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On the singularities of DIRICHLET series

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1. Introduction. Let us put

$$F(s) = \sum_{n=1}^{\infty} a_n \exp(-\lambda_n s) \quad (s = \sigma + it, \quad 0 \leqq \lambda_1 < \lambda_2 < \cdots < \lambda_n \to +\infty) \ . \quad (1.1)$$

The object of this note is to establish the following theorems:

Theorem 1. *Let (1.1) have the finite simple convergence-abscissa σ_s. Then there exists a sequence $\{\varepsilon_n\}$ $(\varepsilon_n = \pm 1)$ such that*

$$\sum_{n=1}^{\infty} \varepsilon_n a_n \exp(-\lambda_n s)$$

has $\sigma = \sigma_s$ as the natural boundary.

Theorem 2. *Let (1.1) have the finite simple convergence-abscissa σ_s. Then there exists a new DIRICHLET series $\sum_{n=1}^{\infty} b_n \exp(-\lambda_n s)$ having $\sigma = \sigma_s$ as the natural boundary such that*

$$|b_n| = |a_n| (n = 1, 2, \ldots), \qquad \lim_{n\to\infty} |\arg(b_n) - \arg(a_n)| = 0 \qquad \text{(i)}$$

or

$$\arg(b_n) = \arg(a_n) \quad (n = 1, 2, \ldots), \qquad \lim_{n\to\infty} |b_n/a_n| = 1 \ . \qquad \text{(ii)}$$

Under the additional conditions $\lim_{n\to\infty} \log n/\lambda_n = 0$, these two theorems have been proved by O. SZASZ ([1], p. 107) and the author ([2], p. 308) respectively. The method of its proofs is based upon A. OSTROWSKI's criterion of singularities.

2. Lemmas. To establish these theorems, we need some lemmas.

Lemma 1. (A. OSTROWSKI [3, 4], pp. 12–16.) *Let (1.1) have the simple convergence-abscissa $\sigma_s = 0$. For $s = 0$ to be the singular point of (1.1), it is necessary and sufficient that we have*

$$\overline{\lim_{m\to\infty}} |O_m(\sigma, \omega)|^{1/m} \geqq 1 \qquad (m: \ \textit{positive integer}),$$

where

$$\sigma(>0), \quad \omega(0<\omega<1): \ \textit{fixed constants}, \qquad \text{(i)}$$

$$O_m(\sigma, \omega) = \sum_{m/\sigma\cdot(1-\omega) \leqq \lambda_n \leqq m/\sigma\cdot(1+\omega)} a_n (\lambda_n e \sigma/m)^m \exp(-\lambda_n \sigma) \ . \qquad \text{(ii)}$$

Lemma 2. *The simple convergence-abscissa σ_s of (1.1) is determined by*

$$\sigma_s = \overline{\lim_{x\to+\infty}}\ 1/x\cdot\log\Big|\sum_{[x]+f(x)\leqq\lambda_n<x} a_n\Big| ,$$

where

$$[x]:\ \textit{GAUSS's symbol}, \tag{i}$$

$$f(x) = 1/m\cdot[m(x-m)] , \qquad m=[x] . \tag{ii}$$

Proof. By T. KOJIMA's theorem [5], σ_s is given by

$$\sigma_s = \overline{\lim_{x\to+\infty}}\ 1/x\cdot\log\Big|\sum_{[x]\leqq\lambda_n<x} a_n\Big|$$
$$= \overline{\lim_{x\to+\infty}}\ 1/[x]\cdot\log\Big|\sum_{[x]\leqq\lambda_n<x} a_n\Big| .$$

Hence, for any given ε (>0),

$$\Big|\sum_{[x]\leqq\lambda_n<x} a_n\Big| < \exp\{[x](\sigma_s+\varepsilon)\} \quad \text{for} \quad [x]>K(\varepsilon) .$$

Therefore

$$\Big|\sum_{[x]+f(x)\leqq\lambda_n<x} a_n\Big| = \Big|\sum_{[x]\leqq\lambda_n<x} - \sum_{[x]\leqq\lambda_n<x+f(x)}\Big|$$
$$<2\exp\{[x](\sigma_s+\varepsilon)\} \quad \text{for} \quad [x]>K(\varepsilon) ,$$

so that

$$\sigma_s^* = \overline{\lim_{x\to+\infty}}\ 1/x\cdot\log\Big|\sum_{[x]+f(x)\leqq\lambda_n<x} a_n\Big| \leqq \sigma_s+\varepsilon .$$

