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Autor: Tanaka, Chuji
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On the singularities of DIRICHLET series

by CHUJI TANAKA, Tokyo

1. Introduction. Let us put

$$F(s) = \sum_{n=1}^{\infty} a_n \exp(-\lambda_n s) \quad (s = \sigma + it, \quad 0 \leq \lambda_1 < \lambda_2 < \dots < \lambda_n \rightarrow +\infty). \quad (1.1)$$

The object of this note is to establish the following theorems:

Theorem 1. *Let (1.1) have the finite simple convergence-abscissa σ_s . Then there exists a sequence $\{\varepsilon_n\}$ ($\varepsilon_n = \pm 1$) such that*

$$\sum_{n=1}^{\infty} \varepsilon_n a_n \exp(-\lambda_n s)$$

has $\sigma = \sigma_s$ as the natural boundary.

Theorem 2. *Let (1.1) have the finite simple convergence-abscissa σ_s . Then there exists a new DIRICHLET series $\sum_{n=1}^{\infty} b_n \exp(-\lambda_n s)$ having $\sigma = \sigma_s$ as the natural boundary such that*

$$|b_n| = |a_n| \quad (n = 1, 2, \dots), \quad \lim_{n \rightarrow \infty} |\arg(b_n) - \arg(a_n)| = 0 \quad (i)$$

or

$$\arg(b_n) = \arg(a_n) \quad (n = 1, 2, \dots), \quad \lim_{n \rightarrow \infty} |b_n/a_n| = 1. \quad (ii)$$

Under the additional conditions $\lim_{n \rightarrow \infty} \log n/\lambda_n = 0$, these two theorems have been proved by O. SZASZ ([1], p. 107) and the author ([2], p. 308) respectively. The method of its proofs is based upon A. OSTROWSKI's criterion of singularities.

2. Lemmas. To establish these theorems, we need some lemmas.

Lemma 1. (A. OSTROWSKI [3, 4], pp. 12–16.) *Let (1.1) have the simple convergence-abscissa $\sigma_s = 0$. For $s = 0$ to be the singular point of (1.1), it is necessary and sufficient that we have*

$$\overline{\lim}_{m \rightarrow \infty} |O_m(\sigma, \omega)|^{1/m} \geq 1 \quad (m : \text{positive integer}),$$

where

$$\sigma (> 0), \quad \omega (0 < \omega < 1) : \text{fixed constants}, \quad (i)$$

$$O_m(\sigma, \omega) = \sum_{m/\sigma \cdot (1-\omega) \leq \lambda_n \leq m/\sigma \cdot (1+\omega)} a_n (\lambda_n e^{\sigma/m})^m \exp(-\lambda_n \sigma). \quad (ii)$$

Lemma 2. *The simple convergence-abscissa σ_s of (1.1) is determined by*

where

$$\sigma_s = \overline{\lim}_{x \rightarrow +\infty} 1/x \cdot \log \left| \sum_{[x]+f(x) \leq \lambda_n < x} a_n \right| ,$$

(i)

$[x]$: GAUSS's symbol,

$$f(x) = 1/m \cdot [m(x - m)] , \quad m = [x] .$$

(ii)

Proof. By T. KOJIMA's theorem [5], σ_s is given by

$$\begin{aligned} \sigma_s &= \overline{\lim}_{x \rightarrow +\infty} 1/x \cdot \log \left| \sum_{[x] \leq \lambda_n < x} a_n \right| \\ &= \overline{\lim}_{x \rightarrow +\infty} 1/[x] \cdot \log \left| \sum_{[x] \leq \lambda_n < x} a_n \right| . \end{aligned}$$

Hence, for any given $\varepsilon (> 0)$,

$$\left| \sum_{[x] \leq \lambda_n < x} a_n \right| < \exp \{ [x](\sigma_s + \varepsilon) \} \quad \text{for } [x] > K(\varepsilon) .$$

Therefore

$$\begin{aligned} \left| \sum_{[x]+f(x) \leq \lambda_n < x} a_n \right| &= \left| \sum_{[x] \leq \lambda_n < x} - \sum_{[x] \leq \lambda_n < x+f(x)} \right| \\ &< 2 \exp \{ [x](\sigma_s + \varepsilon) \} \quad \text{for } [x] > K(\varepsilon) , \end{aligned}$$

so that

$$\sigma_s^* = \overline{\lim}_{x \rightarrow +\infty} 1/x \cdot \log \left| \sum_{[x]+f(x) \leq \lambda_n < x} a_n \right| \leq \sigma_s + \varepsilon .$$

