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# **Cohomology Operations derived from Cyclic Groups\*)**

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### §1. Introduction

In a previous paper [2], STEENROD defined a family of cohomology operations, called reduced powers, each being associated with some permutation group. It was also shown that these operations have a basis, in the sense of composition, consisting of, firstly, four primitive types of operations (which are : addition, cup-product, homomorphisms induced by coefficient homomorphisms, and BOCKSTEIN-WHITNEY coboundary operators) and, secondly, those reduced powers associated with cyclic permutation groups having degree p and order pwhere p ranges over primes.

In this paper, we shall improve the result by showing that there is a smaller basis consisting of the same primitive operations and only particular operations associated with cyclic groups: namely, for each prime p the cyclic reduced powers

$$\mathcal{P}^{i} \colon H^{q}(K; Z_{p}) \to H^{q+2i(p-1)}(K; Z_{p}) \quad , \quad i = 0, 1, \dots$$

and the PONTRJAGIN pth powers

$$\mathfrak{P}_p: \ H^{2q}(K; Z_{pk}) \to H^{2pq}(K; Z_{pk+1}) \ .$$

The latter were defined for p = 2 by PONTRJAGIN [1], and generalized for p > 2 by THOMAS [5]. When p = 2,  $\mathcal{P}^i$  is usually written  $\operatorname{Sq}^{2i}$ .

Throughout the paper an *elementary* cohomology operation will mean one which is a composition of operations of the four primitive types.

#### §2. Recapitulation

Let p be a prime, and  $\pi$  the cyclic permutation group of order p and degree p. The reduced power operations based on  $\pi$  are obtained as elements of cohomology groups  $Hr(W \otimes M^{2} \otimes C)$ (2.1)

$$H^{r}(W \otimes_{\pi} M^{p} \otimes G) \quad . \tag{2.1}$$

In this expression W denotes a  $\pi$ -free acyclic chain complex, G is any coefficient group, and M is a cochain complex having two free generators u, v of dimensions q, q + 1, respectively, and the coboundary relation

$$\delta u = \theta v \tag{2.2}$$

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where  $\theta$  is an integer >1. Finally,  $\pi$  acts on  $M^p = M \otimes \cdots \otimes M$  (p factors) by cyclic permutations of the factors. Then an element  $\xi$  of the group 2.1 determines a cohomology operation

$$\xi: H^{q}(K; \mathbb{Z}_{\theta}) \to H^{r}(K; G) \tag{2.3}$$

defined for all complexes K.

We refer to  $Z_{\theta}, G$  as the *initial* and *terminal* coefficient groups respectively. In a paper by STEENROD [3], it is shown that a basis for cohomology operations is provided by the four primitive types and those operations whose initial and terminal coefficient groups are cyclic of infinite or prime power order. Thus we have only to consider the cases  $\theta = 0$ ,  $\theta = a$  prime power, and  $G = Z_m$ where m = 0 or m = a prime power. In case  $\theta = 0$ , the cochain complex M is simplified by setting v = 0; then  $M^p$  is a cyclic group generated by the cocycle  $u^p$ .

Since the group 2.1 is independent of the choice of W, we shall choose the simplest known  $\pi$ -free acyclic complex W. Let T denote the generator of  $\pi$  which moves each factor of  $M^p$  one step to the right and moves the last factor to the first position. In the group ring  $Z(\pi)$  set

$$\Delta = T - 1$$
 and  $\Sigma = \sum_{j=0}^{p-1} T^{j}$ . (2.4)

The group of r-chains (r = 0, 1, ...) of W is the  $\pi$ -free module having one generator  $e_r$  (i. e. as a complex W has a single r-cell and its distinct transforms in each dimension r). The boundary operator in W is defined by

$$\partial e_{2i+1} = \Delta e_{2i}$$
,  $\partial e_{2i+2} = \Sigma e_{2i+1}$ ,  $i = 0, 1, \dots$  (2.5)

Since each of  $\Delta$ ,  $\Sigma$  generates the annihilator of the other in  $Z(\pi)$ , it follows that W is acyclic.

We have therefore, the problem of computing the cohomology of the specific cochain complex  $W \otimes_{\pi} M^p$ . Speaking roughly our method consists in reducing the complex to normal form and reading off the results. We must distinguish special cases depending on the integers  $p, q, \theta, m, r$ , principally, p = 2 and p > 2, q odd and q even,  $\theta = 0$ ,  $\theta = p^k$  and  $\theta$  prime to p. The case  $\theta = 0$  will be obtained as a subcase of  $\theta = p^k$  by the device of setting v = 0.

## §3. The case $\theta$ prime to p

In case  $\theta$  is prime to p, we will show that each  $\xi \in H^r(W \otimes_{\pi} M^p \otimes G)$  gives an elementary operation. In all subsequent sections it will be assumed that  $\theta$  is a power of p.

Let  $\sigma$  be the subgroup of  $\pi$  consisting of the unit element. Then W is  $\sigma$ -free, and

$$W \otimes_{\sigma} M^{p} = W \otimes M^{p} \xrightarrow{g} W \otimes_{\pi} M^{p}$$

where g is the natural factorization. In [2; 10.4] it is shown that

**3.1.** Each element of  $H^r(W \otimes M^p \otimes G)$  defines an elementary operation. If we now apply the result [2; 3.4], we have

**3.2.** Each element of the image of

$$g^*: H^r(W \otimes M^p \otimes G) \to H^r(W \otimes_\pi M^p \otimes G)$$

defines an elementary operation.

As a corollary we have

**3.3.** Each cocycle of  $W \otimes_{\pi} M^p \otimes G$  of the form  $e_0 \otimes_{\pi} w$ , where w is a cocycle of  $M^p \otimes G$ , defines an elementary operation.

Now let  $\tau$  denote the *transfer* chain transformation defined relative to the subgroup  $\sigma$  of  $\pi$  (see [2; 11.1]

$$\tau: W \otimes_{\pi} M^p \to W \otimes M^p$$

Since p is the index of  $\sigma$  in  $\pi$ , [2; 11.2] gives  $g\tau = p$  where p means multiplication by p. Passing to cohomology with coefficients in G, it follows that  $g^*\tau^* = p$ ; and therefore

3.4. Each element of  $H^r(W \otimes_{\pi} M^p \otimes G)$  which is divisible by p defines an elementary operation.

Assume now that  $\theta$  is prime to p. Then there are integers  $\alpha$ ,  $\beta$  such that

$$lpha heta + eta p = 1$$
 .

Consider now the cochain mappings  $M \to M$  which are multiplications by  $\alpha \theta$ ,  $\beta p$  and 1. We construct a cochain homotopy D of  $\beta p$  into 1 by setting

$$Du=0$$
,  $Dv=\alpha u$ 

The relation  $\delta D + D\delta = \alpha \theta = 1 - \beta p$  follows directly. By [2; 5.2], we have that

$$1 \otimes_{\pi} (\beta p)^{p} \colon W \otimes_{\pi} M^{p} \to W \otimes_{\pi} M^{p}$$

is cochain homotopic to the identity. If we tensor with G and pass to cohomology, it follows that each element of  $H^r(W \otimes_{\pi} M^p \otimes G)$  is divisible by  $(\beta p)^p$ ; and so by 3.4 it defines an elementary operation.

#### §4. The case p = 2 and q even

We shall give a normal form for the complex  $W \otimes_{\pi} M^2$ , i. e. we express it as a direct sum of *elementary* subcomplexes each with two generators, say x and y, and a coboundary relation of the form  $\delta x = ky$ .

In  $M^2$  we shall abbreviate  $u \otimes u$  by  $u^2$ ,  $u \otimes v$  by uv, etc. Then  $M^2$  has the four generators  $u^2$ , uv, vu and  $v^2$ . Since  $(Te_j) \otimes_{\pi} w = e_j \otimes_{\pi} T^{-1}w$ , it follows that  $W \otimes_{\pi} M^2$  has the generators  $e_j \otimes_{\pi} u^2$ ,  $e_j \otimes_{\pi} uv$ ,  $e_j \otimes_{\pi} vu$  and  $e_j \otimes_{\pi} v^2$  for all  $j \ge 0$ . Recall that the definition of the grading of a tensor product of a chain and a cochain complex [2; 2.2] gives  $\dim(e_j \otimes_{\pi} w) = \dim w - j$ . Hence, in the highest non-zero dimension 2(q+1), there is just one generator, and we set

$$\alpha_0 = e_0 \otimes_\pi v^2 \ . \tag{4.1}$$

In the dimension 2q + 1 there are three generators, and we define a unimodular transformation to a new basis  $a_0, \beta_0, \gamma_0$  by the matrix

By virtue of § 3, we are working under the assumption that  $\theta$  is a power of p = 2, so  $-\frac{1}{2}\theta$  is an integer. The determinant is 1, hence  $a_0, \beta_0, \gamma_0$  form a basis in this dimension.