Letting $\varepsilon\to 0$,

$$\sigma_s^* \leqq \sigma_s . \tag{2.1}$$

Putting

$$\begin{cases} x = m + y/m \quad (m=[x],\ 0<y<m) , \\ [x]+f(x) = m + [y]/m , \end{cases}$$

we have easily

$$\sigma_s^* = \overline{\lim_{\substack{m\to+\infty\\ 0<y<m}}}\ 1/m\cdot\log\Big|\sum_{m+[y]/m\leqq\lambda_n<m+y/m} a_n\Big| \qquad (m:\ \text{positive integer}).$$

Hence, for any given ε (>0),

$$\Big|\sum_{\substack{m+[y]/m\leqq\lambda_n<m+y/m\\ 0<y<m}} a_n\Big| < \exp\{m(\sigma_s^*+\varepsilon)\} \quad \text{for} \quad m>K^*(\varepsilon) .$$

Accordingly

$$\Big|\sum_{[x]\leqq\lambda_n<x} a_n\Big|$$
$$= \Big|\sum_{m\leqq\lambda_n<m+1/m} + \sum_{m+1/m\leqq\lambda_n<m+2/m} + \cdots + \sum_{m+[y]/m\leqq\lambda_n<m+y/m}\Big|$$
$$\leqq ([y]+1)\exp\{m(\sigma_s^*+\varepsilon)\} \leqq m\exp\{m(\sigma_s^*+\varepsilon)\} \quad \text{for} \quad m>K^*(\varepsilon) ,$$

so that

$$\sigma_s = \overline{\lim_{x\to+\infty}}\ 1/x\cdot\log\Big|\sum_{[x]\leqq\lambda_n<x} a_n\Big| \leqq \sigma_s^* + \varepsilon\ .$$

Letting $\varepsilon \to 0$,

$$\sigma_s \leqq \sigma_s^*\ . \tag{2.2}$$

Combining (2.1) with (2.2), lemma 2 is completely established.

Lemma 3. *Put*

$$O(x,t) = \sum_{[x]+f(x)\leqq\lambda_n<x} a_n(\lambda_n e/[x])^{[x]}\exp\left(-\lambda_n(1+it)\right)\ .$$

If

$$\Big|\sum_{[x]+f(x)\leqq\lambda_n<x} a_n\Big| \geqq \Big|\sum_{[x]+f(x)\leqq\lambda_n<X} a_n\Big|$$

for every X such that $[x]+f(x)<X\leqq x$, then

$$|O(x,t)| \geqq \Big|\sum_{[x]+f(x)\leqq\lambda_n<x} a_n\Big|\cdot\{1-(|t|+K(x))/[x]\}\ ,$$

where $K(x)=0(1)$.

Proof. Put

$$\begin{cases} x = m + y/m \quad (m=[x],\ 0<y<m)\ , \\ [x]+f(x) = m + [y]/m\ . \end{cases}$$

Let us denote by $\{\lambda_{m,i}\}$ $(i=1,2,\ldots,r)$ the exponents $\lambda_n's$ contained in $m+[y]/m\leqq\lambda_n<m+y/m$ and by $\{a_{m,i}\}$ the coefficient corresponding to $\lambda_{m,i}$. Setting

$$S_k = \sum_{i=1}^{k} a_{m,i} \quad (1\leqq k\leqq r), \qquad f(\lambda) = \lambda^m\exp\left(-\lambda(1+it)\right)\ ,$$

then by ABEL's transformation

$$\begin{aligned} O(x,t) &= (e/m)^m\cdot\sum_{i=1}^{r} a_{m,i}f(\lambda_{m,i}) \\ &= (e/m)^m\cdot\{S_r f(\lambda_{m,r}) + \sum_{i=1}^{r-1} S_i(f(\lambda_{m,i})-f(\lambda_{m,i+1}))\}\ . \end{aligned}$$

Since

$$|S_r| \geqq |S_i| \quad (i=1,2,\ldots,r-1)\ ,$$

we have

$$|O(x,t)| \geqq (e/m)^m\cdot|S_r|\cdot\{|f(\lambda_{m,r})| - \sum_{i=1}^{r-1}|f(\lambda_{m,i})-f(\lambda_{m,i+1})|\}\ . \tag{2.3}$$