Letting $\varepsilon \rightarrow 0$,

$$\sigma_s^* \leq \sigma_s .$$

(2.1)

Putting

$$\begin{cases} x = m + y/m & (m = [x], \quad 0 < y < m) , \\ [x] + f(x) = m + [y]/m , \end{cases}$$

we have easily

$$\sigma_s^* = \overline{\lim}_{\substack{m \rightarrow +\infty \\ 0 < y < m}} 1/m \cdot \log \left| \sum_{m+[y]/m \leq \lambda_n < m+y/m} a_n \right| \quad (m : \text{positive integer}).$$

Hence, for any given $\varepsilon (> 0)$,

$$\left| \sum_{\substack{m+[y]/m \leq \lambda_n < m+y/m \\ 0 < y < m}} a_n \right| < \exp \{ m(\sigma_s^* + \varepsilon) \} \quad \text{for } m > K^*(\varepsilon) .$$

Accordingly

$$\begin{aligned} &\left| \sum_{[x] \leq \lambda_n < x} a_n \right| \\ &= \left| \sum_{m \leq \lambda_n < m+1/m} + \sum_{m+1/m \leq \lambda_n < m+2/m} + \cdots + \sum_{m+[y]/m \leq \lambda_n < m+y/m} \right| \\ &\leq ([y] + 1) \exp \{ m(\sigma_s^* + \varepsilon) \} \leq m \exp \{ m(\sigma_s^* + \varepsilon) \} \quad \text{for } m > K^*(\varepsilon) , \end{aligned}$$

so that

$$\sigma_s = \overline{\lim}_{x \rightarrow +\infty} 1/x \cdot \log \left| \sum_{[x] \leq \lambda_n < x} a_n \right| \leq \sigma_s^* + \varepsilon .$$

Letting $\varepsilon \rightarrow 0$,

$$\sigma_s \leq \sigma_s^* . \quad (2.2)$$

Combining (2.1) with (2.2), lemma 2 is completely established.

Lemma 3. *Put*

$$O(x, t) = \sum_{[x] + f(x) \leq \lambda_n < x} a_n (\lambda_n e/[x])^{[x]} \exp(-\lambda_n(1+it)) .$$

If

$$\left| \sum_{[x] + f(x) \leq \lambda_n < x} a_n \right| \geq \left| \sum_{[x] + f(x) \leq \lambda_n < X} a_n \right|$$

for every X such that $[x] + f(x) < X \leq x$, then

$$|O(x, t)| \geq \left| \sum_{[x] + f(x) \leq \lambda_n < x} a_n \right| \cdot \{1 - (|t| + K(x))/[x]\} ,$$

where $K(x) = O(1)$.

Proof. Put

$$\begin{cases} x = m + y/m & (m = [x], 0 < y < m) , \\ [x] + f(x) = m + [y]/m . \end{cases}$$

Let us denote by $\{\lambda_{m,i}\}$ ($i = 1, 2, \dots, r$) the exponents λ'_n 's contained in $m + [y]/m \leq \lambda_n < m + y/m$ and by $\{a_{m,i}\}$ the coefficient corresponding to $\lambda_{m,i}$. Setting

$$S_k = \sum_{i=1}^k a_{m,i} \quad (1 \leq k \leq r), \quad f(\lambda) = \lambda^m \exp(-\lambda(1+it)) ,$$

then by ABEL's transformation

$$\begin{aligned} O(x, t) &= (e/m)^m \cdot \sum_{i=1}^r a_{m,i} f(\lambda_{m,i}) \\ &= (e/m)^m \cdot \{S_r f(\lambda_{m,r}) + \sum_{i=1}^{r-1} S_i (f(\lambda_{m,i}) - f(\lambda_{m,i+1}))\} . \end{aligned}$$

Since

$$|S_r| \geq |S_i| \quad (i = 1, 2, \dots, r-1) ,$$

we have

$$|O(x, t)| \geq (e/m)^m \cdot |S_r| \cdot \left\{ |f(\lambda_{m,r})| - \sum_{i=1}^{r-1} |f(\lambda_{m,i}) - f(\lambda_{m,i+1})| \right\} . \quad (2.3)$$

