is a composition of elementary operations and the operation corresponding to the cocycle x mod k.*

Let us recall the way in which a cocycle of $W \otimes_\pi M^p$ corresponds to a cohomology operation on an element $\bar{u} \in H^q(K; Z_\theta)$ (see [2; § 2]). A mapping $\psi: M \to K^*$ representing $\bar{u}$ is chosen (i. e. ψu is a cocycle of the class $\bar{u}$). Then ψ determines a mapping

$$\varphi\psi: \ W \otimes_\pi M^p \to K^* \ ,$$

which induces a cohomology homomorphism

$$\Phi: \ H^r(W \otimes_\pi M^p \otimes G) \to H^r(K; G) \ .$$

If $\xi \in H^r(W \otimes_\pi M^p \otimes G)$, then $\xi(\bar{u})$ is defined to be $\Phi(\xi)$. Now let

$$\xi \in H^r(N \otimes Z_k)$$

be the class of the cocycle x mod k. Then $\varphi\psi(x)$ is a cocycle of the class $\xi(\bar{u}) \in H^r(K; Z_k)$. Therefore $\varphi\psi | N: \ N \to K^*$ represents $\xi(\bar{u})$. Apply now [2; Lemma 10.1] which asserts that the image $H^s(N \otimes G) \to H^s(K; G)$ is generated by $\xi(\bar{u})$ and elementary operations. This proves the lemma.

By virtue of the lemma, it suffices to identify the cohomology operations corresponding to the cocycle b_0 mod 2θ, the cocycles b_i mod 2 for $i \geqq 1$, and the cocycles a_i mod 2 for $i \geqq 0$. Now

$$b_0 = e_0 \otimes_\pi u^2 + \theta e_1 \otimes_\pi uv$$

corresponds by definition to the PONTRJAGIN squaring operation $\mathfrak{P}_2$ (see [5; 3.4]). And

$$b_i = e_{2i} \otimes_\pi u^2$$

corresponds by definition to the cyclic reduced square $\mathrm{Sq}_{2i} = \mathrm{Sq}^{q-2i}$ (see [2; p. 6] and [4; § 4]). If it is felt desirable that Sq_{2i} should operate only on cocycles mod 2, then the operation represented by b_i can be written as $\mathrm{Sq}_{2i}\eta_*$ where $\eta: Z_\theta \to Z_2$ is reduction mod 2. Note that Sq^{q-2i} has an even superscript since q is even.

It remains to identify the operation corresponding to the cocycle a_i mod 2. Let δ^* denote the BOCKSTEIN coboundary operator corresponding to the exact sequence $0 \to Z \xrightarrow{\theta} Z \to Z_\theta \to 0$. If $\bar{u} \in H^q(K; Z_\theta)$ and $\psi: M \to K^*$ represents $\bar{u}$, then $\psi(v)$ represents $\delta^*\bar{u}$; and finally $\varphi\psi(e_{2i+1} \otimes_\pi v^2)$ represents

$$\mathrm{Sq}_{2i+1}\delta^*\bar{u} = \mathrm{Sq}^{q-2i}\delta^*\bar{u} \ .$$

Therefore a_i corresponds to the cohomology operation $\mathrm{Sq}^{q-2i}\delta^*$. Note again that the upper index is even.

In the special case $\theta = 0$, we set $v = 0$, and then $W \otimes_\pi M^2$ is in the normal form

$$\begin{aligned} \delta(e_0 \otimes_\pi u^2) &= 0 \\ \delta(e_{2i} \otimes_\pi u^2) &= 2e_{2i-1} \otimes_\pi u^2 \qquad i \geqq 1 \ . \end{aligned}$$

The cocycle $e_0 \otimes_\pi u^2$ corresponds to the operation of squaring in the sense of the cup product (see 3.3 and [2; 10.2]). As before $e_{2i} \otimes_\pi u^2$ mod 2 corresponds to Sq^{q-2i}.

§ 5. The case $p = 2$ and q odd

The change in the parity of q affects both the coboundary operator and the action of T in M^2. For the latter we have

$$Tu^2 = -u^2 \ , \quad Tuv = vu \ , \quad Tvu = uv \ , \quad Tv^2 = v^2 \ .$$

Starting with the same generators of $W \otimes_\pi M^2$ as in § 4, we reduce to normal form as follows. In the highest dimension $2q + 2$, we have one generator, and we set

$$\gamma_0 = e_0 \otimes_\pi v^2 \ .$$

In the dimension $2q + 1$, we define a new basis by the unimodular transformation

	$e_0 \otimes_\pi uv$	$e_0 \otimes_\pi vu$	$e_1 \otimes_\pi v^2$
c_0	1	0	0
δ_0	-1	1	0
α_0	0	0	1

In all dimensions $2q - 2i$ for $i \geqq 0$, we define a new basis by the unimodular transformation

	$e_{2i} \otimes_\pi u^2$	$e_{2i+1} \otimes_\pi uv$	$e_{2i+1} \otimes_\pi vu$	$e_{2i+2} \otimes_\pi v^2$
β_{i+1}	1	$-\frac{1}{2}\theta$	$\frac{1}{2}\theta$	0
d_i	0	1	0	$\frac{1}{2}\theta$
γ_{i+1}	0	1	1	θ
a_i	0	0	0	1

In all dimension $2q - 2i + 1$ for $i \geqq 1$, we define a new basis by the unimodular transformation

	$e_{2i-1} \otimes_\pi u^2$	$e_{2i} \otimes_\pi uv$	$e_{2i} \otimes_\pi vu$	$e_{2i+1} \otimes_\pi v^2$
b_i	1	0	0	0
c_i	0	1	0	0
δ_i	0	-1	1	0
α_i	0	0	0	1

In the terms of the new basis, the coboundary relations become

$$
\begin{aligned}
\delta c_0 &= \theta\gamma_0 & & \\
\delta b_i &= -2\beta_i & & i \geqq 1 , \\
\delta a_i &= 2\alpha_i & & i \geqq 0 , \\
\delta d_i &= \delta_i & & i \geqq 0 , \\
\delta c_i &= \gamma_i & & i \geqq 1 .
\end{aligned}
$$

The last two subcomplexes have zero cohomology. By 4.11, we need only identify the cohomology operations corresponding to the cocycles $c_0 \bmod \theta$, $a_i \bmod 2$, and $b_i \bmod 2$. Now $c_0 = e_0 \otimes_\pi uv$ defines an elementary operation by 3.3 (it is in fact the operation $\bar{u} \to \bar{u} \smile \delta^* \bar{u}$ where δ^* is the obvious BOCKSTEIN).

By definition, $b_i = e_{2i-1} \otimes_\pi u^2$ is the cyclic reduced square

$$\mathrm{Sq}_{2i-1} = \mathrm{Sq}^{q-2i+1} .$$

Since q is odd, $q - 2i + 1$ is even. Again the operation may be written $\mathrm{Sq}^{q-2i+1}\eta_*$ where η is reduction mod 2.