On the other hand, $|f(\lambda)| = \lambda^m\exp(-\lambda)$ is monotone-decreasing for $m\leqq\lambda$. Hence, for $m\leqq\alpha<\beta<m+1$,

$$|f(\alpha)-f(\beta)| \leqq |f(\alpha)| - |f(\beta)| + |f(\beta)|(\beta-\alpha)|t|\ ,$$

so that

$$|f(\alpha) - f(\beta)| < \int_{\alpha}^{\beta} z^m \exp(-z)\cdot(|t| + 1 - m/z)dz \ . \tag{2.4}$$

Therefore, by (2.4)

$$\sum_{i=1}^{r-1} |f(\lambda_{m,i}) - f(\lambda_{m,i+1})| < \int_{\lambda_{m,1}}^{\lambda_{m,r}} z^m \exp(-z)\cdot(|t| + 1 - m/z)dz$$
$$< (|t| + 1)\cdot \int_{m+[y]/m}^{m+([y]+1)/m} z^m \exp(-z)dz \ .$$

Hence, by (2.3) and $[y] + 1 \leqq m$

$$|O(x,t)|$$
$$\geqq |S_r|(e/m)^m\{(m + ([y] + 1)/m)^m\cdot\exp(-(m + ([y] + 1)/m))$$
$$-(|t| + 1)\cdot \int_{m+[y]/m}^{m+([y]+1)/m} z^m \exp(-z)dz\}$$
$$> |S_r|\{(1 + ([y] + 1)/m^2)^m\cdot\exp(-([y] + 1)/m) - (|t| + 1)\cdot 1/m\}$$
$$> |S_r|\{\exp(-0(1/m)) - (|t| + 1)\cdot 1/m\} = |S_r|\cdot\{1 - (|t| + 0(1))/m\} \ ,$$

which proves lemma 3.

Lemma 4. *Let* (1.1) *have the simple convergence-abscissa* $\sigma_s = 0$. *Then there exists a sequence* $\{x_\nu\}$ $([x_\nu] \uparrow +\infty)$ *independent of* t $(-\infty < t < +\infty)$ *such that*

$$\left.\begin{array}{ll} \text{(a)} & [x_\nu](1 + \omega) < [x_{\nu+1}](1 - \omega) \qquad (\nu = 1,2,\ldots,0<\omega<1) \ , \\ \text{(b)} & \lim\limits_{\nu\to+\infty} 1/x_\nu\cdot\log|\sum\limits_{[x_\nu]+f(x_\nu)\leqq\lambda_n<x_\nu} a_n| = 0 \ , \\ \text{(c)} & \varliminf\limits_{\nu\to\infty} |O(x_\nu,t)|^{1/[x_\nu]} \leqq 1 \quad \textit{for arbitrary} \quad t(-\infty<t<+\infty) \ . \end{array}\right\}$$

Proof. By lemma 2, we can find a sequence $\{y_\nu\}$ $([y_\nu] \uparrow +\infty)$ such that

$$\left.\begin{array}{l} [y_\nu](1 + \omega) < [y_{\nu+1}](1 - \omega) \qquad (\nu = 1,2,\ldots,0<\omega<1) \ , \\ \lim\limits_{\nu\to\infty} 1/y_\nu\cdot\log|\sum\limits_{[y_\nu]+f(y_\nu)\leqq\lambda_n<y_\nu} a_n| = 0 \ . \end{array}\right\} \tag{2.5}$$

Put

$$\operatorname*{Max}_{[y_\nu]+f(y_\nu)\leqq x\leqq y_\nu} |\sum_{[x]+f(x)\leqq\lambda_n<x} a_n| = |\sum_{[x_\nu]+f(x_\nu)\leqq\lambda_n<x_\nu} a_n| \ ,$$

where $[x_\nu] = [y_\nu]$, $f(x_\nu) = f(y_\nu)$. Since

$$0 = \lim_{\nu\to\infty} 1/y\cdot\log|\sum_{[y_\nu]+f(y_\nu)\leqq\lambda_n<y_\nu} a_n| \leqq \varlimsup_{\nu\to\infty} 1/x\cdot\log|\sum_{[x_\nu]+f(x_\nu)\leqq\lambda_n<x_\nu} a_n|$$
$$\leqq \varlimsup_{x\to+\infty} 1/x\cdot\log|\sum_{[x]+f(x)\leqq\lambda_n<x} a_n| = 0 \ ,$$