On the other hand, $|f(\lambda)| = \lambda^m \exp(-\lambda)$ is monotone-decreasing for $m \leq \lambda$. Hence, for $m \leq \alpha < \beta < m + 1$,

$$|f(\alpha) - f(\beta)| \leq |f(\alpha)| - |f(\beta)| + |f(\beta)|(\beta - \alpha)|t| ,$$

so that

$$|f(\alpha) - f(\beta)| < \int_{\alpha}^{\beta} z^m \exp(-z) \cdot (|t| + 1 - m/z) dz . \tag{2.4}$$

Therefore, by (2.4)

$$\begin{aligned} \sum_{i=1}^{r-1} |f(\lambda_{m,i}) - f(\lambda_{m,i+1})| &< \int_{\lambda_{m,1}}^{\lambda_{m,r}} z^m \exp(-z) \cdot (|t| + 1 - m/z) dz \\ &< (|t| + 1) \cdot \int_{m+[y]/m}^{m+([y]+1)/m} z^m \exp(-z) dz . \end{aligned}$$

Hence, by (2.3) and $[y] + 1 \leq m$

$$\begin{aligned} &|O(x,t)| \\ &\geq |S_r| (e/m)^m \{ (m + ([y] + 1)/m)^m \cdot \exp(- (m + ([y] + 1)/m)) \\ &\quad - (|t| + 1) \cdot \int_{m+[y]/m}^{m+([y]+1)/m} z^m \exp(-z) dz \} \\ &> |S_r| \{ (1 + ([y] + 1)/m^2)^m \cdot \exp(- ([y] + 1)/m) - (|t| + 1) \cdot 1/m \} \\ &> |S_r| \{ \exp(-0(1/m)) - (|t| + 1) \cdot 1/m \} = |S_r| \cdot \{ 1 - (|t| + 0(1))/m \} , \end{aligned}$$

which proves lemma 3.

Lemma 4. *Let (1.1) have the simple convergence-abscissa $\sigma_s = 0$. Then there exists a sequence $\{x_\nu\}$ ($[x_\nu] \uparrow +\infty$) independent of t ($-\infty < t < +\infty$) such that*

- (a) $[x_\nu](1 + \omega) < [x_{\nu+1}](1 - \omega) \quad (\nu = 1, 2, \dots, 0 < \omega < 1) ,$
- (b) $\lim_{\nu \rightarrow +\infty} 1/x_\nu \cdot \log \left| \sum_{[x_\nu]+f(x_\nu) \leq \lambda_n < x_\nu} a_n \right| = 0 ,$
- (c) $\varliminf_{\nu \rightarrow \infty} |O(x_\nu, t)|^{1/[x_\nu]} \leq 1 \quad \text{for arbitrary } t (-\infty < t < +\infty) .$

Proof. By lemma 2, we can find a sequence $\{y_\nu\}$ ($[y_\nu] \uparrow +\infty$) such that

$$\left. \begin{aligned} &[y_\nu](1 + \omega) < [y_{\nu+1}](1 - \omega) \quad (\nu = 1, 2, \dots, 0 < \omega < 1) , \\ &\lim_{\nu \rightarrow \infty} 1/y_\nu \cdot \log \left| \sum_{[y_\nu]+f(y_\nu) \leq \lambda_n < y_\nu} a_n \right| = 0 . \end{aligned} \right\} \tag{2.5}$$

Put

$$\text{Max}_{[y_\nu]+f(y_\nu) \leq x \leq y_\nu} \left| \sum_{[x]+f(x) \leq \lambda_n < x} a_n \right| = \left| \sum_{[x_\nu]+f(x_\nu) \leq \lambda_n < x_\nu} a_n \right| ,$$

where $[x_\nu] = [y_\nu]$, $f(x_\nu) = f(y_\nu)$. Since

$$\begin{aligned} 0 = \lim_{\nu \rightarrow \infty} 1/y \cdot \log \left| \sum_{[y_\nu]+f(y_\nu) \leq \lambda_n < y_\nu} a_n \right| &\leq \overline{\lim}_{\nu \rightarrow \infty} 1/x \cdot \log \left| \sum_{[x_\nu]+f(x_\nu) \leq \lambda_n < x_\nu} a_n \right| \\ &\leq \overline{\lim}_{x \rightarrow +\infty} 1/x \cdot \log \left| \sum_{[x]+f(x) \leq \lambda_n < x} a_n \right| = 0 , \end{aligned}$$

by (2.5) and (2.6), selecting a suitable subsequence, if necessary, we have easily