The cocycle $a_i = e_{2i+2} \otimes_\pi v^2 \bmod 2$ can be identified with

$$\mathrm{Sq}_{2i+2}\, \delta^* = \mathrm{Sq}^{q+1-(2i+2)}\, \delta^*$$

exactly as the cocycle a_i of § 4. Again the upper index of the square is even.

In the special case $\theta = 0$, we set $v = 0$ and then $W \otimes_\pi M^2$ is in the normal form

$$\delta(e_{2i+1} \otimes_\pi u^2) = -\, 2 e_{2i} \otimes_\pi u^2 \ , \qquad i \geqq 0 \ .$$

As above the cocycle $e_{2i+1} \otimes_\pi u^2 \bmod 2$ corresponds to the operation Sq^{q-2i-1}.

We may summarize our results in the case $p = 2$ as follows. The only operations needed in addition to the elementary operations are the squares Sq^{2i} $(i > 0)$ when q is odd or $\theta = 0$. When q is even and $\theta = 2^k$, the PONTRJAGIN square is also needed.

§ 6. The automorphism g_* of $H(W \otimes_\pi M^p)$

We assume henceforth that p is an odd prime. Our analysis must take account now of a phenomenon not present when $p{=}2$, namely: π is a *proper* subgroup of the symmetric group $\mathcal{S}_p$ of degree p. If $\pi \subset \varrho \subset \mathcal{S}_p$, and U is a ϱ-free acyclic complex, then the inclusion $\pi \subset \varrho$ induces a homomorphism

$$h_* :\ H^r(W \otimes_\pi M^p \otimes G) \to H^r(U \otimes_\varrho M^p \otimes G) \ ; \qquad (6.1)$$

and, for any element ξ on the left, the cohomology operations corresponding to ξ and $h_*(\xi)$ coincide (see [2 ; 3.4]). In particular, if ξ is in the kernel of h_*, the corresponding cohomology operation is zero. *The aim of this section is to show that certain explicit elements belong to* $\ker h_*$ *when* ϱ *is the normalizor of* π *in* $\mathcal{S}_p$. It is a fact that these elements generate the kernel even for $\varrho = \mathcal{S}_p$; but we omit the proof of this since the proof is complicated and the fact is not needed.

We need at this point of the discussion a special case of a rather general proposition. Because it is just as easy and less confusing to present the latter, we shall do so. We shall consider objects (ϱ, A) where ϱ is a group, A is a cochain complex, and ϱ operates as automorphisms of A. By a *mapping* f of (ϱ, A) into another such (σ, B), we mean a homomorphism $\varrho \to \sigma$ and a cochain mapping $A \to B$, both denoted by f, such that

$$f(xa) = f(x) f(a) \quad \text{for} \quad x \in \varrho,\ a \in A. \qquad (6.2)$$

These objects and mappings form a category. A pair (ϱ, G) where G is a ϱ-module may be regarded as an object of the category by treating G as a cochain complex having just one non-zero cochain group in the dimension zero. Now the ordinary homology theory of groups, developed for this subcategory of pairs (ϱ, G), can be extended to the entire category in a fairly obvious way. We shall review this extension briefly.

Let U be a ϱ-free acyclic chain complex. Defining the cochain complex $U \otimes_\varrho A$ as in [2; 2.2], we proceed to show that $H^r(U \otimes_\varrho A)$ is independent of the choice of U. If $f: (\varrho, A) \to (\sigma, B)$ is a mapping, and V is a σ-free acyclic chain complex, let ϱ operate on V through $f: \varrho \to \sigma$. Then the fundamental lemma (see [2; 2.7]) gives a chain mapping $f_\#: U \to V$ satisfying the equivariance condition

$$f_\#(xc) = f(x) f_\#(c) , \quad x \epsilon \varrho, \; c \epsilon U. \tag{6.3}$$

It follows that $f_\# \otimes f: U \otimes A \to V \otimes B$ induces a chain mapping

$$f^\#: U \otimes_\varrho A \to V \otimes_\sigma B , \tag{6.4}$$

and thereby induces homomorphisms of cohomology

$$f_*: H^r(U \otimes_\varrho A) \to H^r(V \otimes_\sigma B) . \tag{6.5}$$

The second part of the fundamental lemma asserts that any two equivariant chain maps $f_\#, f'_\#$ of U into V are connected by an equivariant chain homotopy D. Then $D \otimes_\varrho f$ gives a cochain homotopy of $f^\#$ into $f^{\#\prime}$. Therefore f_* is independent of the choice of $f_\#$.

An obvious property of f_* is

$$f = \text{identity map of } (\varrho, A) \text{ implies } f_* = \text{identity.} \tag{6.6}$$

For $f_\#$ can be taken as the identity.

Let $f: (\varrho, A) \to (\sigma, B)$ and $g: (\sigma, B) \to (\tau, C)$ be mappings. Then

$$(gf)_* = g_* f_* . \tag{6.7}$$

For, having chosen $g_\#$ and $f_\#$, we may choose $(gf)_\#$ to be the composition $g_\# f_\#$.

Now let U, V be two ϱ-free acyclic complexes. Corresponding to the identity map f of (ϱ, A), we obtain two induced homomorphisms

$$H^r(U \otimes_\varrho A) \to H^r(V \otimes_\varrho A) \to H^r(U \otimes_\varrho A) ,$$

whose compositions in either order again correspond to f by the property 6.7. Then 6 6 asserts that both compositions give identity maps. Therefore the various choices of the ϱ-free acyclic complex U give a family of cohomology groups, any two connected by an isomorphism, and the family of these isomorphisms is transitive by virtue of 6.7. As is customary in such a case, we identify this family of groups with a single group. To emphasize its analogy with the ordinary homology group of a group, we shall call it the rth homology group of ϱ with coefficients in A, thus:

$$H_r(\varrho\,;A) = H^r(U\otimes_\varrho A)\ . \tag{6.8}$$

If $f\colon\ (\varrho,A)\to(\sigma,B)$, then 6.5 becomes

$$f_*\colon\ H_r(\varrho\,;A)\to H_r(\sigma\,;B)\ . \tag{6.9}$$

It is clear that 6.6 and 6.7 continue to hold for the induced homomorphisms taken in this more general sense.

A mapping $f\colon\ (\varrho,A)\to(\varrho,A)$ is called an *automorphism* if both mappings $\varrho\to\varrho$ and $A\to A$ are automorphisms. Then f has an inverse mapping, and we may apply 6.7 and 6.6 to conclude that f_* is an automorphism of $H_r(\varrho\,;A)$.

The *inner* automorphism f corresponding to an element $y\in\varrho$ is defined by

$$f(x) = yxy^{-1}\ ,\quad f(a) = ya\ ,\quad x\in\varrho,\quad a\in A. \tag{6.10}$$

Then we have

$$f = \text{an inner automorphism implies } f_* = \text{identity.} \tag{6.11}$$

To see this, let $f_\#$ be the chain mapping $U\to U$ defined by $f_\#(c) = yc$. Since

$$f_\#(xc) = yxc = yxy^{-1}yc = f(x)f_\#(c)\ ,$$

the equivariance condition 6.3 is fulfilled. Then

$$(f_\#\otimes f)(c\otimes a) = f_\# c\otimes fa = yc\otimes ya = y(c\otimes a)\ .$$

This implies that the induced mapping f^* of $U\otimes_\varrho A$ into itself is the identity; and so $f_* =$ identity.