In the dimension 2q, we have four generators, and we define a new basis  $b_0, c_0, \alpha_1, \delta_1$  by the unimodular transformation

	$e_0 \otimes_{\pi} u^2$	$e_1 \otimes_{\pi} uv$	$e_1 \otimes_{\pi} v u$	$e_2 \otimes_{\pi} v^2$	
b <sub>o</sub>	1	θ	0	0	
c <sub>0</sub>	0	1	0	0	(4.9)
$\delta_1$	0	1	1	0	(4.3)
α1	0	0	0	1	
	1				

In the dimensions 2q - 2i for  $i \ge 1$ , we define new generators  $b_i, c_i, \delta_{i+1}, \alpha_{i+1}$  by the unimodular matrix

	$e_{2i} \otimes_{\pi} u^2$	$e_{2i+1} \otimes_{\pi} uv$	$e_{2i+1} \otimes_{\pi} v u$	$e_{2i+2} \otimes_{\pi} v^2$
b <sub>i</sub>	1	0	0	0
c <sub>i</sub>	0	1	0	0
$\delta_{i+1}$	0	1	1	0
$\alpha_{i+1}$	0	0	0	1
	1			

Finally, in dimensions 2q - 2i + 1, for  $i \ge 1$ , we define new generators  $\beta_i, \gamma_i, d_i, a_i$  by

	$e_{2i-1} \otimes_{\pi} u^2$	$e_{2i} \otimes_{\pi} uv$	$e_{2i} \otimes_{\pi} v u$	$e_{2i+1} \otimes_{\pi} v^2$	
$\beta_i$	1	$\frac{1}{2}\theta$	$\frac{1}{2} \theta$	0	
$d_i$	0	1	0	$\frac{1}{2} \theta$	(4 5)
Yi	0	- 1	1	— <b>θ</b>	(4.0)
$a_i$	0	0	0	1	

With this new basis the coboundary in  $W \otimes_{\pi} M^2$  takes the normal form

$$\delta a_i = -2 \alpha_i$$
 ,  $i \ge 0$  , (4.6)

$$\delta b_0 = 2\theta \beta_0 \quad , \tag{4.7}$$

$$\delta b_i = 2 eta_i$$
 ,  $i \ge 1$  , (4.8)

$$\delta c_i = \gamma_i$$
 ,  $i \ge 0$  , (4.9)

$$\delta d_i = \delta_i , \qquad i \ge 1 . \qquad (4.10)$$

As an example, we compute 4.7 in detail.

$$\delta(e_0 \otimes_\pi u^2) = (\partial e_0) \otimes_\pi u^2 + e_0 \otimes_\pi \delta(u^2)$$
$$= \theta e_0 \otimes_\pi (vu + uv)$$

since q is even.

$$\delta(e_1 \otimes_\pi uv) = (\partial e_1) \otimes_\pi uv - e_1 \otimes_\pi \delta(uv)$$
  
=  $(\Delta e_0) \otimes_\pi uv - \theta e_1 \otimes_\pi v^2$ .

Now use  $\Delta = T - 1$ , and  $(Te_0) \otimes_{\pi} uv = e_0 \otimes_{\pi} T^{-1}uv = e_0 \otimes_{\pi} vu$ . Then

$$(\varDelta e_0) \otimes_{\pi} uv = e_0 \otimes_{\pi} (vu - uv) .$$

Therefore

$$\begin{split} \delta b_0 &= \delta \left( e_0 \otimes_{\pi} u^2 + \theta e_1 \otimes_{\pi} u v \right) \\ &= \theta e_0 \otimes_{\pi} \left( v u + u v \right) + \theta e_0 \otimes_{\pi} \left( v u - u v \right) - \theta^2 e_1 \otimes_{\pi} v^2 \\ &= 2 \theta e_0 \otimes_{\pi} u v - \theta^2 e_1 \otimes_{\pi} v^2 = 2 \theta \beta_0 \end{split}$$

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The computations of the other coboundaries are similar. It should be pointed out that the assumption that q is even is used in obtaining the relations

$$Tu^2 = u^2$$
,  $Tv^2 = -v^2$ ,  $\delta(u^2) = \theta(vu + uv)$ .

Using this normal form, we can now read off the cohomology of  $W \otimes_{\pi} M^2$ ; it is the direct sum of the cohomologies of the subcomplexes 4.6 to 4.10. Obviously 4.9 and 4.10 are acyclic complexes, and their cohomologies are zero. Before treating the other three, we will prove a lemma which greatly reduces the task of showing that the cohomology operations corresponding to the various cocycles are compositions of the operations specified in § 1. In particular the lemma eliminates the need of considering various terminal coefficient groups  $Z_m$ .

4.11. Lemma. Let N be an elementary subcomplex of  $W \otimes_{\pi} M^p$  generated by x, y with  $\delta x = ky$ . Then the cohomology operation which corresponds to any cocycle of  $N \otimes G$  is a composition of elementary operations and the operation corresponding to the cocycle  $x \mod k$ .

Let us recall the way in which a cocycle of  $W \otimes_{\pi} M^{p}$  corresponds to a cohomology operation on an element  $\overline{u} \, \epsilon \, H^q(K; Z_{\theta})$  (see [2; § 2]). A mapping  $\psi: M \to K^*$  representing  $\overline{u}$  is chosen (i. e.  $\psi u$  is a cocycle of the class  $\overline{u}$ ). Then  $\psi$  determines a mapping

$$\varphi \psi: W \otimes_{\pi} M^p \to K^*$$
,

which induces a cohomology homomorphism

$$\Phi: H^{r}(W \otimes_{\pi} M^{p} \otimes G) \to H^{r}(K;G)$$

If  $\xi \in H^r(W \otimes_{\pi} M^p \otimes G)$ , then  $\xi(\overline{u})$  is defined to be  $\Phi(\xi)$ . Now let

$$\xi \epsilon H^r(N \otimes Z_k)$$

be the class of the cocycle  $x \mod k$ . Then  $\varphi \psi(x)$  is a cocycle of the class  $\xi(\overline{u}) \in H^r(K; Z_k)$ . Therefore  $\varphi \psi | N: N \to K^*$  represents  $\xi(\overline{u})$ . Apply now [2; Lemma 10.1] which asserts that the image  $H^{\mathfrak{s}}(N \otimes G) \to H^{\mathfrak{s}}(K;G)$  is generated by  $\xi(\bar{u})$  and elementary operations. This proves the lemma.

By virtue of the lemma, it suffices to identify the cohomology operations corresponding to the cocycle  $b_0 \mod 2\theta$ , the cocycles  $b_i \mod 2$  for  $i \ge 1$ , and the cocycles  $a_i \mod 2$  for  $i \ge 0$ . Now

$$b_0 = e_0 \otimes_\pi u^2 + \theta e_1 \otimes_\pi u v$$

corresponds by definition to the PONTRJAGIN squaring operation  $\mathfrak{P}_2$  (see [5; 3.4]). And

$$b_i = e_{2i} \otimes_\pi u^2$$

corresponds by definition to the cyclic reduced square  $\operatorname{Sq}_{2i} = \operatorname{Sq}^{q-2i}$  (see [2; p. 6] and [4; § 4]). If it is felt desirable that  $\operatorname{Sq}_{2i}$  should operate only on cocycles mod 2, then the operation represented by  $b_i$  can be written as  $\operatorname{Sq}_{2i}\eta_*$  where  $\eta: Z_0 \to Z_2$  is reduction mod 2. Note that  $\operatorname{Sq}^{q-2i}$  has an even superscript since q is even.

It remains to identify the operation corresponding to the cocycle  $a_i \mod 2$ . Let  $\delta^*$  denote the BOCKSTEIN coboundary operator corresponding to the exact  $\theta$ sequence  $0 \to Z \to Z \to Z_{\theta} \to 0$ . If  $\bar{u} \in H^q(K; Z_{\theta})$  and  $\psi: M \to K^*$  represents  $\bar{u}$ , then  $\psi(v)$  represents  $\delta^* \bar{u}$ ; and finally  $\varphi \psi(e_{2i+1} \otimes_{\pi} v^2)$  represents

$$\operatorname{Sq}_{2i+1}\delta^*\overline{u} = \operatorname{Sq}^{q-2i}\delta^*\overline{u}$$
.

Therefore  $a_i$  corresponds to the cohomology operation  $\operatorname{Sq}^{q-2i} \delta^*$ . Note again that the upper index is even.

In the special case  $\theta = 0$ , we set v = 0, and then  $W \otimes_{\pi} M^2$  is in the normal form

$$\delta(e_0\otimes_\pi u^2) = 0$$
  
 $\delta(e_{2i}\otimes_\pi u^2) = 2e_{2i-1}\otimes_\pi u^2 \qquad i \ge 1$ .

The cocycle  $e_0 \otimes_{\pi} u^2$  corresponds to the operation of squaring in the sense of the cup product (see 3.3 and [2; 10.2]). As before  $e_{2i} \otimes_{\pi} u^2 \mod 2$  corresponds to  $\operatorname{Sq}^{q-2i}$ .

### §5. The case p = 2 and q odd

The change in the parity of q affects both the coboundary operator and the action of T in  $M^2$ . For the latter we have

$$Tu^2=-u^2$$
 ,  $Tuv=vu$  ,  $Tvu=uv$  ,  $Tv^2=v^2$  .