$$\left. \begin{aligned}
 & [x_\nu](1 + \omega) < [x_{\nu+1}](1 - \omega) \quad (\nu = 1, 2, \dots, 0 < \omega < 1) , \\
 & \lim_{\nu \rightarrow \infty} 1/x_\nu \cdot \log \left| \sum_{[x_\nu] + f(x_\nu) \leq \lambda_n < x_\nu} a_n \right| = 0 , \\
 & \left| \sum_{[x_\nu] + f(x_\nu) \leq \lambda_n < x_\nu} a_n \right| \geq \left| \sum_{[x_\nu] + f(x_\nu) \leq \lambda_n < X} a_n \right| \\
 & \text{for every } X \text{ such that } [x_\nu] + f(x_\nu) < X \leq x_\nu .
 \end{aligned} \right\} \tag{2.7}$$

On account of (2.7) and lemma 3

$$\left. \begin{aligned}
 & [x_\nu](1 + \omega) < [x_{\nu+1}](1 - \omega) \quad (\nu = 1, 2, \dots, 0 < \omega < 1) , \\
 & \lim_{\nu \rightarrow \infty} 1/x_\nu \cdot \log \left| \sum_{[x_\nu] + f(x_\nu) \leq \lambda_n < x_\nu} a_n \right| = 0 , \\
 & \underline{\lim}_{\nu \rightarrow \infty} |O(x_\nu, t)|^{1/[x_\nu]} \geq \lim_{\nu \rightarrow \infty} \left| \sum_{[x_\nu] + f(x_\nu) \leq \lambda_n < x_\nu} a_n \right|^{1/[x_\nu]} = 1 \\
 & \text{for arbitrary } t \text{ } (-\infty < t < +\infty) ,
 \end{aligned} \right\}$$

which is to be proved.

Lemma 5. *Let (1.1) have the simple convergence-abscisse $\sigma_s = 0$. Then, for any sequence $\{x_\nu\}$ ($[x_\nu] \uparrow +\infty$), $\sum_{\nu=1}^{+\infty} u_\nu(s)$ converges absolutely for $\sigma > 0$, where*

$$u_\nu(s) = \sum_{[x_\nu] + f(x_\nu) \leq \lambda_n < x_\nu} a_n \exp(-\lambda_n s) .$$

Proof. By lemma 2

$$0 = \overline{\lim}_{x \rightarrow +\infty} 1/x \cdot \log \left| \sum_{[x] + f(x) \leq \lambda_n < x} a_n \right| = \overline{\lim}_{x \rightarrow +\infty} 1/[x] \cdot \log \left| \sum_{[x] + f(x) \leq \lambda_n < x} a_n \right| .$$

Hence, for any given $\varepsilon (> 0)$

$$\left| \sum_{[x] + f(x) \leq \lambda_n < x} a_n \right| < \exp(\varepsilon[x]) \quad \text{for } [x] > X(\varepsilon) . \tag{2.8}$$

Using the same notations as in lemma 3, by ABEL's transformation

$$\begin{aligned}
 u_\nu(s) &= \sum_{i=1}^{r_\nu} a_{m_\nu, i} \exp(-\lambda_{m_\nu, i} s) \\
 &= S_{r_\nu} \exp(-\lambda_{m_\nu, r_\nu} s) \\
 &\quad + \sum_{i=1}^{r_\nu-1} S_i \{ \exp(-\lambda_{m_\nu, i} s) - \exp(-\lambda_{m_\nu, i+1} s) \} ,
 \end{aligned}$$

where

$$x_\nu = m_\nu + y_\nu/m_\nu \quad (m_\nu = [x_\nu], \quad 0 < y_\nu < m_\nu),$$

Since

$$[x_\nu] + f(x_\nu) = m_\nu + [y_\nu]/m_\nu.$$

$$|\exp(-\alpha s) - \exp(-\beta s)| \leq |s|/\sigma \cdot \{\exp(-\alpha\sigma) - \exp(-\beta\sigma)\}$$

for $\alpha < \beta$, $\sigma > 0$, by (2.8)

$$\begin{aligned} |u_\nu(s)| &< |s|/\sigma \cdot \exp(\varepsilon[x_\nu]) \cdot \{\exp(-\lambda_{m_\nu, r_\nu}\sigma) \\ &+ \sum_{i=1}^{r_\nu-1} \exp(-\lambda_{m_\nu, i}\sigma) - \exp(-\lambda_{m_\nu, i+1}\sigma)\} \\ &\leq |s|/\sigma \cdot \exp([x_\nu](\varepsilon - \sigma)) \quad \text{for } \sigma > 0, \quad [x_\nu] > X(\varepsilon). \end{aligned}$$

Therefore, putting $\varepsilon = \sigma/2$

$$|u_\nu(s)| \leq |s|/\sigma \cdot \exp(-\sigma/2 \cdot [x_\nu]),$$

so that $\sum_{\nu=1}^{+\infty} u_\nu(s)$ is absolutely convergent for $\sigma > 0$. q. e. d.