This completes the discussion of the general theory, and we return now to the special case with which we began this section. In applying the above results, we take

$$A = M^p\otimes G\ ;$$

in this case π, its normalizor ϱ, and the symmetric group $\mathcal{S}_p$ operate in A by permuting the factors of M^p and acting as the identity in G. Let y be any element of ϱ, let f be the corresponding inner automorphism given by 6.10, and let g be the automorphism of (π,A) obtained by restricting f. Let h: $(\pi,A)\to(\varrho,A)$ be the inclusion $\pi\subset\varrho$ and the identity on A. Obviously

$$hg = fh\ . \tag{6.12}$$

If we pass to the induced homomorphisms and apply 6.7 and 6.11, we obtain

$$h_*g_* = f_*h_* = h_*\ . \tag{6.13}$$

Thus we have proved

6.14 Lemma. *If g_* is the automorphism of $H^r(W \otimes_\pi M^p \otimes G)$ determined by any element y of the normalizor ϱ of π, then $g_* \xi - \xi$ belongs to the kernel of h_* where h is the inclusion $\pi \subset \varrho$ and $\xi \in H^r(W \otimes_\pi M^p \otimes G)$. Thus, as remarked after* 6.1, *the cohomology operation corresponding to $g_* \xi - \xi$ is zero.*

In order to use the lemma effectively in computations, we shall choose an explicit y and a corresponding chain mapping $g_\#$. Let the factors of M^p be numbered $0, 1, \ldots, p-1$ so that the generator T of π can be described as the transformation $T(i) \equiv i + 1 \bmod p$ in terms of integers. Let k be a primitive root of the prime p (i. e. $k^j \equiv 1 \bmod p$ implies that j is a multiple of $p-1$). Let y be the permutation of $0, 1, \ldots, p-1$ defined by

$$y(i) \equiv ki \bmod p \ . \tag{6.15}$$

Then $y^{-1}(i) \equiv k^{p-2} i \bmod p$, and this gives

$$yTy^{-1} = T^k \ . \tag{6.16}$$

Thus y belongs to the normalizor ϱ of π. (Since the order of y is $p-1$, it is a generator of ϱ/π.) If we arrange the integers 0 to $p-1$ in the order

$$0, 1, k, k^2, \ldots, k^{p-2} \quad \bmod p \ ,$$

it is seen that y leaves 0 fixed and permutes the remaining $p-1$ elements cyclically. This shows that y is an *odd* permutation because $p-1$ is even. Therefore

$$yu^p = (-1)^q u^p \ , \quad yv^p = (-1)^{q+1} v^p \ . \tag{6.17}$$

Letting W be as in § 2, we define a chain mapping $g_\# : W \to W$ by specifying first its values on the π-basis $\{e_j\}$:

$$g_\# e_{2i} = k^i e_{2i} \ , \quad g_\# e_{2+1} = k^i \sum_{m=0}^{k-1} T^m e_{2i+1} \ , \quad i \geqq 0 \ . \tag{6.18}$$

Then, for each $s = 1, \ldots, p-1$, we set

$$g_\# T^s e_j = T^{ks} g_\# e_j \ , \quad j \geqq 0 \ . \tag{6.19}$$

From this it follows that $g_\#$ satisfies the equivariance condition 6.3 for the automorphism 6.16 of π. It is now an easy matter to verify $g_\# \partial = \partial g_\#$. Then the resulting chain transformation $g^\#$ of $W \otimes_\pi M^p$ (see 6.4) is defined by

$$g^\# (e \otimes_\pi c) = (g_\# e) \otimes_\pi yc \ . \tag{6.20}$$

These specific calculations will be needed in sections 8, 9, and 10.

§ 7. The decomposition: $W \otimes_\pi M^p = L_1 + L + L_2$

As a first step in analysing the structure of $W \otimes_\pi M^p$ (p odd), we shall decompose it into a direct sum of three cochain subcomplexes as indicated above. Of importance is the fact that each is transformed into itself by the $g^\#$ of 6.20.

A cochain of $W \otimes_\pi M^p$ is said to be in *canonical form* if it is written $\sum_j e_j \otimes_\pi c_j$ where c_j is a cochain of M^p. Since $Te \otimes_\pi c = e \otimes_\pi T^{-1} c$, each cochain has one and only one canonical form.

Recall that M^p has all cochain groups equal to zero save in the range pq to $p(q+1)$ inclusive, $C^{pq}(M^p)$ has one generator u^p, and $C^{p(q+1)}(M^p)$ has one generator v^p. In the dimensions $pq + j$ for $0 < j < p$, M^p is generated by products having $p - j$ factors u and j factors v. Therefore π operates freely in these dimensions.

Let us adopt the convention that the index j of the canonical cochain $e_i \otimes_\pi c_j$ signifies that c_j has dimension $pq + j$. Thus c_j is zero unless $0 \leqq j \leqq p$, c_0 is a multiple of u^p, and c_p is a multiple of v^p. In the highest non-zero dimension, a canonical cochain has a single term $e_0 \otimes_\pi c_p$. In dimensions $pq + j$ for $0 < j < p$, a canonical cochain has $p - j + 1$ terms

$$e_0 \otimes_\pi c_j + e_1 \otimes_\pi c_{j+1} + \cdots + e_{p-j} \otimes_\pi c_p \tag{7.1}$$

and in all dimensions $\leqq pq$, it has $p + 1$ terms

$$e_s \otimes_\pi c_0 + e_{s+1} \otimes_\pi c_1 + \cdots + e_{s+p} \otimes_\pi c_p \ . \tag{7.2}$$

We define L_1 to consist of all cochains having canonical forms of one of the two following types for some $i \geqq 0$:

$$e_{2i+2} \otimes_\pi c_0 \quad \text{or} \quad e_{2i+1} \otimes_\pi c_0 + e_{2i+2} \otimes_\pi \frac{1}{p} \delta c_0 \ . \tag{7.3}$$

The second type is described explicitly by requiring $c_j = 0$ for $j > 1$, and $c_1 = \dfrac{1}{p} \delta c_0$ (recall that any coboundary in M^p is divisible by θ, and θ is a power of p). Since $Tu^p = u^p$, it follows that $\Sigma c_0 = pc_0$; hence

$$\begin{aligned} \delta (e_{2i+2} \otimes_\pi c_0) &= \Sigma e_{2i+1} \otimes_\pi c_0 + e_{2i+2} \otimes_\pi \delta c_0 \\ &= e_{2i+1} \otimes_\pi \Sigma c_0 + e_{2i+2} \otimes_\pi \delta c_0 \\ &= p \left(e_{2i+1} \otimes_\pi c_0 + e_{2i+2} \otimes_\pi \frac{1}{p} \delta c_0 \right) . \end{aligned} \tag{7.4}$$

This shows that a cochain of the first type has a coboundary of the second type, and each cochain of the second type is a cocycle. Therefore L_1 is a cochain subcomplex. Clearly L_1 is generated by the cochains $e_{2i+2} \otimes_\pi u^p$ and

$\frac{1}{p}\delta(e_{2i+2}\otimes_\pi u^p)$ for all $i\geqq 0$, and is already in normal form with respect to these generators. Applying 6.20, 6.18 and 6.17, we obtain

$$\begin{aligned} g^{\#}(e_{2i+2}\otimes_\pi u^p) &= g_{\#}e_{2i+2}\otimes_\pi y u^p \\ &= (-1)^q k^{i+1} e_{2i+2}\otimes_\pi u^p \ . \end{aligned} \tag{7.5}$$

From this it follows that $g^{\#}$ transforms L_1 into itself.