Starting with the same generators of  $W \otimes_{\pi} M^2$  as in § 4, we reduce to normal form as follows. In the highest dimension 2q + 2, we have one generator, and we set

$$\gamma_0=e_0\otimes_\pi v^2$$
 .

In the dimension 2q + 1, we define a new basis by the unimodular transformation

	$e_0 \otimes_\pi uv$	$e_0 \otimes_\pi v u$	$e_1 \otimes_{\pi} v^2$
<i>c</i> 0	1	0	0
$\delta_0$	-1	1	0
α0	0	0	1

In all dimensions 2q - 2i for  $i \ge 0$ , we define a new basis by the unimodular transformation

	$e_{2i} \otimes_{\pi} u^2$	$e_{2i+1} \otimes_{\pi} uv$	$e_{2i+1} \otimes_{\pi} v u$	$e_{2i+2} \otimes_{\pi} v^2$
$\beta_{i+1}$	1	$-\frac{1}{2}\theta$	$\frac{1}{2}\theta$	0
$d_i$	0	1	0	$\frac{1}{2}\theta$
$\gamma_{i+1}$	0	1	1	θ
<i>ai</i>	0	0	0	1

In all dimension 2q - 2i + 1 for  $i \ge 1$ , we define a new basis by the unimodular transformation

	$e_{2i-1} \otimes_{\pi} u^2$	$e_{2i} \otimes_{\pi} uv$	$e_{2i} \otimes_{\pi} v u$	$e_{2i+1} \otimes_{\pi} v^2$
b <sub>i</sub>	1	0	0	0
c,	0	1	0	0
$\delta_i$	0	— 1	1	0
α,	0	0	0	1

In the terms of the new basis, the coboundary relations become

$\delta c_0 =  heta \gamma_0$	
$\delta b_i = -2eta_i$	$i\!\geq\!1$ ,
$\delta a_i = 2 \alpha_i$	$i\!\geq\!0$ ,
$\delta d_{i} = \delta_{i}$	$i\!\ge\! 0$ ,
$\delta c_i = \gamma_i$	$i\!\geq\!1$ .

The last two subcomplexes have zero cohomology. By 4.11, we need only identify the cohomology operations corresponding to the cocycles  $c_0 \mod \theta$ ,  $a_i \mod 2$ , and  $b_i \mod 2$ . Now  $c_0 = e_0 \otimes_{\pi} uv$  defines an elementary operation by 3.3 (it is in fact the operation  $\overline{u} \to \overline{u} \cup \delta^* \overline{u}$  where  $\delta^*$  is the obvious BOCK-STEIN).

By definition,  $b_i = e_{2i-1} \otimes_{\pi} u^2$  is the cyclic reduced square

$$\mathrm{Sq}_{2i-1} = \mathrm{Sq}^{q-2i+1} \ .$$

Since q is odd, q - 2i + 1 is even. Again the operation may be written  $\operatorname{Sq}^{q-2i+1}\eta_*$  where  $\eta$  is reduction mod 2.

The cocycle  $a_i = e_{2i+2} \otimes_{\pi} v^2 \mod 2$  can be identified with

$$\operatorname{Sq}_{2i+2}\delta^* = \operatorname{Sq}_{2i+2}\delta^*$$

exactly as the cocycle  $a_i$  of § 4. Again the upper index of the square is even.

In the special case  $\theta = 0$ , we set v = 0 and then  $W \otimes_{\pi} M^2$  is in the normal form

$$\delta(e_{2i+1} \otimes_{\pi} u^2) = - 2 e_{2i} \otimes_{\pi} u^2$$
 ,  $i \ge 0$  .

As above the cocycle  $e_{2i+1} \otimes_{\pi} u^2 \mod 2$  corresponds to the operation  $\operatorname{Sq}^{q-2i-1}$ .

We may summarize our results in the case p = 2 as follows. The only operations needed in addition to the elementary operations are the squares  $\operatorname{Sq}^{2i}(i>0)$  when q is odd or  $\theta = 0$ . When q is even and  $\theta = 2^k$ , the PON-TRJAGIN square is also needed.

## §6. The automorphism $g_*$ of $H(W \otimes_{\pi} M^p)$

We assume henceforth that p is an odd prime. Our analysis must take account now of a phenomenon not present when p=2, namely:  $\pi$  is a proper subgroup of the symmetric group  $S_p$  of degree p. If  $\pi \subset \varrho \subset S_p$ , and U is a  $\varrho$ -free acyclic complex, then the inclusion  $\pi \subset \varrho$  induces a homomorphism

$$h_*: H^r(W \otimes_{\pi} M^p \otimes G) \to H^r(U \otimes_{\varrho} M^p \otimes G) ; \qquad (6.1)$$

and, for any element  $\xi$  on the left, the cohomology operations corresponding to  $\xi$  and  $h_*(\xi)$  coincide (see [2; 3.4]). In particular, if  $\xi$  is in the kernel of  $h_*$ , the corresponding cohomology operation is zero. The aim of this section is to show that certain explicit elements belong to ker  $h_*$  when  $\varrho$  is the normalizor of  $\pi$ in  $\mathcal{S}_p$ . It is a fact that these elements generate the kernel even for  $\varrho = \mathcal{S}_p$ ; but we omit the proof of this since the proof is complicated and the fact is not needed.

We need at this point of the discussion a special case of a rather general proposition. Because it is just as easy and less confusing to present the latter, we shall do so. We shall consider objects  $(\varrho, A)$  where  $\varrho$  is a group, A is a cochain complex, and  $\varrho$  operates as automorphisms of A. By a mapping f of  $(\varrho, A)$  into another such  $(\sigma, B)$ , we mean a homomorphism  $\varrho \to \sigma$  and a cochain mapping  $A \to B$ , both denoted by f, such that

$$f(xa) = f(x)f(a) \quad \text{for} \quad x \in \varrho, \quad a \in A.$$
(6.2)

These objects and mappings form a category. A pair  $(\varrho, G)$  where G is a  $\varrho$ -module may be regarded as an object of the category by treating G as a cochain complex having just one non-zero cochain group in the dimension zero. Now

the ordinary homology theory of groups, developed for this subcategory of pairs  $(\varrho, G)$ , can be extended to the entire category in a fairly obvious way. We shall review this extension briefly.

Let U be a  $\varrho$ -free acyclic chain complex. Defining the cochain complex  $U \otimes_{\varrho} A$  as in [2; 2.2], we proceed to show that  $H^{r}(U \otimes_{\varrho} A)$  is independent of the choice of U. If  $f: (\varrho, A) \to (\sigma, B)$  is a mapping, and V is a  $\sigma$ -free acyclic chain complex, let  $\varrho$  operate on V through  $f: \varrho \to \sigma$ . Then the fundamental lemma (see [2; 2.7]) gives a chain mapping  $f_{\#}: U \to V$  satisfying the equivariance condition

$$f_{*}(xc) = f(x)f_{*}(c) , \quad x \in \varrho, \ c \in U.$$
 (6.3)

It follows that  $f_* \otimes f: U \otimes A \to V \otimes B$  induces a chain mapping

$$f^*: U \otimes_{\varrho} A \to V \otimes_{\sigma} B$$
, (6.4)

and thereby induces homomorphisms of cohomology

$$f_*: H^r(U \otimes_{\varrho} A) \to H^r(V \otimes_{\sigma} B) . \tag{6.5}$$

The second part of the fundamental lemma asserts that any two equivariant chain maps  $f_*$ ,  $f'_*$  of U into V are connected by an equivariant chain homotopy D. Then  $D \otimes_{e} f$  gives a cochain homotopy of  $f^*$  into  $f^{*'}$ . Therefore  $f_*$  is independent of the choice of  $f_*$ .

An obvious property of  $f_*$  is

$$f =$$
identity map of  $(\varrho, A)$  implies  $f_* =$ identity. (6.6)

For  $f_*$  can be taken as the identity.

Let  $f: (\varrho, A) \to (\sigma, B)$  and  $g: (\sigma, B) \to (\tau, C)$  be mappings. Then

$$(gf)_* = g_*f_* \ . \tag{6.7}$$

For, having chosen  $g_{\#}$  and  $f_{\#}$ , we may choose  $(gf)_{\#}$  to be the composition  $g_{\#}f_{\#}$ .

Now let U, V be two  $\varrho$ -free acyclic complexes. Corresponding to the identity map f of  $(\varrho, A)$ , we obtain two induced homomorphisms

$$H^r(U \otimes_{\varrho} A) \to H^r(V \otimes_{\varrho} A) \to H^r(U \otimes_{\varrho} A)$$
,

whose compositions in either order again correspond to f by the property 6.7. Then 6 6 asserts that both compositions give identity maps. Therefore the various choices of the  $\varrho$ -free acyclic complex U give a family of cohomology groups, any two connected by an isomorphism, and the family of these isomorphisms is transitive by virtue of 6.7. As is customary in such a case, we identify this family of groups with a single group. To emphasize its analogy with the ordinary homology group of a group, we shall call it the rth homology group of  $\rho$  with coefficients in A, thus:

$$H_r(\varrho; A) = H^r(U \otimes_{\varrho} A) . \tag{6.8}$$

If  $f: (\varrho, A) \to (\sigma, B)$ , then 6.5 becomes

$$f_*: H_r(\varrho; A) \to H_r(\sigma; B) . \tag{6.9}$$

It is clear that 6.6 and 6.7 continue to hold for the induced homomorphisms taken in this more general sense.