Lemma 6. *Let (1.1) have the simple convergence-abscissa $\sigma_s = 0$.*

(A) *Let us put*

$$\sum_{\nu=1}^{+\infty} \delta(m_\nu) u_\nu(s) = \sum_{n=1}^{+\infty} \varepsilon_n a_n \exp(-\lambda_n s), \tag{2.9}$$

where

$$(a) \quad \lim_{\nu \rightarrow +\infty} \overline{1/x_\nu \cdot \log \left| \sum_{[x_\nu] + f(x_\nu) \leq \lambda_n < x_\nu} a_n \right|} = 0^1 \quad ([x_\nu] = m_\nu \uparrow +\infty)$$

$$(b) \quad \lim_{x \rightarrow +\infty} 1/x \cdot \log |\delta(x)| = 0,$$

$$(c) \quad u_\nu(s) = \sum_{[x_\nu] + f(x_\nu) \leq \lambda_n < x_\nu} a_n \exp(-\lambda_n s)$$

$$(d) \quad \begin{cases} \varepsilon_n = \delta(m_\nu) & \text{for } \lambda_n \in \{I_\nu\} \quad (I_\nu: [x_\nu] + f(x_\nu) \leq x < x_\nu) \\ \varepsilon_n = 0 & \text{for } \lambda_n \bar{\in} \{I_\nu\}. \end{cases}$$

Then (2.9) has the simple convergence-abscissa $\sigma = 0$.

(B) *Set*

$$f_0(s) + \sum_{\nu=1}^{+\infty} \delta(m_\nu) u_\nu(s) = \sum_{n=1}^{+\infty} \varepsilon'_n a_n \exp(-\lambda_n s), \tag{2.10}$$

where

$$(e) \quad f_0(s) = \sum_{\lambda_n \bar{\in} \{I_\nu\}} a_n \exp(-\lambda_n s)$$

$$(f) \quad \begin{cases} \varepsilon'_n = \delta(m_\nu) & \text{for } \lambda_n \in \{I_\nu\}, \\ \varepsilon'_n = 1 & \text{for } \lambda_n \bar{\in} \{I_\nu\}. \end{cases}$$

Then (2.10) has also the simple convergence-abscissa $\sigma = 0$.

¹⁾ By lemma 1, there exists a sequence $\{x_\nu\}$ satisfying (a).

Proof. (A) By lemma 2

$$\overline{\lim}_{x \rightarrow +\infty} 1/x \cdot \log \left| \sum_{[x]+f(x) \leq \lambda_n < x} a_n \right| = 0 . \quad (2.11)$$

Taking account of (a), (b), (c) and (d), we get

$$\overline{\lim}_{\substack{x \rightarrow +\infty \\ [x] \neq m_\nu}} 1/x \cdot \log \left| \sum_{[x]+f(x) \leq \lambda_n < x} \varepsilon_n a_n \right| = -\infty ,$$

$$\begin{aligned} \overline{\lim}_{\substack{x \rightarrow +\infty \\ [x] = m_\nu \\ x_\nu \leq x < m_\nu + 1}} 1/x \cdot \log \left| \sum_{[x]+f(x) \leq \lambda_n < x} \varepsilon_n a_n \right| &= \overline{\lim}_{\nu \rightarrow +\infty} 1/x_\nu \cdot \log \left\{ |\delta(m_\nu)| \cdot \left| \sum_{m_\nu + f(x_\nu) \leq \lambda_n < x_\nu} a_n \right| \right\}, \\ &= 0 , \end{aligned}$$

$$\begin{aligned} \overline{\lim}_{\substack{x \rightarrow +\infty \\ [x] = m_\nu \\ m_\nu + f(x_\nu) < x < x_\nu}} 1/x \cdot \log \left| \sum_{[x]+f(x) \leq \lambda_n < x} \varepsilon_n a_n \right| &= \overline{\lim}_{\nu \rightarrow +\infty} 1/x_\nu \cdot \log \left\{ |\delta(m_\nu)| \cdot \left| \sum_{[x_\nu + f(x_\nu) \leq \lambda_n < x_\nu} a_n \right| \right\}, \\ &\leq 0 \quad (\text{by (2.11)}) , \end{aligned}$$