Define L_2 to consist of all cochains having canonical forms of the type

$$e_s\otimes_\pi c_p \quad \text{for} \quad s>0 \ . \tag{7.6}$$

It is in normal form:

$$\delta(e_{2i}\otimes_\pi v^p) = p e_{2i-1}\otimes_\pi v^p \ , \qquad i\geqq 1 \ . \tag{7.7}$$

It is obvious from 6.18 and 6.20 that $g^{\#}$ maps L_2 into itself.

We shall describe L by imposing conditions on the initial and final terms of a cochain in canonical form as follows:

(7.8) If the initial term is $e_s\otimes_\pi c_0$ with $s>0$, see 7.2, we require that $c_0=0$.

(7.9) If the final term is $e_t\otimes_\pi c_p$ with t odd, we require that $c_p=0$.

(7.10) If the final term is $e_t\otimes_\pi c_p$ with t even and positive, we require that $c_p=\frac{1}{p}\delta c_{p-1}$.

Of course, all cochains $e_0\otimes_\pi c_p$ are in L. The condition 7.8 is obviously stable under δ. If t is odd and >2

$$\begin{aligned} \delta(e_{t-1}\otimes_\pi c_{p-1}) &= \Sigma e_{t-2}\otimes_\pi c_{p-1} + e_{t-1}\otimes_\pi \delta c_{p-1} \\ &= e_{t-2}\otimes_\pi \Sigma c_{p-1} + e_{t-1}\otimes_\pi \frac{1}{p}\delta\Sigma c_{p-1} \end{aligned}$$

because δc_{p-1} is a multiple of v^p. Therefore a cochain satisfying 7.9 has a coboundary satisfying 7.10. If t is even and >1

$$\begin{aligned} &\delta\left(e_{t-1}\otimes_\pi c_{p-1} + e_t\otimes_\pi \frac{1}{p}\,\delta c_{p-1}\right) \\ &= \Delta e_{t-2}\otimes_\pi c_{p-1} - e_{t-1}\otimes_\pi \delta c_{p-1} + \Sigma e_{t-1}\otimes_\pi \frac{1}{p}\,\delta c_{p-1} \\ &= e_{t-2}\otimes_\pi (T^{-1}-1)c_{p-1} + e_{t-1}\otimes_\pi\left(-\,\delta c_{p-1} + \Sigma\frac{1}{p}\,\delta c_{p-1}\right) . \end{aligned}$$

The last term is zero since δc_{p-1} is a multiple of v^p. Therefore a cochain satisfying 7.10 has a coboundary satisfying 7.9. This proves that L is a cochain subcomplex.

The conditions 7.8, 7.9 are obviously stable under the chain mapping g^*. As for 7.10, we have

$$g^*\left(e_{2i-1}\otimes_\pi c_{p-1} + e_{2i}\otimes_\pi \frac{1}{p}\,\delta c_{p-1}\right)$$
$$= (k^{i-1}\sum_{m=0}^{k-1} T^m e_{2i-1})\otimes_\pi y c_{p-1} + k^i e_{2i}\otimes_\pi \frac{1}{p}\, y\,\delta c_{p-1}$$
$$= e_{2i-1}\otimes_\pi\{k^{i-1}\sum_{m=0}^{k-1} T^{p-m} y c_{p-1}\} + e_{2i}\otimes_\pi \frac{1}{p}\, k^i y\,\delta c_{p-1}\ .$$

Since δc_{p-1} is a multiple of v^p, 6.17 gives

$$T^{p-m} y\,\delta c_{p-1} = (-1)^{q+1}\delta c_{p-1}$$

and therefore

$$\frac{1}{p}\,\delta\{k^{i-1}\sum_{m=0}^{k-1} T^{p-m} y c_{p-1}\} = \frac{1}{p}\, k^i y\,\delta c_{p-1}\ .$$

This shows that condition 7.10 is stable under g^*. Therefore g^* maps L into itself.

It remains to show that the entire complex is the direct sum of the three subcomplexes. That $L_1\cap L_2 = 0$ is clear by comparing 7.3 and 7.6, i. e. $c_p = 0$ for any element of L_1, and $c_p \neq 0$ for a non-zero element of L_2. A non-zero element of $L_1 + L_2$ has a non-zero c_0 if its component in L_1 is non-zero, or else it lies in L_2 and then $c_{p-1} = 0$ and $c_p \neq 0$. In the first case 7.8 does not hold, in the second neither 7.9 nor 7.10 could hold. Thus

$$L\cap(L_1 + L_2) = 0\ .$$

Given any cochain in normal form, if it has an initial term $e_s\otimes_\pi c_0$ with $s>0$ and $c_0\neq 0$, we may subtract from it an element of L_1 (the first or second element of 7.3 according as s is even or odd) and obtain a cochain satisfying 7.8. If the resulting cochain has a final term $e_t\otimes_\pi c_p$ with t odd and $c_p\neq 0$, we subtract $e_t\otimes_\pi c_p$ in L_2, and obtain a cochain satisfying both 7.8 and 7.9 which is therefore a cochain of L. On the other hand, if the final term $e_t\otimes_\pi c_p$ has t even and >0, we subtract $e_t\otimes_\pi\left(c_p - \frac{1}{p}\,\delta c_{p-1}\right)$ in L_2, and obtain a cochain satisfying both 7.8 and 7.10 which is therefore in L. This completes the proof of the direct sum decomposition.

§ 8. Cohomology operations obtained from L_1 and L_2

The subcomplex L_1 defined in 7.3 is in the normal form

$$\delta(e_{2i}\otimes_\pi u^p) = p\left(e_{2i-1}\otimes_\pi u^p + e_{2i}\otimes_\pi \frac{1}{p}\,\delta u^p\right)\ , \quad i>1\ . \tag{8.1}$$

According to 4.11, we have only to identify the cohomology operation corresponding to each of the cocycles $e_{2i}\otimes_\pi u^p \bmod p$. By 6.20, 18 and 17

$$g^{\#}(e_{2i}\otimes_\pi u^p) = g_{\#}e_{2i}\otimes_\pi y u^p = (-1)^q k^i e_{2i}\otimes_\pi u^p\ .$$

Therefore, by 6.4, the cocycle

$$(g^{\#}-1)(e_{2i}\otimes_\pi u^p) = [(-1)^q k^i - 1]e_{2i}\otimes_\pi u^p \tag{8.2}$$

belongs to the kernel of h_*, and therefore is zero as a cohomology operation.