A mapping  $f: (\varrho, A) \to (\varrho, A)$  is called an *automorphism* if both mappings  $\varrho \to \varrho$  and  $A \to A$  are automorphisms. Then f has an inverse mapping, and we may apply 6.7 and 6.6 to conclude that  $f_*$  is an automorphism of  $H_r(\varrho; A)$ .

The inner automorphism f corresponding to an element  $y \epsilon \rho$  is defined by

$$f(x) = y x y^{-1}$$
,  $f(a) = y a$ ,  $x \in \varrho$ ,  $a \in A$ . (6.10)  
re

Then we have

$$f =$$
 an inner automorphism implies  $f_* =$  identity. (6.11)

To see this, let  $f_*$  be the chain mapping  $U \to U$  defined by  $f_*(c) = yc$ . Since

$$f_{*}(xc) = yxc = yxy^{-1}yc = f(x)f_{*}(c)$$
,

the equivariance condition 6.3 is fulfilled. Then

$$(f_* \otimes f)(c \otimes a) = f_* c \otimes fa = yc \otimes ya = y(c \otimes a)$$
.

This implies that the induced mapping  $f^*$  of  $U \otimes_{\varrho} A$  into itself is the identity; and so  $f_* =$ identity.

This completes the discussion of the general theory, and we return now to the special case with which we began this section. In applying the above results, we take

$$A = M^{p} \otimes G ;$$

in this case  $\pi$ , its normalizor  $\varrho$ , and the symmetric group  $\mathcal{S}_p$  operate in A by permuting the factors of  $M^p$  and acting as the identity in G. Let y be any element of  $\varrho$ , let f be the corresponding inner automorphism given by 6.10, and let g be the automorphism of  $(\pi, A)$  obtained by restricting f. Let h:  $(\pi, A) \to (\varrho, A)$  be the inclusion  $\pi \subset \varrho$  and the identity on A. Obviously

$$hg = fh \quad . \tag{6.12}$$

If we pass to the induced homomorphisms and apply 6.7 and 6.11, we obtain

$$h_*g_* = f_*h_* = h_*$$
 (6.13)

Thus we have proved

6.14 Lemma. If  $g_*$  is the automorphism of  $H^r(W \otimes_{\pi} M^p \otimes G)$  determined by any element y of the normalizor  $\varrho$  of  $\pi$ , then  $g_* \xi - \xi$  belongs to the kernel of  $h_*$ where h is the inclusion  $\pi \subset \varrho$  and  $\xi \in H^r(W \otimes_{\pi} M^p \otimes G)$ . Thus, as remarked after 6.1, the cohomology operation corresponding to  $g_* \xi - \xi$  is zero.

In order to use the lemma effectively in computations, we shall choose an explicit y and a corresponding chain mapping  $g_{\#}$ . Let the factors of  $M^p$  be numbered  $0,1,\ldots,p-1$  so that the generator T of  $\pi$  can be described as the transformation  $T(i) \equiv i + 1 \mod p$  in terms of integers. Let k be a primitive root of the prime p (i. e.  $k^j \equiv 1 \mod p$  implies that j is a multiple of p-1). Let y be the permutation of  $0, 1, \ldots, p-1$  defined by

$$y(i) \equiv k i \mod p \quad . \tag{6.15}$$

Then  $y^{-1}(i) \equiv k^{p-2}i \mod p$ , and this gives

$$yTy^{-1} = T^k . (6.16)$$

Thus y belongs to the normalizor  $\rho$  of  $\pi$ . (Since the order of y is p-1, it is a generator of  $\rho/\pi$ .) If we arrange the integers 0 to p-1 in the order

$$0, 1, k, k^2, \ldots, k^{p-2} \mod p$$
,

it is seen that y leaves 0 fixed and permutes the remaining p-1 elements cyclically. This shows that y is an odd permutation because p-1 is even. Therefore

$$yu^{p} = (-1)^{q}u^{p}$$
,  $yv^{p} = (-1)^{q+1}v^{p}$ . (6.17)

Letting W be as in § 2, we define a chain mapping  $g_* : W \to W$  by specifying first its values on the  $\pi$ -basis  $\{e_i\}$ :

$$g_{\#}e_{2i} = k^{i}e_{2i}$$
,  $g_{\#}e_{2+1} = k^{i}\sum_{m=0}^{k-1}T^{m}e_{2i+1}$ ,  $i \ge 0$ . (6.18)

Then, for each  $s = 1, \ldots, p - 1$ , we set

$$g_{\#}T^{s}e_{j} = T^{ks}g_{\#}e_{j}, \quad j \ge 0$$
 . (6.19)

From this it follows that  $g_{\#}$  satisfies the equivariance condition 6.3 for the automorphism 6.16 of  $\pi$ . It is now an easy matter to verify  $g_{\#} \partial = \partial g_{\#}$ . Then the resulting chain transformation  $g^{\#}$  of  $W \otimes_{\pi} M^{p}$  (see 6.4) is defined by

$$g^*(e \otimes_{\pi} c) = (g_* e) \otimes_{\pi} yc \quad . \tag{6.20}$$

These specific calculations will be needed in sections 8, 9, and 10.

# §7. The decomposition: $W \otimes_{\pi} M^p = L_1 + L + L_2$

As a first step in analysing the structure of  $W \otimes_{\pi} M^{p}$  (p odd), we shall decompose it into a direct sum of three cochain subcomplexes as indicated above. Of importance is the fact that each is transformed into itself by the  $g^{*}$  of 6.20.

A cochain of  $W \otimes_{\pi} M^{p}$  is said to be in *canonical form* if it is written  $\sum_{j} e_{j} \otimes_{\pi} c_{j}$ where  $c_{j}$  is a cochain of  $M^{p}$ . Since  $T e \otimes_{\pi} c = e \otimes_{\pi} T^{-1}c$ , each cochain has one and only one canonical form.

Recall that  $M^p$  has all cochain groups equal to zero save in the range pq to p(q+1) inclusive,  $C^{pq}(M^p)$  has one generator  $u^p$ , and  $C^{p(q+1)}(M^p)$  has one generator  $v^p$ . In the dimensions pq + j for 0 < j < p,  $M^p$  is generated by products having p - j factors u and j factors v. Therefore  $\pi$  operates freely in these dimensions.

Let us adopt the convention that the index j of the canonical cochain  $e_i \otimes_{\pi} c_j$  signifies that  $c_j$  has dimension pq + j. Thus  $c_j$  is zero unless  $0 \leq j \leq p$ ,  $c_0$  is a multiple of  $u^p$ , and  $c_p$  is a multiple of  $v^p$ . In the highest non-zero dimension, a canonical cochain has a single term  $e_0 \otimes_{\pi} c_p$ . In dimensions pq + j for 0 < j < p, a canonical cochain has p - j + 1 terms

$$e_0 \otimes_{\pi} c_j + e_1 \otimes_{\pi} c_{j+1} + \dots + e_{p-j} \otimes_{\pi} c_p \tag{7.1}$$

and in all dimensions  $\leq pq$ , it has p+1 terms

$$e_s \otimes_{\pi} c_0 + e_{s+1} \otimes_{\pi} c_1 + \dots + e_{s+p} \otimes_{\pi} c_p$$
 (7.2)

We define  $L_1$  to consist of all cochains having canonical forms of one of the two following types for some  $i \ge 0$ :

$$e_{2i+2} \otimes_{\pi} c_0 \quad \text{or} \quad e_{2i+1} \otimes_{\pi} c_0 + e_{2i+2} \otimes_{\pi} \frac{1}{p} \, \delta c_0 \, .$$
 (7.3)

The second type is described explicitly by requiring  $c_j = 0$  for j > 1, and  $c_1 = \frac{1}{p} \delta c_0$  (recall that any coboundary in  $M^p$  is divisible by  $\theta$ , and  $\theta$  is a power of p). Since  $Tu^p = u^p$ , it follows that  $\Sigma c_0 = pc_0$ ; hence

$$\delta(e_{2i+2}\otimes_{\pi}c_{0}) = \Sigma e_{2i+1}\otimes_{\pi}c_{0} + e_{2i+2}\otimes_{\pi}\delta c_{0}$$
  
=  $e_{2i+1}\otimes_{\pi}\Sigma c_{0} + e_{2i+2}\otimes_{\pi}\delta c_{0}$   
=  $p\left(e_{2i+1}\otimes_{\pi}c_{0} + e_{2i+2}\otimes_{\pi}\frac{1}{p}\delta c_{0}\right)$ . (7.4)

This shows that a cochain of the first type has a coboundary of the second type, and each cochain of the second type is a cocycle. Therefore  $L_1$  is a cochain subcomplex. Clearly  $L_1$  is generated by the cochains  $e_{2i+2} \otimes_{\pi} u^p$  and

 $\frac{1}{p}\delta(e_{2i+2}\otimes_{\pi}u^p)$  for all  $i \ge 0$ , and is already in normal form with respect to these generators. Applying 6.20, 6.18 and 6.17, we obtain

$$g^{\#}(e_{2i+2} \otimes_{\pi} u^{p}) = g_{\#} e_{2i+2} \otimes_{\pi} y u^{p}$$
  
=  $(-1)^{q} k^{i+1} e_{2i+2} \otimes_{\pi} u^{p}$ . (7.5)

From this it follows that  $g^*$  transforms  $L_1$  into itself.