by (2.5) and (2.6), selecting a suitable subsequence, if necessary, we have easily

$$\left.\begin{array}{l} [x_\nu](1+\omega)<[x_{\nu+1}](1-\omega) \quad (\nu=1,2,\ldots,0<\omega<1)\ , \\ \lim\limits_{\nu\to\infty} 1/x_\nu\cdot\log\Big|\sum\limits_{[x_\nu]+f(x_\nu)\leqq\lambda_n<x_\nu} a_n\Big| = 0\ , \\ \Big|\sum\limits_{[x_\nu]+f(x_\nu)\leqq\lambda_n<x_\nu} a_n\Big| \geqq \Big|\sum\limits_{[x_\nu]+f(x_\nu)\leqq\lambda_n<X} a_n\Big| \\ \text{for every } X \text{ such that } [x_\nu]+f(x_\nu)<X\leqq x_\nu\ . \end{array}\right\} \tag{2.7}$$

On account of (2.7) and lemma 3

$$\left.\begin{array}{l} [x_\nu](1+\omega)<[x_{\nu+1}](1-\omega) \quad (\nu=1,2,\ldots,0<\omega<1)\ , \\ \lim\limits_{\nu\to\infty} 1/x_\nu\cdot\log\Big|\sum\limits_{[x_\nu]+f(x_\nu)\leqq\lambda_n<x_\nu} a_n\Big| = 0\ , \\ \varliminf\limits_{\nu\to\infty} |O(x_\nu,t)|^{1/[x_\nu]} \geqq \lim\limits_{\nu\to\infty} \Big|\sum\limits_{[x_\nu]+f(x_\nu)\leqq\lambda_n<x_\nu} a_n\Big|^{1/[x_\nu]} = 1 \\ \text{for arbitrary } t\ (-\infty<t<+\infty)\ , \end{array}\right\}$$

which is to be proved.

Lemma 5. *Let* (1.1) *have the simple convergence-abscisse* $\sigma_s = 0$. *Then, for any sequence* $\{x_\nu\}$ $([x_\nu]\uparrow+\infty)$, $\sum\limits_{\nu=1}^{+\infty} u_\nu(s)$ *converges absolutely for* $\sigma>0$, *where* $u_\nu(s) = \sum\limits_{[x_\nu]+f(x_\nu)\leqq\lambda_n<x_\nu} a_n \exp(-\lambda_n s)$.

Proof. By lemma 2

$$0 = \varlimsup_{x\to+\infty} 1/x\cdot\log\Big|\sum_{[x]+f(x)\leqq\lambda_n<x} a_n\Big| = \varlimsup_{x\to+\infty} 1/[x]\cdot\log\Big|\sum_{[x]+f(x)\leqq\lambda_n<x} a_n\Big|\ .$$

Hence, for any given ε (>0)

$$\Big|\sum_{[x]+f(x)\leqq\lambda_n<x} a_n\Big| < \exp(\varepsilon[x]) \quad \text{for} \quad [x]>X(\varepsilon)\ . \tag{2.8}$$

Using the same notations as in lemma 3, by ABEL's transformation

$$\begin{aligned} u_\nu(s) &= \sum_{i=1}^{r_\nu} a_{m_\nu,i}\exp(-\lambda_{m_\nu,i}s) \\ &= S_{r_\nu}\exp(-\lambda_{m_\nu,r_\nu}s) \\ &\quad + \sum_{i=1}^{r_\nu-1} S_i\{\exp(-\lambda_{m_\nu,i}s) - \exp(-\lambda_{m_\nu,i+1}s)\}\ , \end{aligned}$$

where
$$x_\nu = m_\nu + y_\nu/m_\nu \qquad (m_\nu = [x_\nu]\ , \qquad 0<y_\nu<m_\nu)\ ,$$
$$[x_\nu] + f(x_\nu) = m_\nu + [y_\nu]/m_\nu\ .$$
Since
$$|\exp(-\alpha s) - \exp(-\beta s)| \leqq |s|/\sigma\cdot\{\exp(-\alpha\sigma) - \exp(-\beta\sigma)\}$$
for $\alpha<\beta$, $\sigma>0$, by (2.8)
$$|u_\nu(s)| < |s|/\sigma\cdot\exp(\varepsilon[x_\nu]\cdot\{\exp(-\lambda_{m_\nu,r_\nu}\sigma)$$
$$+\sum_{i=1}^{r_\nu-1}\exp(-\lambda_{m_\nu,i}\sigma) - \exp(-\lambda_{m_\nu,i+1}\sigma)\}$$
$$\leqq |s|/\sigma\cdot\exp([x_\nu](\varepsilon-\sigma)) \qquad \text{for} \qquad \sigma>0, \qquad [x_\nu]>X(\varepsilon)\ .$$
Therefore, putting $\varepsilon = \sigma/2$
$$|u_\nu(s)| \leqq |s|/\sigma\cdot\exp(-\sigma/2\cdot[x_\nu])\ ,$$
so that $\sum_{\nu=1}^{+\infty} u_\nu(s)$ is absolutely convergent for $\sigma>0$. q. e. d.