If q is even and i is not a multiple of $p-1$, the coefficient $(-1)^q k^i - 1$ is non-zero $\bmod p$ because k is a primitive root. Working $\bmod p$, we may divide 8.2 by this coefficient, and conclude that $e_{2i}\otimes_\pi u^p$ represents zero as a cohomology operation.

Again let q be even, and suppose $i = s(p-1)$. In this case the coefficient in 8.2 is zero $\bmod p$, so it imposes no relation on the cohomology operation corresponding to $e_{2i}\otimes_\pi u^p$. The operation is in fact a suitable multiple of the cyclic reduced power $\mathcal{P}^{\frac{1}{2}q-s}$, namely:

$$\Phi\{e_{2s(p-1)}\otimes_\pi u^p\} = (-1)^s\,\mathcal{P}^{\frac{1}{2}q-s}\bar{u}\ . \tag{8.3}$$

In this formula Φ is as defined in [2; 2.11], and the braces $\{\}$ mean to take the cohomology class of the cocycle enclosed. To prove 8.3, we must recall the definition [4; 6.8] of $\mathcal{P}^t$ namely

$$\mathcal{P}^t\bar{u} = (-1)^{mt+mq(q-1)/2}(m\,!)^{2t-q}\,\bar{u}^p/e_{(q-2t)(p-1)}\ . \tag{8.4}$$

In this formula $m = \frac{1}{2}(p-1)$, and the coëfficient is computed in the field Z_p. Also, by [4; 2.8]

$$(\bar{u}^p/e)\cdot\sigma = \bar{u}^p\cdot\varphi'(e\otimes\sigma)$$

where $\varphi'\colon\ W\otimes K\to K^p$ has the same meaning as in [2; 2.6]. Comparing this with [2; 2.8], we obtain

$$\Phi\{e_{2i}\otimes_\pi u^p\} = (-1)^i\,\bar{u}^p/e_{2i}\ . \tag{8.5}$$

In 8.4, we take $q-2t = 2s$ (i. e. $t = \frac{1}{2}q - s$), in 8.5 we take $i = s(p-1)$, then we eliminate $\bar{u}^p/e$ between the two equations, and obtain 8.3. In the computation one must use properties of $m\,!$, namely, by WILSON's theorem,

$m!$ is non-zero mod p; if m is even, $(m!)^2 \equiv -1$; if m is odd, $(m!)^2 \equiv 1$; so in either case $(m!)^2 \equiv (-1)^{m+1}$.

Now let q be odd. The coefficient in 8.2 becomes $-(k^i+1)$. Since k is a primitive root of p, $k^i + 1 \equiv 0 \mod p$ if and only if i is an odd multiple of $\frac{1}{2}(p-1)$. If this is not the case, we may divide 8.2 by k^i+1, and conclude as before that $e_{2i} \otimes_\pi u^p$ represents zero as a cohomology operation.

Let q be odd, and suppose $i = (2s+1)\frac{1}{2}(p-1)$. In this case, the cocycle corresponds again to a suitable $\mathcal{P}^t$, namely

$$\Phi\{e_{(2s+1)(p-1)} \otimes_\pi u^p\} = (-1)^{s+m}(m!)\,\mathcal{P}^{\frac{1}{2}(q-1)-s}\bar{u} \; . \tag{8.6}$$

As in the case of 8.3, this is derived from 8.4 and 8.5 by setting

$$t = \tfrac{1}{2}(q-1) - s$$

in 8.4, $i = (2s+1)m$ in 8.5, and eliminating $\bar{u}^p/e$. This completes the analysis of the cohomology operations derived from L_1.

The subcomplex L_2 defined in 7.6 is already in normal form (see 7.7). Let M' be the subcomplex of M generated by v. Let $\psi\colon M \to K^*$ be a cochain map representing the class $\bar{u}$. Then $\psi\,|\,M' = \psi'\colon M' \to K^*$ represents the cohomology class $\bar{v} \in H^{q+1}(K; Z)$ containing the cocycle $\psi'(v)$. Then $\bar{v} = \delta^*\bar{u}$ where δ^* is the Bockstein coboundary for the coefficient sequence

$$0 \to Z \xrightarrow{\theta} Z \to Z_\theta \to 0$$

(see [2; 10.1]). Define L_1' in $W \otimes_\pi M'^p$ in the same manner as L_1 in $W \otimes_\pi M^p$, replacing u by v and q by $q+1$. It is seen that, under the inclusion mapping $W \otimes_\pi M'^p \subset W \otimes_\pi M^p$, we have $L_1' = L_2$. The analysis given above for L_1 applies to L_1' and hence to L_2. It follows that each $e_{2i} \otimes_\pi v^p$ corresponds to a cohomology operation which is zero or to a suitable multiple of $\mathcal{P}^t\delta^*$.

In the special case $\theta = 0$, we set $v = 0$. Then $W \otimes_\pi M^p$ reduces to the normal form: $\delta(e_0 \otimes_\pi u^p) = 0$ and

$$\delta(e_{2i} \otimes_\pi u^p) = p e_{2i-1} \otimes_\pi u^p \;, \qquad i \geqq 1 \; .$$

Now $e_0 \otimes_\pi u^p$ corresponds to the pth power operation in the sense of cup products with integer coefficients (see [2; 10.1–4]). The remaining cocycles lie in L_1 and have already been analysed. This concludes the case $\theta = 0$, and we may suppose $\theta = p^k$ henceforth.

§9. The equivalence of L and ΣM^p

The analysis of L (see 7.8–10) is more complicated and devious. The remaining two sections are devoted to the task. The conclusion however is not complicated to state:

9.1. *If q is odd, each cohomology operation derived from L is elementary. If q is even, the only non-elementary operations derivable from L are obtained from cocycles of the elementary subcomplex*

$$\delta[e_0 \otimes_\pi u^p + e_1 \otimes_\pi \Sigma^* u^{p-1} v] = p\theta \left[e_0 \otimes_\pi u^{p-1} v - e_1 \otimes_\pi \Sigma^* \frac{1}{p} \delta(u^{p-1} v) \right], \tag{9.2}$$

where Σ^* in the group ring of π is defined by

$$\Sigma^* = \sum_{k=1}^{p-1} k T^{p-k} . \tag{9.3}$$

In checking the formula 9.2, the following identity is useful:

$$(T^{-1} - 1) \Sigma^* = -\Sigma + p\,1 . \tag{9.4}$$

The cochain on the left of 9.2 is a cocycle mod $p\theta$, and its corresponding cohomology operation is the PONTRJAGIN pth power as defined by THOMAS [5; 3.3, 3.4]. Once 9.1 is proved the proof of the main result of this paper will be complete.