$$\overline{\lim}_{\substack{x \rightarrow +\infty \\ [x] = m_\nu \\ m_\nu \leq x \leq m_\nu + f(x_\nu)}} 1/x \cdot \log \left| \sum_{[x]+f(x) \leq \lambda_n < x} \varepsilon_n a_n \right| = -\infty ,$$

so that

$$\overline{\lim}_{x \rightarrow +\infty} 1/x \cdot \log \left| \sum_{[x]+f(x) \leq \lambda_n < x} \varepsilon_n a_n \right| = 0 ,$$

which proves that the simple convergence-abscissa of (2.9) is $\sigma = 0$.

(B) Since $f_0(s) = f(s) - \sum_{\nu=1}^{+\infty} u_\nu(s)$, by lemma 5 $f_0(s)$ is simply convergent at least for $\sigma > 0$. On account of (A), $\sum_{\nu=1}^{+\infty} \delta(m_\nu) \cdot u_\nu(s) = \sum_{n=1}^{+\infty} \varepsilon_n a_n \exp(-\lambda_n s)$ is simply convergent exactly for $\sigma > 0$, so that (2.10) is simply convergent at least for $\sigma > 0$. In other words, by lemma 2

$$\overline{\lim}_{x \rightarrow +\infty} 1/x \cdot \log \left| \sum_{[x]+f(x) \leq \lambda_n < x} \varepsilon'_n a_n \right| \leq 0 .$$

On the other hand, by (a), (b), and (f)

$$\begin{aligned} \overline{\lim}_{\nu \rightarrow +\infty} 1/x_\nu \cdot \log \left| \sum_{[x_\nu] + f(x_\nu) \leq \lambda_n < x_\nu} \varepsilon'_n a_n \right| &= \overline{\lim}_{\nu \rightarrow +\infty} 1/x_\nu \cdot \log \left\{ |\delta(m_\nu)| \cdot \left| \sum_{[x_\nu] + f(x_\nu) \leq \lambda_n < x_\nu} a_n \right| \right\} . \\ &= 0 . \end{aligned}$$

Therefore

$$\overline{\lim}_{x \rightarrow +\infty} 1/x \cdot \log \left| \sum_{[x]+f(x) \leq \lambda_n < x} \varepsilon'_n a_n \right| = 0 ,$$

which proves that (2.10) has also the simple convergence-abscissa $\sigma = 0$.
q. e. d.

3. Proofs of theorems

Proof of theorem 1. Without any loss of generality, we can assume $\sigma_s = 0$. Let us denote by $\{x_\nu\}$ the sequence of lemma 4. Then, by lemma 5, we can put

$$\sum_{\nu=1}^{+\infty} u_\nu(s) = f_1(s) + f_2(s) + f_3(s) \cdots + f_n(s) + \cdots,$$

where

- (i) $u_\nu(s) = \sum_{[x_\nu] + f(x_\nu) \leq \lambda_n < x_\nu} a_n \exp(-\lambda_n s),$
- (ii) each $f_n(s)$ ($n = 1, 2, \dots$) contains infinite number of $\{u_\nu(s)\}$.

For any given sequence $\{\delta_n\}$ ($\delta_n = \pm 1$), set

$$\begin{aligned} f(s; \{\delta_n\}) &= f_0(s) + \delta_1 f_1(s) + \cdots + \delta_n f_n(s) + \cdots \\ &= f_0(s) + \sum_{\nu=1}^{+\infty} \alpha_\nu u_\nu(s), \\ &= \sum_{n=1}^{+\infty} \varepsilon'_n s_n \exp(-\lambda_n s), \end{aligned} \tag{3.1}$$

where

- (i) $f_0(s) = \sum_{\lambda_n \in \bar{I}_\nu} a_n \exp(-\lambda_n s) \quad (I_\nu: [x_\nu] + f(x_\nu) \leq x < x_\nu),$
- (ii) $\alpha_\nu = \pm 1,$
- (iii) $\varepsilon'_n = \alpha_\nu \quad \text{for } \lambda_n \in \{I_\nu\},$
 $\varepsilon'_n = 1 \quad \text{for } \lambda_n \in \bar{I}_\nu.$

Then, by lemma 6 (B), (3.1) has the simple convergence-abscissa $\sigma = 0$.