Define  $L_2$  to consist of all cochains having canonical forms of the type

$$e_s \otimes_{\pi} c_p \quad \text{for} \quad s > 0 \quad .$$
 (7.6)

It is in normal form :

$$\delta(e_{2i} \otimes_{\pi} v^p) = p e_{2i-1} \otimes_{\pi} v^p \quad , \quad i \ge 1 \quad . \tag{7.7}$$

It is obvious from 6.18 and 6.20 that  $g^*$  maps  $L_2$  into itself.

We shall describe L by imposing conditions on the initial and final terms of a cochain in canonical form as follows:

- (7.8) If the initial term is  $e_s \otimes_{\pi} c_0$  with s > 0, see 7.2, we require that  $c_0 = 0$ .
- (7.9) If the final term is  $e_t \otimes_{\pi} c_p$  with t odd, we require that  $c_p = 0$ .
- (7.10) If the final term is  $e_t \otimes_{\pi} c_p$  with t even and positive, we require that  $c_p = \frac{1}{n} \delta c_{p-1}$ .

Of course, all cochains  $e_0 \otimes_{\pi} c_p$  are in L. The condition 7.8 is obviously stable under  $\delta$ . If t is odd and >2

$$\delta(e_{t-1}\otimes_{\pi}c_{p-1}) = \Sigma e_{t-2}\otimes_{\pi}c_{p-1} + e_{t-1}\otimes_{\pi}\delta c_{p-1}$$
$$= e_{t-2}\otimes_{\pi}\Sigma c_{p-1} + e_{t-1}\otimes_{\pi}\frac{1}{p}\delta\Sigma c_{p-1}$$

because  $\delta c_{p-1}$  is a multiple of  $v^p$ . Therefore a cochain satisfying 7.9 has a coboundary satisfying 7.10. If t is even and >1

$$\begin{split} \delta \Big( e_{i-1} \otimes_{\pi} c_{p-1} + e_i \otimes_{\pi} \frac{1}{p} \, \delta c_{p-1} \Big) \\ &= \varDelta e_{i-2} \otimes_{\pi} c_{p-1} - e_{i-1} \otimes_{\pi} \delta c_{p-1} + \varSigma e_{i-1} \otimes_{\pi} \frac{1}{p} \, \delta c_{p-1} \\ &= e_{i-2} \otimes_{\pi} (T^{-1} - 1) c_{p-1} + e_{i-1} \otimes_{\pi} \Big( - \, \delta c_{p-1} + \varSigma \frac{1}{p} \, \delta c_{p-1} \Big) \; . \end{split}$$

The last term is zero since  $\delta c_{p-1}$  is a multiple of  $v^p$ . Therefore a cochain satisfying 7.10 has a coboundary satisfying 7.9. This proves that L is a cochain subcomplex.

The conditions 7.8, 7.9 are obviously stable under the chain mapping  $g^*$ . As for 7.10, we have

$$g^{*}\left(e_{2i-1}\otimes_{\pi}c_{p-1} + e_{2i}\otimes_{\pi}\frac{1}{p}\,\delta c_{p-1}\right)$$
  
=  $\left(k^{i-1}\sum_{m=0}^{k-1}T^{m}e_{2i-1}\right)\otimes_{\pi}yc_{p-1} + k^{i}e_{2i}\otimes_{\pi}\frac{1}{p}\,y\,\delta c_{p-1}$   
=  $e_{2i-1}\otimes_{\pi}\left\{k^{i-1}\sum_{m=0}^{k-1}T^{p-m}yc_{p-1}\right\} + e_{2i}\otimes_{\pi}\frac{1}{p}\,k^{i}y\,\delta c_{p-1}$ 

Since  $\delta c_{p-1}$  is a multiple of  $v^p$ , 6.17 gives

$$T^{p-m}y\,\delta c_{p-1} = (-1)^{q+1}\,\delta c_{p-1}$$

and therefore

$$\frac{1}{p}\,\delta\{k^{i-1}\sum_{m=0}^{k-1}T^{p-m}yc_{p-1}\} = \frac{1}{p}\,k^iy\,\delta c_{p-1}$$

This shows that condition 7.10 is stable under  $g^{*}$ . Therefore  $g^{*}$  maps L into itself.

It remains to show that the entire complex is the direct sum of the three subcomplexes. That  $L_1 \cap L_2 = 0$  is clear by comparing 7.3 and 7.6, i.e.  $c_p = 0$  for any element of  $L_1$ , and  $c_p \neq 0$  for a non-zero element of  $L_2$ . A non-zero element of  $L_1 + L_2$  has a non-zero  $c_0$  if its component in  $L_1$  is nonzero, or else it lies in  $L_2$  and then  $c_{p-1} = 0$  and  $c_p \neq 0$ . In the first case 7.8 does not hold, in the second neither 7.9 nor 7.10 could hold. Thus

$$L \cap (L_1 + L_2) = 0$$
.

Given any cochain in normal form, if it has an initial term  $e_s \otimes_{\pi} c_0$  with s > 0and  $c_0 \neq 0$ , we may subtract from it an element of  $L_1$  (the first or second element of 7.3 according as s is even or odd) and obtain a cochain satisfying 7.8. If the resulting cochain has a final term  $e_t \otimes_{\pi} c_p$  with t odd and  $c_p \neq 0$ , we subtract  $e_t \otimes_{\pi} c_p$  in  $L_2$ , and obtain a cochain satisfying both 7.8 and 7.9 which is therefore a cochain of L. On the other hand, if the final term  $e_t \otimes_{\pi} c_p$ has t even and >0, we subtract  $e_t \otimes_{\pi} \left(c_p - \frac{1}{p} \delta c_{p-1}\right)$  in  $L_2$ , and obtain a cochain satisfying both 7.8 and 7.10 which is therefore in L. This completes the proof of the direct sum decomposition.

# §8. Cohomology operations obtained from $L_1$ and $L_2$

The subcomplex  $L_1$  defined in 7.3 is in the normal form

$$\delta(e_{2i}\otimes_{\pi}u^p) = p\left(e_{2i-1}\otimes_{\pi}u^p + e_{2i}\otimes_{\pi}\frac{1}{p}\,\delta u^p\right) , \quad i > 1 . \tag{8.1}$$

According to 4.11, we have only to identify the cohomology operation corresponding to each of the cocycles  $e_{2i} \otimes_{\pi} u^p \mod p$ . By 6.20, 18 and 17

$$g^{*}(e_{2i}\otimes_{\pi}u^{p})=g_{*}e_{2i}\otimes_{\pi}yu^{p}=(-1)^{q}k^{i}e_{2i}\otimes_{\pi}u^{p}$$

Therefore, by 6.4, the cocycle

$$(g^{*}-1)(e_{2i}\otimes_{\pi}u^{p}) = [(-1)^{q}k^{i}-1]e_{2i}\otimes_{\pi}u^{p}$$
(8.2)

belongs to the kernel of  $h_*$ , and therefore is zero as a cohomology operation.

If q is even and i is not a multiple of p-1, the coefficient  $(-1)^q k^i - 1$  is non-zero mod p because k is a primitive root. Working mod p, we may divide 8.2 by this coefficient, and conclude that  $e_{2i} \otimes_{\pi} u^p$  represents zero as a cohomology operation.

Again let q be even, and suppose i = s(p-1). In this case the coefficient in 8.2 is zero mod p, so it imposes no relation on the cohomology operation corresponding to  $e_{2i} \otimes_{\pi} u^{p}$ . The operation is in fact a suitable multiple of the cyclic reduced power  $\mathcal{P}^{\frac{1}{2}q-s}$ , namely:

$$\Phi\{e_{2s(p-1)}\otimes_{\pi}u^{p}\}=(-1)^{s}\mathcal{P}^{\frac{1}{2}q-s}\overline{u} \quad . \tag{8.3}$$

In this formula  $\Phi$  is as defined in [2; 2.11], and the braces  $\{\}$  mean to take the cohomology class of the cocycle enclosed. To prove 8.3, we must recall the definition [4;6.8] of  $\mathcal{P}^t$  namely

$$\mathcal{P}^{t}\overline{u} = (-1)^{mt + mq(q-1)/2} (m!)^{2t-q} \overline{u}^{p} / e_{(q-2t)(p-1)} .$$
(8.4)

In this formula  $m = \frac{1}{2}(p-1)$ , and the coefficient is computed in the field  $Z_p$ . Also, by [4; 2.8]

$$(\overline{u}^{p}/e) \cdot \sigma = \overline{u}^{p} \cdot \varphi' (e \otimes \sigma)$$

where  $\varphi': W \otimes K \to K^p$  has the same meaning as in [2; 2.6]. Comparing this with [2; 2.8], we obtain

$$\Phi\{e_{2i} \otimes_{\pi} u^{p}\} = (-1)^{i} \overline{u}^{p} / e_{2i} \quad . \tag{8.5}$$

In 8.4, we take q - 2t = 2s (i. e.  $t = \frac{1}{2}q - s$ ), in 8.5 we take i = s(p - 1), then we eliminate  $\overline{u}^{p}/e$  between the two equations, and obtain 8.3. In the computation one must use properties of m!, namely, by WILSON's theorem,

m! is non-zero mod p; if m is even,  $(m !)^2 \equiv -1$ ; if m is odd,  $(m !)^2 \equiv 1$ ; so in either case  $(m !)^2 \equiv (-1)^{m+1}$ .