Lemma 6. *Let* (1.1) *have the simple convergence-abscissa* $\sigma_s = 0$.

(A) *Let us put*
$$\sum_{\nu=1}^{+\infty}\delta(m_\nu)u_\nu(s) = \sum_{n=1}^{+\infty}\varepsilon_n a_n\exp(-\lambda_n s)\ , \tag{2.9}$$
where

(a) $\overline{\lim\limits_{\nu\to+\infty}}\ 1/x_\nu\cdot\log\Big|\sum\limits_{[x_\nu]+f(x_\nu)\leqq\lambda_n<x_\nu} a_n\Big| = 0$ [1] $\quad([x_\nu] = m_\nu\uparrow+\infty)$

(b) $\lim\limits_{x\to+\infty}\ 1/x\cdot\log|\delta(x)| = 0$,

(c) $u_\nu(s) = \sum\limits_{[x_\nu]+f(x_\nu)\leqq\lambda_n<x_\nu} a_n\exp(-\lambda_n s)$

(d) $\begin{cases}\varepsilon_n = \delta(m_\nu) & \textit{for} \quad \lambda_n\in\{I_\nu\} \quad (I_\nu\colon [x_\nu]+f(x_\nu)\leqq x<x_\nu)\\ \varepsilon_n = 0 & \textit{for} \quad \lambda_n\bar{\in}\{I_\nu\}\ .\end{cases}$

Then (2.9) *has the simple convergence-abcissa* $\sigma = 0$.

(B) *Set*
$$f_0(s) + \sum_{\nu=1}^{+\infty}\delta(m_\nu)u_\nu(s) = \sum_{n=1}^{+\infty}\varepsilon'_n a_n\exp(-\lambda_n s)\ , \tag{2.10}$$
where

(e) $f_0(s) = \sum\limits_{\lambda_n\bar{\in}\{I_\nu\}} a_n\exp(-\lambda_n s)$

(f) $\begin{cases}\varepsilon'_n = \delta(m_\nu) & \textit{for} \quad \lambda_n\in\{I_\nu\}\ ,\\ \varepsilon'_n = 1 & \textit{for} \quad \lambda_n\bar{\in}\{I_\nu\}\ .\end{cases}$

Then (2.10) *has also the simple convergence-abscissa* $\sigma = 0$.

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[1]) By lemma 1, there exists a sequence $\{x_\nu\}$ satisfying (a).

Proof. (A) By lemma 2

$$\overline{\lim_{x\to+\infty}}\ 1/x\cdot\log\Big|\sum_{[x]+f(x)\leqq\lambda_n<x} a_n\Big| = 0\ . \tag{2.11}$$

Taking account of (a), (b), (c) and (d), we get

$$\overline{\lim_{\substack{x\to+\infty\\ [x]\neq m_\nu}}}\ 1/x\cdot\log\Big|\sum_{[x]+f(x)\leqq\lambda_n<x}\varepsilon_n a_n\Big| = -\infty\ ,$$

$$\overline{\lim_{\substack{x\to+\infty\\ [x]=m_\nu\\ x_\nu\leqq x<m_\nu+1}}}\ 1/x\cdot\log\Big|\sum_{[x]+f(x)\leqq\lambda_n<x}\varepsilon_n a_n\Big| = \overline{\lim_{\nu\to+\infty}}\ 1/x_\nu\cdot\log\Big\{|\delta(m_\nu)|\cdot\Big|\sum_{m_\nu+f(x_\nu)\leqq\lambda_n<x_\nu} a_n\Big|\Big\},$$
$$= 0\ ,$$