For $\{\delta_n\} \not\equiv \{\delta'_n\}$ ($\delta_n, \delta'_n = \pm 1$), we have

$$\left. \begin{aligned} f(s; \{\delta_n\}) - f(s; \{\delta'_n\}) &= \sum_{n=1}^{+\infty} \delta''_n f_n(s) \\ &= \sum_{\nu=1}^{+\infty} \pm 2 \cdot u_\nu^*(s) \\ &= \sum_{n=1}^{+\infty} \varepsilon_n a_n \exp(-\lambda_n s), \end{aligned} \right\} \tag{3.2}$$

where

- (i) $\delta''_n = \delta_n - \delta'_n$ (Since $\{\delta_n\} \not\equiv \{\delta'_n\}$, there exists at least one n such that $\delta''_n = \pm 2$),
- (ii) $u_\nu^*(s) = \sum_{[x'_\nu] + f(x'_\nu) \leq \lambda_n < x'_\nu} a_n \exp(-\lambda_n s) \quad (\{x'_\nu\}: \text{subsequence of } \{x_\nu\}),$
- (iii) $\varepsilon_n = \pm 2 \quad \text{for } \lambda_n \in \{I'_\nu\} \quad (I'_\nu: [x'_\nu] + f(x'_\nu) \leq x < x'_\nu)$
 $\varepsilon_n = 0 \quad \text{for } \lambda_n \in \bar{I}_\nu.$

On account of lemma 4 (b) and lemma 6 (A), (3.2) has also the simple convergence-abscissa $\sigma = 0$.

(3.2) has $\sigma = 0$ as the natural boundary. To prove this, by lemma 1, it is sufficient to show that

$$\overline{\lim}_{m \rightarrow +\infty} |O_m(1, \omega; t)|^{1/m} \geq 1 \quad \text{for arbitrary } t \quad (-\infty < t < +\infty), \quad (3.3)$$

where

$$O_m(1, \omega; t) = \sum_{m(1-\omega) \leq \lambda_n \leq m(1+\omega)} \varepsilon_n a_n (\lambda_n e/m)^m \cdot \exp(-\lambda_n(1+it)).$$

Since $O_{[x'_\nu]}(1, \omega; t) = \pm 2O(x'_\nu, t)$ by lemma 4 (a), we have by lemma 4 (c)

$$\begin{aligned} \overline{\lim}_{m \rightarrow \infty} |O_m(1, \omega; t)|^{1/m} &\geq \overline{\lim}_{\nu \rightarrow +\infty} |O_{[x'_\nu]}(1, \omega; t)|^{1/[x'_\nu]} \\ &= \overline{\lim}_{\nu \rightarrow +\infty} |O(x'_\nu, t)|^{1/[x'_\nu]} \\ &\geq \underline{\lim}_{\nu \rightarrow +\infty} |O(x_\nu, t)|^{1/[x_\nu]} \geq 1, \end{aligned}$$

which proves (3.3).

Let us denote by $E(\{\delta_n\})$ the set of regular points on $\sigma = 0$ of (3.1), which is evidently an open set. Then we have

$$E(\{\delta_n\}) \cap E(\{\delta'_n\}) = \emptyset \quad \text{for} \quad \{\delta_n\} \neq \{\delta'_n\}. \quad (3.4)$$

In fact, if $s_0 \in E(\{\delta_n\}) \cap E(\{\delta'_n\}) \neq \emptyset$, (3.2), would be regular at $s = s_0$, which is impossible by (3.3). If $E(\{\delta_n\}) \neq \emptyset$ for all $\{\delta_n\}$, by (3.4) the set of all functions $\{f(s; \{\delta_n\})\}$ is at most enumerable, which contradicts the power of continuum of the set of all $\{\delta_n\}$. Hence $E(\{\delta_n\}) = \emptyset$ for at least one $\{\delta_n\}$, which shows that $f(s; \{\delta_n\})$ has $\sigma = 0$ as the natural boundary.

(3.1) is evidently of the form $\sum_{n=1}^{+\infty} \varepsilon'_n a_n \exp(-\lambda_n s)$ ($\varepsilon'_n = \pm 1$). q. e. d.