Now let q be odd. The coefficient in 8.2 becomes  $-(k^i + 1)$ . Since k is a primitive root of p,  $k^i + 1 \equiv 0 \mod p$  if and only if i is an odd multiple of  $\frac{1}{2}(p-1)$ . If this is not the case, we may divide 8.2 by  $k^i + 1$ , and conclude as before that  $e_{2i} \otimes_{\pi} u^p$  represents zero as a cohomology operation.

Let q be odd, and suppose  $i = (2s + 1)\frac{1}{2}(p - 1)$ . In this case, the cocycle corresponds again to a suitable  $\mathcal{P}^t$ , namely

$$\Phi\left\{e_{(2s+1)(p-1)}\otimes_{\pi}u^{p}\right\} = (-1)^{s+m}(m!)\mathcal{P}^{\frac{1}{2}(q-1)-s}\overline{u} \quad . \tag{8.6}$$

As in the case of 8.3, this is derived from 8.4 and 8.5 by setting

$$t=\frac{1}{2}(q-1)-s$$

in 8.4, i = (2s + 1)m in 8.5, and eliminating  $\overline{u}^{p}/e$ . This completes the analysis of the cohomology operations derived from  $L_1$ .

The subcomplex  $L_2$  defined in 7.6 is already in normal form (see 7.7). Let M' be the subcomplex of M generated by v. Let  $\psi: M \to K^*$  be a cochain map representing the class  $\overline{u}$ . Then  $\psi \mid M' = \psi': M' \to K^*$  represents the cohomology class  $\overline{v} \in H^{q+1}(K; Z)$  containing the cocycle  $\psi'(v)$ . Then  $\overline{v} = \delta^* \overline{u}$  where  $\delta^*$  is the BOCKSTEIN coboundary for the coefficient sequence

$$0 \to Z \xrightarrow{\theta} Z \to Z_{\theta} \to 0$$

(see [2; 10.1]). Define  $L'_1$  in  $W \otimes_{\pi} M'^p$  in the same manner as  $L_1$  in  $W \otimes_{\pi} M^p$ , replacing u by v and q by q + 1. It is seen that, under the inclusion mapping  $W \otimes_{\pi} M'^p \subset W \otimes_{\pi} M^p$ , we have  $L'_1 = L_2$ . The analysis given above for  $L_1$ applies to  $L'_1$  and hence to  $L_2$ . It follows that each  $e_{2i} \otimes_{\pi} v^p$  corresponds to a cohomology operation which is zero or to a suitable multiple of  $\mathcal{P}^t \delta^*$ .

In the special case  $\theta = 0$ , we set v = 0. Then  $W \otimes_{\pi} M^{p}$  reduces to the normal form:  $\delta(e_{0} \otimes_{\pi} u^{p}) = 0$  and

$$\delta(e_{2i}\otimes_{\pi} u^p) = pe_{2i-1}\otimes_{\pi} u^p$$
,  $i \ge 1$ .

Now  $e_0 \otimes_{\pi} u^p$  corresponds to the *p*th power operation in the sense of cup products with integer coefficients (see [2; 10.1-4]). The remaining cocycles lie in  $L_1$  and have already been analysed. This concludes the case  $\theta = 0$ , and we may suppose  $\theta = p^k$  henceforth.

### §9. The equivalence of L and $\Sigma M^p$

The analysis of L (see 7.8–10) is more complicated and devious. The remaining two sections are devoted to the task. The conclusion however is not complicated to state:

9.1. If q is odd, each cohomology operation derived from L is elementary. If q is even, the only non-elementary operations derivable from L are obtained from cocycles of the elementary subcomplex

$$\delta[e_0 \otimes_\pi u^p + e_1 \otimes_\pi \Sigma^* u^{p-1} v] = p\theta \left[ e_0 \otimes_\pi u^{p-1} v - e_1 \otimes_\pi \Sigma^* \frac{1}{p} \,\delta(u^{p-1} v) \right], \quad (9.2)$$

where  $\Sigma^*$  in the group ring of  $\pi$  is defined by

$$\Sigma^* = \sum_{k=1}^{p-1} k T^{p-k} .$$
 (9.3)

In checking the formula 9.2, the following identity is useful :

$$(T^{-1} - 1) \Sigma^* = -\Sigma + p 1 . (9.4)$$

The cochain on the left of 9.2 is a cocycle mod  $p\theta$ , and its corresponding cohomology operation is the PONTRJAGIN *pth* power as defined by THOMAS [5; 3.3, 3.4]. Once 9.1 is proved the proof of the main result of this paper will be complete.

Define  $\Sigma M^p$  to be the subcomplex of  $M^p$  consisting of cochains of the form  $\Sigma c$  where c is a cochain of  $M^p$ . Define a cochain mapping

$$f: L \to \Sigma M^p \tag{9.5}$$

as follows. If the initial term of a cochain in canonical form is  $e_j \otimes_{\pi} c_0$  where j > 0, its image under f is 0. If its initial term is  $e_0 \otimes_{\pi} c_j$ , its image is  $\Sigma c_j$ :

$$f(e_j \otimes_{\pi} c_0 + \cdots + e_{j+p} \otimes_{\pi} c_p) = 0 \quad , \qquad j > 0 \tag{9.6}$$

$$f(e_0 \otimes_{\pi} c_j + \dots + e_{p-j} \otimes_{\pi} c_p) = \Sigma c_j \quad . \tag{9.7}$$

Clearly f is a homomorphism. To prove  $\delta f = f\delta$ , we suppose in the first case j = 2i > 0. Then the initial term of  $\delta(e_{2i} \otimes_{\pi} c_0 + \cdots)$  is  $e_{2i-1} \otimes_{\pi} \Sigma c_0$ ; and so its f-image is zero. Suppose next that j = 2i + 1 > 0. Then the initial term of  $\delta(e_{2i+1} \otimes_{\pi} c_0 + \cdots)$  is  $e_{2i} \otimes_{\pi} (T^{-1} - 1)c_0$ . If i > 0, 9.6 applies, and its f-image is zero. If i = 0, 9.7 applies, and its f-image is  $\Sigma(T^{-1} - 1)c_0 = 0$ . In the second case, the initial term of  $\delta(e_0 \otimes_{\pi} c_j + \cdots)$  is

$$e_0 \otimes_{\pi} [\delta c_j + (T^{-1} - 1)c_{j+1}]$$
.

Thus, its *f*-image is

$$\Sigma \delta c_j = \delta \Sigma c_j = \delta f(e_0 \otimes_{\pi} c_j + \cdots)$$

Therefore,  $\delta f = f \delta$  in all cases.

9.8. The automorphism y of  $M^p$  (defined by 6.15) transforms  $\Sigma M^p$  into itself. If  $g^*$  is the cochain mapping 6.20 restricted to L, then  $fg^*(z) = yf(z)$  for all cochains z of L.

The first assertion follows from  $y\Sigma = \Sigma y$  which is an immediate consequence of 6.16. To prove the second, suppose z is such that 9.6 applies. Then yf(z) = y(0) = 0. By 6.18 and 6.20,  $g^{\#}(z)$  has the same type of initial term as z, so  $fg^{\#}(z) = 0$ . If z is as in 9.7, then  $yf(z) = y(\Sigma c_j) = \Sigma y c_j$ . But  $g^{\#}(z)$  has the initial term  $e_0 \otimes_{\pi} y c_j$  so  $fg^{\#}(z)$  is  $\Sigma y c_j$ .

# 9.9. If J denotes the kernel of f, then J is acyclic. This implies

$$f_*: H^r(L \otimes G) \approx H^r(\Sigma M^p \otimes G)$$

Recall that the cochains of L are defined by conditions 7.8-10. In particular, then, a cochain of J in canonical form has a first non-zero term of the form  $e_i \otimes_{\pi} c_j$ , where  $i \ge 0$ , and 0 < j < p. If i = 0, then  $\Sigma c_j = 0$ , since the cochain is in J. Suppose that i is even and >0, and that the cochain is a cocycle. This again implies that  $\Sigma c_j = 0$ . Thus, in either case, we must have  $c_j = (T^{-1} - 1)d$  for some d, since  $M^p$  is free in the dimension j. Then

$$\delta(e_{i+1} \otimes_{\pi} d) = e_i \otimes_{\pi} c_j - e_{i+1} \otimes_{\pi} \delta d .$$

Subtracting this from the cocycle gives a cohomologous cocycle whose first non-zero term has an index >i. If i is odd, then we have  $(T^{-1}-1)c_j = 0$ . The freeness of  $M^p$  implies that  $c_j = \Sigma d$ , and

$$\delta(e_{i+1} \otimes_{\pi} d) = e_i \otimes_{\pi} c_j + e_{i+1} \otimes_{\pi} \delta d$$

Subtracting this gives again a cohomologous cocycle whose first non-zero term has an index >i.