Define ΣM^p to be the subcomplex of M^p consisting of cochains of the form Σc where c is a cochain of M^p. Define a cochain mapping

$$f\colon L \to \Sigma M^p \tag{9.5}$$

as follows. If the initial term of a cochain in canonical form is $e_j \otimes_\pi c_0$ where $j > 0$, its image under f is 0. If its initial term is $e_0 \otimes_\pi c_j$, its image is Σc_j:

$$f(e_j \otimes_\pi c_0 + \cdots + e_{j+p} \otimes_\pi c_p) = 0 , \qquad j > 0 \tag{9.6}$$

$$f(e_0 \otimes_\pi c_j + \cdots + e_{p-j} \otimes_\pi c_p) = \Sigma c_j . \tag{9.7}$$

Clearly f is a homomorphism. To prove $\delta f = f \delta$, we suppose in the first case $j = 2i > 0$. Then the initial term of $\delta(e_{2i} \otimes_\pi c_0 + \cdots)$ is $e_{2i-1} \otimes_\pi \Sigma c_0$; and so its f-image is zero. Suppose next that $j = 2i + 1 > 0$. Then the initial term of $\delta(e_{2i+1} \otimes_\pi c_0 + \cdots)$ is $e_{2i} \otimes_\pi (T^{-1} - 1) c_0$. If $i > 0$, 9.6 applies, and its f-image is zero. If $i = 0$, 9.7 applies, and its f-image is $\Sigma(T^{-1} - 1) c_0 = 0$. In the second case, the initial term of $\delta(e_0 \otimes_\pi c_j + \cdots)$ is

$$e_0 \otimes_\pi [\delta c_j + (T^{-1} - 1) c_{j+1}] .$$

Thus, its f-image is

$$\Sigma \delta c_j = \delta \Sigma c_j = \delta f(e_0 \otimes_\pi c_j + \cdots) .$$

Therefore, $\delta f = f \delta$ in all cases.

9.8. *The automorphism y of M^p (defined by 6.15) transforms ΣM^p into itself. If g^* is the cochain mapping 6.20 restricted to L, then $f g^*(z) = y f(z)$ for all cochains z of L.*

The first assertion follows from $y\Sigma = \Sigma y$ which is an immediate consequence of 6.16. To prove the second, suppose z is such that 9.6 applies. Then $yf(z) = y(0) = 0$. By 6.18 and 6.20, $g^{\#}(z)$ has the same type of initial term as z, so $fg^{\#}(z) = 0$. If z is as in 9.7, then $yf(z) = y(\Sigma c_j) = \Sigma y c_j$. But $g^{\#}(z)$ has the initial term $e_0 \otimes_\pi y c_j$ so $fg^{\#}(z)$ is $\Sigma y c_j$.

9.9. *If J denotes the kernel of f, then J is acyclic. This implies*

$$f_*:\ H^r(L\otimes G) \approx H^r(\Sigma M^p \otimes G)\ .$$

Recall that the cochains of L are defined by conditions 7.8–10. In particular, then, a cochain of J in canonical form has a first non-zero term of the form $e_i \otimes_\pi c_j$, where $i \geqq 0$, and $0<j<p$. If $i = 0$, then $\Sigma c_j = 0$, since the cochain is in J. Suppose that i is even and >0, and that the cochain is a cocycle. This again implies that $\Sigma c_j = 0$. Thus, in either case, we must have $c_j = (T^{-1} - 1)d$ for some d, since M^p is free in the dimension j. Then

$$\delta(e_{i+1} \otimes_\pi d) = e_i \otimes_\pi c_j - e_{i+1} \otimes_\pi \delta d\ .$$

Subtracting this from the cocycle gives a cohomologous cocycle whose first non-zero term has an index $>i$. If i is odd, then we have $(T^{-1} - 1)c_j = 0$. The freeness of M^p implies that $c_j = \Sigma d$, and

$$\delta(e_{i+1} \otimes_\pi d) = e_i \otimes_\pi c_j + e_{i+1} \otimes_\pi \delta d\ .$$

Subtracting this gives again a cohomologous cocycle whose first non-zero term has an index $>i$.

Repeating the process we obtain eventually a cohomologous cocycle having one of the two forms (see 7.9, 10)

$$e_{2i} \otimes_\pi c_{p-1} \quad \text{or} \quad e_{2i-1} \otimes_\pi c_{p-1} + e_{2i} \otimes_\pi \frac{1}{p}\,\delta c_{p-1}\ .$$

In the first case, we alter the first method of the preceding paragraph by observing that

$$\delta\left(e_{2i+1} \otimes_\pi d + e_{2i+2} \otimes_\pi \frac{1}{p}\,\delta d\right) = e_{2i} \otimes_\pi c_{p-1}\ .$$

In the second case, we apply the second method unaltered, and observe that $\delta d = \frac{1}{p}\,\delta c_{p-1}$ because $Tv^p = v^p$. This shows that every cocycle of J is a coboundary, and completes the proof of 9.9.

§10. The cohomology of ΣM^p

By [2; 11.7], each element of $H^r(W \otimes_\pi M^p)$ has an order dividing $p\theta$. The same must be true of $H^r(L)$ and, by 9.9, of $H^r(\Sigma M^p)$. It follows that a normal form for ΣM^p will consist of elementary subcomplexes whose torsion numbers divide $p\theta$.

10.1. *If P is an elementary subcomplex of ΣM^p whose torsion number m divides θ, then there is a mapping ζ: $P \to M^p$ such that $\Sigma\zeta$ is the identity. Therefore each cohomology operation corresponding to a cohomology class of P is elementary.*

We may denote the generators of P by Σc and Σd with $\delta\Sigma c = m\Sigma d$. Since each coboundary in M^p is divisible by θ, we have $\delta c = \theta d'$, and d' is a cocycle. It follows that $\theta\Sigma d' = m\Sigma d$. If we set

$$\zeta(\Sigma c) = c \ , \qquad \zeta(\Sigma d) = \frac{\theta}{m} d'$$

we obtain the required ζ. If w is any cocycle of $P \otimes G$, then $e_0 \otimes_\pi \zeta w$ is a cocycle of $L \otimes G$ whose image in $\Sigma M^p \otimes G$ under f is w. By 3.3, $e_0 \otimes_\pi \zeta w$ and therefore w corresponds to an elementary operation.

Because of this result, we have only to analyse the torsions of order exactly $p\theta$ in ΣM^p. To this end, we define a category $\boldsymbol{N}$ of cochain complexes having certain properties of M^p. A cochain complex N belongs to $\boldsymbol{N}$ if

(10.2) π operates as automorphisms of N.

(10.3) $C^j(N) = 0$ if $j < pq$ or $j > p(q+1)$.

(10.4) $C^{pq}(N)$ has a single generator α_0 fixed under π.

(10.5) $C^{p(q+1)}(N)$ has a single generator α_p fixed under π.