Proof of theorem 2. Without any loss of generality, we can assume $\sigma_s = 0$. Let us denote by $\{x_\nu\}$ the sequence of lemma 4. Let us put

$$f(s, \theta, \alpha) = f_0(s) + f_r(s, \theta, \alpha) = \sum_{n=1}^{\infty} \varepsilon'_n a_n \exp(-\lambda_n s), \quad (3.5)$$

where

$$(i) \quad f_0(s) = \sum_{\lambda_n \in \overline{I_\nu}} a_n \exp(-\lambda_n s) \quad (I_\nu: [x_\nu] + f(x_\nu) \leq x < x_\nu),$$

$$(ii) \quad f_1(s, \theta, \alpha) = \sum_{\nu=1}^{+\infty} \exp(\alpha \theta / [x_\nu]) \cdot u_\nu(s),$$

$$u_\nu(s) = \sum_{\lambda_n \in I_\nu} a_n \exp(-\lambda_n s),$$

θ : a real constant, α : a parameter ($= \sqrt{-1}$ or 1),

$$(iii) \quad \begin{aligned} \varepsilon'_n &= \exp(\alpha\theta/[x_\nu]) & \text{for } \lambda_n \in \{I_\nu\} , \\ \varepsilon'_n &= 1 & \text{for } \lambda_n \bar{\in} \{I_\nu\} . \end{aligned}$$

Then, by lemma 6 (B), (3.5) has the simple convergence-abscissa $\sigma = 0$. Put

$$\begin{aligned} & f(s, \theta, \alpha) - f(s, \theta', \alpha) & (3.6) \\ &= \sum_{\nu=1}^{+\infty} \{ \exp(\alpha\theta/[x_\nu]) - \exp(\alpha\theta'/[x_\nu]) \} \cdot u_\nu(s) \\ &= \sum_{n=1}^{+\infty} \varepsilon_n a_n \exp(-\lambda_n s) \quad \text{for } \theta \neq \theta' , \end{aligned}$$

where

$$\begin{aligned} \varepsilon_n &= 0 & \text{for } \lambda_n \bar{\in} \{I_\nu\} , \\ \varepsilon_n &= \exp(\alpha\theta/[x_\nu]) - \exp(\alpha\theta'/[x_\nu]) \\ &= \alpha(\theta - \theta')/[x_\nu] + O(1/x_\nu^2) & \text{for } \lambda_n \in \{I_\nu\} . \end{aligned}$$

Then, by lemma 4 (b) and lemma 6 (A), (3.6) has also the simple convergence-abscissa $\sigma = 0$.

(3.6) has $\sigma = 0$ as the natural boundary. To establish this fact, by lemma 1, it is sufficient to show that

$$\overline{\lim}_{m \rightarrow \infty} |O_m(1, \omega, t)|^{1/m} \geq 1 \quad \text{for arbitrary } t \quad (-\infty < t < +\infty) . \quad (3.7)$$

Since

$$O_{[x_\nu]}(1, \omega, t) = \{ \alpha(\theta - \theta')/[x_\nu] + O(1/x_\nu^2) \} \cdot O(x_\nu, t) ,$$

by lemma 4 (c)

$$\begin{aligned} \overline{\lim}_{m \rightarrow \infty} |O_m(1, \omega, t)|^{1/m} &\geq \overline{\lim}_{\nu \rightarrow \infty} |O_{[x_\nu]}(1, \omega, t)|^{1/[x_\nu]} \\ &= \overline{\lim}_{\nu \rightarrow \infty} |O(x_\nu, t)|^{1/[x_\nu]} \\ &\geq \overline{\lim}_{\nu \rightarrow \infty} |O(x_\nu, t)|^{1/[x_\nu]} \geq 1 , \end{aligned}$$

from which follows (3.7).

Let us denote by $E(\theta, \alpha)$ the set of regular points on $\sigma = 0$ of (3.5), which is an open set. Then we have

$$E(\theta, \alpha) \cap E(\theta', \alpha) = \emptyset \quad \text{for } \theta \neq \theta' . \quad (3.8)$$

In fact, if $s_0 \in E(\theta, \alpha) \cap E(\theta', \alpha) \neq \emptyset$ (3.6), would be regular at $s = s_0$, which is impossible by (3.7). If $E(\theta, \alpha) \neq \emptyset$ for all θ ($0 < \theta < \gamma$, γ : a fixed constant), by (3.8) the set of all functions $\{f(s, \theta, \alpha)\}$ is at most enumerable, which

contradicts the power of continuum of the set of θ . Hence $E(\theta, \alpha) = 0$ for at least one θ , which shows that $f(s, \theta, \alpha)$ has $\sigma = 0$ as the natural boundary. If $\alpha = \sqrt{-1}$ (or $= 1$), (i) (or (ii)) of theorem 2 is established. q. e. d.

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