Repeating the process we obtain eventually a cohomologous cocycle having one of the two forms (see 7.9, 10)

$$e_{2i} \otimes_{\pi} c_{p-1}$$
 or  $e_{2i-1} \otimes_{\pi} c_{p-1} + e_{2i} \otimes_{\pi} \frac{1}{p} \delta c_{p-1}$ .

In the first case, we alter the first method of the preceding paragraph by observing that

$$\delta\left(e_{2i+1}\otimes_{\pi}d + e_{2i+2}\otimes_{\pi}\frac{1}{p}\,\delta d\right) = e_{2i}\otimes_{\pi}c_{p-1}$$

In the second case, we apply the second method unaltered, and observe that  $\delta d = \frac{1}{p} \, \delta c_{p-1}$  because  $T v^p = v^p$ . This shows that every cocycle of J is a coboundary, and completes the proof of 9.9.

## §10. The cohomology of $\Sigma M^p$

By [2; 11.7], each element of  $H^r(W \otimes_{\pi} M^p)$  has an order dividing  $p\theta$ . The same must be true of  $H^r(L)$  and, by 9.9, of  $H^r(\Sigma M^p)$ . It follows that a

normal form for  $\Sigma M^p$  will consist of elementary subcomplexes whose torsion numbers divide  $p\theta$ .

10.1. If P is an elementary subcomplex of  $\Sigma M^p$  whose torsion number m divides  $\theta$ , then there is a mapping  $\zeta: P \to M^p$  such that  $\Sigma \zeta$  is the identity. Therefore each cohomology operation corresponding to a cohomology class of P is elementary.

We may denote the generators of P by  $\Sigma c$  and  $\Sigma d$  with  $\delta \Sigma c = m \Sigma d$ . Since each coboundary in  $M^p$  is divisible by  $\theta$ , we have  $\delta c = \theta d'$ , and d' is a cocycle. It follows that  $\theta \Sigma d' = m \Sigma d$ . If we set

$$\zeta(\Sigma c) = c \; , \quad \zeta(\Sigma d) = \frac{\theta}{m} \, d'$$

we obtain the required  $\zeta$ . If w is any cocycle of  $P \otimes G$ , then  $e_0 \otimes_{\pi} \zeta w$  is a cocycle of  $L \otimes G$  whose image in  $\Sigma M^p \otimes G$  under f is w. By 3.3,  $e_0 \otimes_{\pi} \zeta w$  and therefore w corresponds to an elementary operation.

Because of this result, we have only to analyse the torsions of order exactly  $p\theta$  in  $\Sigma M^p$ . To this end, we define a category N of cochain complexes having certain properties of  $M^p$ . A cochain complex N belongs to N if

- (10.2)  $\pi$  operates as automorphisms of N.
- (10.3)  $C^{j}(N) = 0$  if j < pq or j > p(q + 1).
- (10.4)  $C^{pq}(N)$  has a single generator  $\alpha_0$  fixed under  $\pi$ .
- (10.5)  $C^{p(q+1)}(N)$  has a single generator  $\alpha_p$  fixed under  $\pi$ .
- (10.6)  $C^{pq+j}(N)$  is  $\pi$ -free if 0 < j < p.
- (10.7) An integral cocycle of N is a coboundary if and only if it is divisible by  $\theta$ .

A mapping  $\lambda: N \to N'$  of the category N is a cochain mapping which is  $\pi$ -equivariant. Let  $k_{\lambda}$  be the integer such that  $\lambda \alpha_{p} = k_{\lambda} \alpha'_{p}$ . Under compositions of two mappings  $\lambda, \mu$ , we have

$$k_{\mu\lambda} = k_{\mu}k_{\lambda} \ . \tag{10.8}$$

10.9. If N, N' are in N, and k is an integer, then there exists a mapping  $\lambda$ :  $N \rightarrow N'$  such that  $k_{\lambda} = k$ .

This is proved by a downward induction on the dimension. Start by setting  $\lambda \alpha_p = k \alpha'_p$ . Suppose  $\lambda$  has been properly defined in dimensions > pq + j. If j > 0, by 10.6 we can choose a  $\pi$ -basis  $\{\beta_i\}$  of  $C^{pq+j}$ . By 10.7,  $\delta \beta_i = \theta \gamma_i$ . Then  $\lambda \gamma_i$  is defined. Since  $\delta \lambda \gamma_i = \lambda \delta \gamma_i = 0$ ,  $\lambda \gamma_i$  is a cocycle. By 10.7,  $\theta \lambda \gamma_i$  is the coboundary of some cochain. We select one such and denote it by  $\lambda \beta_i$ . For any  $x \epsilon \pi$ , we define  $\lambda x \beta_i = x \lambda \beta_i$ . Since  $\lambda$  is  $\pi$ -equivariant in the dimension pq + j + 1, it follows that  $\lambda \delta x \beta_i = \delta \lambda x \beta_i$ . In this fashion the induction continues down to the dimension pq. To define  $\lambda$  on  $\alpha_0$ , we prove as above that  $\lambda \delta \alpha_0$  is a cocycle and it is divisible by  $\theta$ . Hence we may choose a cochain  $\lambda \alpha_0$  such that  $\delta \lambda \alpha_0 = \lambda \delta \alpha_0$ . By 10.4,  $C^{pq}(N')$  has a single generator fixed under  $\pi$ . Therefore  $\lambda \alpha_0$  is fixed and so  $\lambda$  is equivariant.

10.10. If N, N' are in N and  $\lambda_0, \lambda_1$  are two mappings  $N \to N'$  of N such that

$$k_{\lambda_0} \equiv k_{\lambda_1} \bmod p$$
 ,

then there exists an equivariant cochain homotopy

$$D: \ \theta \, \lambda_0 \simeq \theta \, \lambda_1$$
 .

i. e. for each j, D is a  $\pi$ -homomorphism of  $C^{j}(N)$  into  $C^{j-1}(N')$  such that

$$\delta D \alpha = \theta \lambda_1 \alpha - \theta \lambda_0 \alpha - D \delta \alpha$$
,  $\alpha \epsilon C^j(N)$ . (10.11)

This is also proved by a downward induction. By hypothesis

$$\theta \lambda_1 \alpha_p - \theta \lambda_0 \alpha_p = \theta (k_{\lambda_1} - k_{\lambda_0}) \alpha'_p = pr \theta \alpha'_p$$

for some r. By 10.7,  $r\theta \alpha'_p$  is a coboundary of some cochain, say  $\gamma$ . Then

$$\delta \Sigma \gamma = \Sigma r \theta \alpha'_p = p r \theta \alpha'_p$$
.

So we may set  $D\alpha_p = \Sigma \gamma$ , and D is equivariant and satisfies 10.11 with  $\alpha = \alpha_p$ .

Suppose *D* has been defined properly in dimensions >pq + j. Let  $\{\beta_i\}$  be a  $\pi$ -free basis in dimension pq + j (assuming j > 0). Then the right side of 10.11 is defined for  $\alpha = \beta_i$ . The standard calculation shows that it is a cocycle. It is also divisible by  $\theta$  because  $\delta\beta_i$  is divisible by  $\theta$ . Hence it is a coboundary of some cochain, we define  $D\beta_i$  to be one such. We extend *D* to be a  $\pi$ -homomorphism, and then verify that 10.11 still holds in the dimension pq + j. When j = 0, we set  $D\alpha_0 = 0$ . This is clearly equivariant. Also the right side of 10.11 with  $\alpha = \alpha_0$  must be zero. For it is a cocycle divisible by  $\theta$ , and hence it is a coboundary; but by 10.3,  $C^{pq-1}(N') = 0$ .

We introduced the category N for the purpose of studying the subcomplex  $\Sigma M^p$ . Now if N and N' are any complexes in N, we compare the cohomology of  $\Sigma N$  and  $\Sigma N'$  as follows:

10.12. If  $\lambda: N \to N'$  is in N and  $k_{\lambda}$  is prime to p, then  $\lambda$  induces an isomorphism  $\lambda_*: \theta H^j(\Sigma N) \approx \theta H^j(\Sigma N')$ .

By 10.9, there is a  $\mu: N' \to N$  such that  $k_{\mu}k_{\lambda} \equiv 1 \mod p$ . By 10.8 and 10.10, there is a cochain homotopy D of  $\theta \mu \lambda$  into  $\theta 1$ , where 1 is the identity map. Then if  $\Sigma_{\alpha}$  is any cocycle of  $\Sigma N$ , 10.11 gives

$$\delta D \Sigma \alpha = \delta \Sigma D \alpha = \theta \mu \lambda \Sigma \alpha - \theta \Sigma \alpha$$

Therefore  $(\mu \lambda)_* = \mu_* \lambda_*$  induces the identity mapping of  $\theta H^j(\Sigma N)$ . By the symmetry of the situation,  $\lambda_* \mu_*$  induces the identity in  $\theta H^j(\Sigma N')$ . This proves 10.12.