(10.6) $C^{pq+j}(N)$ is π-free if $0 < j < p$.

(10.7) An integral cocycle of N is a coboundary if and only if it is divisible by θ.

A mapping λ: $N \to N'$ of the category $\boldsymbol{N}$ is a cochain mapping which is π-equivariant. Let k_λ be the integer such that $\lambda\alpha_p = k_\lambda\alpha'_p$. Under compositions of two mappings λ, μ, we have

$$k_{\mu\lambda} = k_\mu k_\lambda \ . \tag{10.8}$$

10.9. *If N, N' are in $\boldsymbol{N}$, and k is an integer, then there exists a mapping λ: $N \to N'$ such that $k_\lambda = k$.*

This is proved by a downward induction on the dimension. Start by setting $\lambda\alpha_p = k\alpha'_p$. Suppose λ has been properly defined in dimensions $> pq + j$. If $j > 0$, by 10.6 we can choose a π-basis $\{\beta_i\}$ of C^{pq+j}. By 10.7, $\delta\beta_i = \theta\gamma_i$. Then $\lambda\gamma_i$ is defined. Since $\delta\lambda\gamma_i = \lambda\delta\gamma_i = 0$, $\lambda\gamma_i$ is a cocycle. By 10.7,

$\theta\lambda\gamma_i$ is the coboundary of some cochain. We select one such and denote it by $\lambda\beta_i$. For any $x\epsilon\pi$, we define $\lambda x\beta_i = x\lambda\beta_i$. Since λ is π-equivariant in the dimension $pq+j+1$, it follows that $\lambda\delta x\beta_i = \delta\lambda x\beta_i$. In this fashion the induction continues down to the dimension pq. To define λ on α_0, we prove as above that $\lambda\delta\alpha_0$ is a cocycle and it is divisible by θ. Hence we may choose a cochain $\lambda\alpha_0$ such that $\delta\lambda\alpha_0 = \lambda\delta\alpha_0$. By 10.4, $C^{pq}(N')$ has a single generator fixed under π. Therefore $\lambda\alpha_0$ is fixed and so λ is equivariant.

10.10. *If N, N' are in $\boldsymbol{N}$ and λ_0, λ_1 are two mappings $N \to N'$ of $\boldsymbol{N}$ such that*

$$k_{\lambda_0} \equiv k_{\lambda_1} \bmod p \ ,$$

then there exists an equivariant cochain homotopy

$$D\colon\ \theta\lambda_0 \simeq \theta\lambda_1 \ .$$

i. e. for each j, D is a π-homomorphism of $C^j(N)$ into $C^{j-1}(N')$ such that

$$\delta D\alpha = \theta\lambda_1\alpha - \theta\lambda_0\alpha - D\delta\alpha \ , \qquad \alpha\epsilon C^j(N) \ . \tag{10.11}$$

This is also proved by a downward induction. By hypothesis

$$\theta\lambda_1\alpha_p - \theta\lambda_0\alpha_p = \theta(k_{\lambda_1} - k_{\lambda_0})\alpha'_p = pr\theta\alpha'_p$$

for some r. By 10.7, $r\theta\alpha'_p$ is a coboundary of some cochain, say γ. Then

$$\delta\Sigma\gamma = \Sigma r\theta\alpha'_p = pr\theta\alpha'_p \ .$$

So we may set $D\alpha_p = \Sigma\gamma$, and D is equivariant and satisfies 10.11 with $\alpha = \alpha_p$.

Suppose D has been defined properly in dimensions $> pq+j$. Let $\{\beta_i\}$ be a π-free basis in dimension $pq+j$ (assuming $j>0$). Then the right side of 10.11 is defined for $\alpha = \beta_i$. The standard calculation shows that it is a cocycle. It is also divisible by θ because $\delta\beta_i$ is divisible by θ. Hence it is a coboundary of some cochain, we define $D\beta_i$ to be one such. We extend D to be a π-homomorphism, and then verify that 10.11 still holds in the dimension $pq+j$. When $j=0$, we set $D\alpha_0 = 0$. This is clearly equivariant. Also the right side of 10.11 with $\alpha = \alpha_0$ must be zero. For it is a cocycle divisible by θ, and hence it is a coboundary; but by 10.3, $C^{pq-1}(N') = 0$.

We introduced the category $\boldsymbol{N}$ for the purpose of studying the subcomplex ΣM^p. Now if N and N' are any complexes in $\boldsymbol{N}$, we compare the cohomology of ΣN and $\Sigma N'$ as follows:

10.12. *If $\lambda\colon\ N \to N'$ is in $\boldsymbol{N}$ and k_λ is prime to p, then λ induces an isomorphism*

$$\lambda_*\colon\ \theta H^j(\Sigma N) \approx \theta H^j(\Sigma N') \ .$$

By 10.9, there is a $\mu\colon\ N' \to N$ such that $k_\mu k_\lambda \equiv 1 \bmod p$. By 10.8 and 10.10, there is a cochain homotopy D of $\theta\mu\lambda$ into $\theta 1$, where 1 is the identity map. Then if $\Sigma\alpha$ is any cocycle of ΣN, 10.11 gives

$$\delta D\Sigma\alpha = \delta\Sigma D\alpha = \theta\mu\lambda\Sigma\alpha - \theta\Sigma\alpha \ .$$

Therefore $(\mu\lambda)_* = \mu_*\lambda_*$ induces the identity mapping of $\theta H^j(\Sigma N)$. By the symmetry of the situation, $\lambda_*\mu_*$ induces the identity in $\theta H^j(\Sigma N')$. This proves 10.12.

By 10.9, there are maps λ with $k_\lambda = 1$. It follows that $\theta H^j(\Sigma N)$ has the same structure for all N in $\boldsymbol{N}$. To compute these groups we construct a simplest N_0 in $\boldsymbol{N}$ as follows. It has a single fixed generator α_0 for $C^{pq}(N_0)$ and α_p for $C^{p(q+1)}(N_0)$, and has a single π-free generator α_j for $C^{pq+j}(N_0)$, $0<j<p$. Define δ by

$$\delta\alpha_{2i} = \theta\Sigma\alpha_{2i+1} \ , \qquad \delta\alpha_{2i+1} = \theta\Delta\alpha_{2i+2} \ , \tag{10.13}$$

for $0 \leqq i \leqq (p-1)/2$. The conditions 10.2 to 10.6 are trivially true. The truth of 10.7 follows from the fact that each of Δ, Σ generates the annihilator of the other in the group ring of π.

The complex ΣN_0 has a single generator $\Sigma\alpha_j$ for each $0 \leqq j \leqq p$. The coboundary relations are

$$\delta\Sigma\alpha_{2i} = p\theta\Sigma\alpha_{2i+1} \ , \qquad \delta\Sigma\alpha_{2i+1} = 0 \ . \tag{10.14}$$

Thus ΣN_0 is in normal form, and it has torsion of order $p\theta$ in every other dimension from pq to $p(q+1)$. Since M^p is in $\boldsymbol{N}$, 10.12 implies that the same conclusion holds for ΣM^p. These results are summarized in:

10.15. *The torsion numbers* $= p\theta$ *of* ΣM^p *occur just once in every other dimension from* pq *to* $p(q+1)$. *One obtains elementary subcomplexes of* ΣM^p *containing these torsions by taking the* λ*-image of* ΣN_0 *where* $\lambda\colon\ N_0 \to M^p$ *is an equivariant mapping such that* k_λ *is prime to* p.

Cocycles of order $p\theta$ obtained in this way are not generally in the image $H(M^p) \to H(\Sigma M^p)$ under Σ. However in all but one exceptional case they correspond to elementary cohomology operations. To see this we must study the behavior of such cocycles under the automorphism y of ΣM^p which, by 9.8, corresponds to the chain mapping $g^{\#}$ of L.

Let $\lambda\colon\ N_0 \to M^p$ be a fixed equivariant mapping such that $k_\lambda = 1$. Now $y\lambda$ is a cochain mapping $N_0 \to M^p$ but it is not equivariant because $yT = T^k y$ where k is a primitive root of p. However for each integer i in the range 0 to $\frac{1}{2}(p-1)$ we shall construct an equivariant mapping μ depending on i such that $y\lambda$ and μ coincide on $C^{pq+2i}(\Sigma N_0)$. Let the integer m be the inverse mod p of the primitive root k. Set

$$\Lambda = \sum_{j=0}^{k-1} T^j , \quad \Gamma = \sum_{j=0}^{m-1} T^j$$

in $Z(\pi)$. In the group ring of the normalizor of π, we obtain readily the relations

$$y\Sigma = \Sigma y, \quad y\Delta = \Delta\Lambda y, \quad \Delta y = y\Gamma\Delta . \tag{10.16}$$

and therefore

$$\Lambda^s y\Delta = \Delta\Lambda^{s+1}y, \quad y\Gamma^s\Delta = \Delta y\Gamma^{s+1}, \quad s \geqq 0 . \tag{10.17}$$

Define an equivariant mapping $\mu : N_0 \to M^p$ by specifying its values on basis elements as follows

$$\begin{aligned}
\mu(\alpha_{2i+2s}) &= \Lambda^s y\lambda(\alpha_{2i+2s}) , & s &= 0, 1, \ldots, \tfrac{1}{2}(p-1-2i) , \\
\mu(\alpha_{2i+2s+1}) &= \Lambda^s y\lambda(\alpha_{2i+2s+1}) , & s &= 0, 1, \ldots, \tfrac{1}{2}(p-1-2i) , \\
\mu(\alpha_{2i-2s}) &= y\Gamma^s\lambda(\alpha_{2i-2s}) , & s &= 0, 1, \ldots, i , \\
\mu(\alpha_{2i-2s+1}) &= y\Gamma^s\lambda(\alpha_{2i-2s+1}) , & s &= 0, 1, \ldots, i .
\end{aligned}$$

The relation $\delta\mu = \mu\delta$ follows directly from 10.13, 10.16, 10.17. When restricted to ΣN_0, μ takes the form

$$\begin{aligned}
\mu(\Sigma\alpha_{2i+2s}) &= k^s y\lambda(\Sigma\alpha_{2i+2s}) , \\
\mu(\Sigma\alpha_{2i+2s+1}) &= k^s y\lambda(\Sigma\alpha_{2i+2s+1}) , \\
\mu(\Sigma\alpha_{2i-2s}) &= m^s y\lambda(\Sigma\alpha_{2i-2s}) , \\
\mu(\Sigma\alpha_{2i-2s+1}) &= m^s y\lambda(\Sigma\alpha_{2i-2s+1}) .
\end{aligned} \tag{10.18}$$

The reason for this is that $y\Sigma = \Sigma y$, $\Lambda\Sigma = k\Sigma$, and $\Gamma\Sigma = m\Sigma$. Taking $s = \frac{1}{2}(p-1) - i$, we have

$$\mu(\Sigma\alpha_p) = k^{\frac{1}{2}(p-1)-i} y\lambda(\Sigma\alpha_p) .$$

Now α_p and $\lambda\alpha_p = v^p$ are fixed under π; so, by 6.17

$$\mu(\alpha_p) = (-1)^{q+1}k^{\frac{1}{2}(p-1)-i}v^p .$$

Consider now the mapping $\mu - \lambda$ of N_0 into M^p. It is obviously equivariant; and

$$(\mu - \lambda)\alpha_p = [(-1)^{q+1}k^{\frac{1}{2}(p-1)-i} - 1]v^p . \tag{10.19}$$

Because k is a primitive root of p, the only case where the coefficient of v^p is divisible by p is the case q even and $i = 0$. In any other case, $k_{\mu-\lambda}$ is prime to p; and so, by 10.15, $\mu - \lambda$ applied to ΣN_0 gives the torsions of order $p\theta$ of ΣM^p. By 10.18, we have

$$(\mu - \lambda)\Sigma\alpha_j = (y-1)\lambda\Sigma\alpha_j \quad \text{for} \quad j = 2i \quad \text{and} \quad 2i+1 . \tag{10.20}$$

For convenience set
$$\lambda\Sigma\alpha_{2i} = \beta \quad \text{and} \quad \lambda\Sigma\alpha_{2i+1} = \gamma \ .$$

Both γ and $(y-1)\gamma$ are cocycles generating torsion of order $p\theta$ in ΣM^p. Since $\theta H^{pq+2i+1}(\Sigma M^p)$ is cyclic of order p, we must have $\theta\gamma \sim b(y-1)\theta\gamma$ for some integer b prime to p; thus

$$\delta w = \theta\gamma - b(y-1)\theta\gamma \quad \text{for} \quad w \in \Sigma M^p \ .$$

Since $\delta\beta = p\theta\gamma$, it follows that $\beta - b(y-1)\beta - pw$ is a cocycle. By 10.15, $\theta H^{pq+2i}(\Sigma M^p) = 0$. Therefore there is a cochain $w_1 \in \Sigma M^p$ such that

$$\delta w_1 = \theta(\beta - b(y-1)\beta - pw) \ .$$

By 10.1, any cocycle of this elementary complex corresponds to an elementary cohomology operation. Taking $Z_{p\theta}$ as coefficient group, it follows that $\beta - b(y-1)\beta$ is a cocycle and corresponds to an elementary operation. However $(y-1)\beta$ corresponds to zero as a cohomology operation; this is seen by assembling 9.8, 6.20, 6.14 and 6.1. It follows that β corresponds to an elementary operation.

In the case q even and $i = 0$, it is clear that

$$\delta\Sigma u^p = p\delta u^p = p\theta\Sigma u^{p-1}v \tag{10.21}$$

is an elementary subcomplex giving the torsion of order $p\theta$ in this case. By the preceding argument, the only non-elementary cohomology operations derivable from ΣM^p are obtained from this subcomplex. If we apply f of 9.5 to 9.2, we obtain 10.21. This completes the proof of 9.1 and, hence, our main result.

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