By 10.9, there are maps  $\lambda$  with  $k_{\lambda} = 1$ . It follows that  $\theta H^{j}(\Sigma N)$  has the same structure for all N in N. To compute these groups we construct a simplest  $N_{0}$  in N as follows. It has a single fixed generator  $\alpha_{0}$  for  $C^{pq}(N_{0})$  and  $\alpha_{p}$  for  $C^{p(q+1)}(N_{0})$ , and has a single  $\pi$ -free generator  $\alpha_{j}$  for  $C^{pq+j}(N_{0})$ , 0 < j < p. Define  $\delta$  by

$$\delta \alpha_{2i} = \theta \Sigma \alpha_{2i+1} , \quad \delta \alpha_{2i+1} = \theta \Delta \alpha_{2i+2} , \qquad (10.13)$$

for  $0 \le i \le (p-1)/2$ . The conditions 10.2 to 10.6 are trivially true. The truth of 10.7 follows from the fact that each of  $\Delta$ ,  $\Sigma$  generates the annihilator of the other in the group ring of  $\pi$ .

The complex  $\Sigma N_0$  has a single generator  $\Sigma \alpha_j$  for each  $0 \leq j \leq p$ . The coboundary relations are

$$\delta \Sigma \alpha_{2i} = p \theta \Sigma \alpha_{2i+1}$$
,  $\delta \Sigma \alpha_{2i+1} = 0$ . (10.14)

Thus  $\Sigma N_0$  is in normal form, and it has torsion of order  $p\theta$  in every other dimension from pq to p(q+1). Since  $M^p$  is in N, 10.12 implies that the same conclusion holds for  $\Sigma M^p$ . These results are summarized in :

10.15. The torsion numbers  $= p\theta$  of  $\Sigma M^p$  occur just once in every other dimension from pq to p(q+1). One obtains elementary subcomplexes of  $\Sigma M^p$ containing these torsions by taking the  $\lambda$ -image of  $\Sigma N_0$  where  $\lambda : N_0 \to M^p$  is an equivariant mapping such that  $k_{\lambda}$  is prime to p.

Cocycles of order  $p\theta$  obtained in this way are not generally in the image  $H(M^p) \to H(\Sigma M^p)$  under  $\Sigma$ . However in all but one exceptional case they correspond to elementary cohomology operations. To see this we must study the behavior of such cocycles under the automorphism y of  $\Sigma M^p$  which, by 9.8, corresponds to the chain mapping  $g^*$  of L.

Let  $\lambda: N_0 \to M^p$  be a fixed equivariant mapping such that  $k_{\lambda} = 1$ . Now  $y\lambda$  is a cochain mapping  $N_0 \to M^p$  but it is not equivariant because  $yT = T^k y$  where k is a primitive root of p. However for each integer i in the range 0 to  $\frac{1}{2}(p-1)$  we shall construct an equivariant mapping  $\mu$  depending on i such

that  $y\lambda$  and  $\mu$  coincide on  $C^{pq+2i}(\Sigma N_0)$ . Let the integer *m* be the inverse mod *p* of the primitive root *k*. Set

$$\Lambda = \sum_{j=0}^{k-1} T^j , \quad \Gamma = \sum_{j=0}^{m-1} T^j$$

in  $Z(\pi)$ . In the group ring of the normalizor of  $\pi$ , we obtain readily the relations

$$y\Sigma = \Sigma y, \quad y\Delta = \Delta \Lambda y, \quad \Delta y = y\Gamma\Delta$$
 (10.16)

and therefore

$$\Lambda^{s} y \Delta = \Delta \Lambda^{s+1} y, \quad y \Gamma^{s} \Delta = \Delta y \Gamma^{s+1}, \quad s \ge 0 .$$
 (10.17)

Define an equivariant mapping  $\mu: N_0 \to M^p$  by specifying its values on basis elements as follows

$$\begin{split} \mu(\alpha_{2i+2s}) &= \Lambda^s y \lambda(\alpha_{2i+2s}) , \qquad s = 0, 1, \dots, \frac{1}{2}(p-1-2i) , \\ \mu(\alpha_{2i+2s+1}) &= \Lambda^s y \lambda(\alpha_{2i+2s+1}) , \qquad s = 0, 1, \dots, \frac{1}{2}(p-1-2i) , \\ \mu(\alpha_{2i-2s}) &= y \Gamma^s \lambda(\alpha_{2i-2s}) , \qquad s = 0, 1, \dots, i , \\ \mu(\alpha_{2i-2s+1}) &= y \Gamma^s \lambda(\alpha_{2i-2s+1}) , \qquad s = 0, 1, \dots, i . \end{split}$$

The relation  $\delta \mu = \mu \delta$  follows directly from 10.13, 10.16, 10.17. When restricted to  $\Sigma N_0$ ,  $\mu$  takes the form

$$\mu(\Sigma \alpha_{2i+2s}) = k^{s} y \lambda(\Sigma \alpha_{2i+2s}) ,$$
  

$$\mu(\Sigma \alpha_{2i+2s+1}) = k^{s} y \lambda(\Sigma \alpha_{2i+2s+1}) ,$$
  

$$\mu(\Sigma \alpha_{2i-2s}) = m^{s} y \lambda(\Sigma \alpha_{2i-2s}) ,$$
  

$$\mu(\Sigma \alpha_{2i-2s+1}) = m^{s} y \lambda(\Sigma \alpha_{2i-2s+1}) .$$
(10.18)

The reason for this is that  $y\Sigma = \Sigma y$ ,  $\Lambda\Sigma = k\Sigma$ , and  $\Gamma\Sigma = m\Sigma$ . Taking  $s = \frac{1}{2}(p-1) - i$ , we have

$$\mu(\Sigma \alpha_p) = k^{\frac{1}{2}(p-1)-i} y \lambda(\Sigma \alpha_p)$$

Now  $\alpha_p$  and  $\lambda \alpha_p = v^p$  are fixed under  $\pi$ ; so, by 6.17

$$\mu(\alpha_p) = (-1)^{q+1} k^{\frac{1}{2}(p-1)-i} v^p$$

Consider now the mapping  $\mu - \lambda$  of  $N_0$  into  $M^p$ . It is obviously equivariant; and  $(\mu - \lambda)\alpha_p = [(-1)^{q+1}k^{\frac{1}{2}(p-1)-i} - 1]v^p$ . (10.19)

Because k is a primitive root of p, the only case where the coefficient of  $v^p$  is divisible by p is the case q even and i = 0. In any other case,  $k_{\mu-\lambda}$  is prime to p; and so, by 10.15,  $\mu - \lambda$  applied to  $\Sigma N_0$  gives the torsions of order  $p\theta$  of  $\Sigma M^p$ . By 10.18, we have

$$(\mu - \lambda)\Sigma \alpha_j = (y - 1)\lambda\Sigma \alpha_j$$
 for  $j = 2i$  and  $2i + 1$ . (10.20)

For convenience set

$$\lambda \Sigma \alpha_{2i} = \beta$$
 and  $\lambda \Sigma \alpha_{2i+1} = \gamma$ .

Both  $\gamma$  and  $(y-1)\gamma$  are cocycles generating torsion of order  $p\theta$  in  $\Sigma M^p$ . Since  $\theta H^{pq+2i+1}(\Sigma M^p)$  is cyclic of order p, we must have  $\theta\gamma \sim b(y-1)\theta\gamma$  for some integer b prime to p; thus

$$\delta w = \theta \gamma - b(y-1)\theta \gamma$$
 for  $w \in \Sigma M^p$ 

Since  $\delta\beta = p\theta\gamma$ , it follows that  $\beta - b(y-1)\beta - pw$  is a cocycle. By 10.15,  $\theta H^{pq+2i}(\Sigma M^p) = 0$ . Therefore there is a cochain  $w_1 \epsilon \Sigma M^p$  such that

$$\delta w_1 = \theta \left(\beta - b(y-1)\beta - pw\right) \; .$$

By 10.1, any cocycle of this elementary complex corresponds to an elementary cohomology operation. Taking  $Z_{p\theta}$  as coefficient group, it follows that  $\beta - b(y-1)\beta$  is a cocycle and corresponds to an elementary operation. However  $(y-1)\beta$  corresponds to zero as a cohomology operation; this is seen by assembling 9.8, 6.20, 6.14 and 6.1. It follows that  $\beta$  corresponds to an elementary operation.

In the case q even and i = 0, it is clear that

$$\delta \Sigma u^{p} = p \,\delta u^{p} = p \,\theta \Sigma u^{p-1} v \tag{10.21}$$

is an elementary subcomplex giving the torsion of order  $p\theta$  in this case. By the preceding argument, the only non-elementary cohomology operations derivable from  $\Sigma M^p$  are obtained from this subcomplex. If we apply f of 9.5 to 9.2, we obtain 10.21. This completes the proof of 9.1 and, hence, our main result.

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