$$\overline{\lim_{\substack{x\to+\infty\\ [x]=m_\nu\\ m_\nu+f(x_\nu)<x<x_\nu}}}\ 1/x\cdot\log\Big|\sum_{[x]+f(x)\leqq\lambda_n<x}\varepsilon_n a_n\Big| = \overline{\lim_{\nu\to+\infty}}\ 1/x_\nu\cdot\log\Big\{|\delta(m_\nu)|\cdot\Big|\sum_{[x_\nu+f(x_\nu)\leqq\lambda_n<x} a_n\Big|\Big\},$$
$$\leqq 0 \quad \text{(by (2.11))}\ ,$$

$$\overline{\lim_{\substack{x\to+\infty\\ [x]=m_\nu\\ m_\nu\leqq x\leqq m_\nu+f(x_\nu)}}}\ 1/x\cdot\log\Big|\sum_{[x]+f(x)\leqq\lambda_n<x}\varepsilon_n a_n\Big| = -\infty\ ,$$

so that

$$\overline{\lim_{x\to+\infty}}\ 1/x\cdot\log\Big|\sum_{[x]+f(x)\leqq\lambda_n<x}\varepsilon_n a_n\Big| = 0\ ,$$

which proves that the simple convergence-abscissa of (2.9) is $\sigma = 0$.

(B) Since $f_0(s) = f(s) - \sum_{\nu=1}^{+\infty} u_\nu(s)$, by lemma 5 $f_0(s)$ is simply convergent at least for $\sigma>0$. On account of (A), $\sum_{\nu=1}^{+\infty}\delta(m_\nu)\cdot u_\nu(s) = \sum_{n=1}^{+\infty}\varepsilon_n a_n \exp(-\lambda_n s)$ is simply convergent exactly for $\sigma>0$, so that (2.10) is simply convergent at least for $\sigma>0$. In other words, by lemma 2

$$\overline{\lim_{x\to+\infty}}\ 1/x\cdot\log\Big|\sum_{[x]+f(x)\leqq\lambda_n<x}\varepsilon'_n a_n\Big|\leqq 0\ .$$

On the other hand, by (a), (b), and (f)

$$\overline{\lim_{\nu\to+\infty}}\ 1/x_\nu\cdot\log\Big|\sum_{[x_\nu]+f(x_\nu)\leqq\lambda_n<x_\nu}\varepsilon'_n a_n\Big| = \overline{\lim_{\nu\to+\infty}}\ 1/x_\nu\cdot\log\Big\{|\delta(m_\nu)|\cdot\Big|\sum_{[x_\nu]+f(x_\nu)\leqq\lambda_n<x_\nu} a_n\Big|\Big\}\ .$$
$$= 0\ .$$

Therefore

$$\overline{\lim_{x\to+\infty}}\ 1/x\cdot\log\Big|\sum_{[x]+f(x)\leqq\lambda_n<x}\varepsilon'_n a_n\Big| = 0\ ,$$

which proves that (2.10) has also the simple convergence-abscissa $\sigma = 0$. q. e. d.

3. Proofs of theorems

Proof of theorem 1. Without any loss of generality, we can assume $\sigma_s = 0$. Let us denote by $\{x_\nu\}$ the sequence of lemma 4. Then, by lemma 5, we can put

$$\sum_{\nu=1}^{+\infty} u_\nu(s) = f_1(s) + f_2(s) + f_3(s) \cdots + f_n(s) + \cdots ,$$

where

(i) $u_\nu(s) = \sum_{[x_\nu]+f(x_\nu) \leqq \lambda_n < x_\nu} a_n \exp(-\lambda_n s)$,

(ii) each $f_n(s)$ $(n = 1, 2, \ldots)$ contains infinite number of $\{u_\nu(s)\}$.

For any given sequence $\{\delta_n\}$ $(\delta_n = \pm 1)$, set

$$\begin{aligned} f(s : \{\delta_n\}) &= f_0(s) + \delta_1 f_1(s) + \cdots + \delta_n f_n(s) + \cdots \qquad (3.1) \\ &= f_0(s) + \sum_{\nu=1}^{+\infty} \alpha_\nu u_\nu(s) , \\ &= \sum_{n=1}^{+\infty} \varepsilon'_n s_n \exp(-\lambda_n s) , \end{aligned}$$

where

(i) $f_0(s) = \sum_{\lambda_n \bar{\in} \{I_\nu\}} a_n \exp(-\lambda_n s) \quad (I_\nu : \ [x_\nu] + f(x_\nu) \leqq x < x_\nu)$,

(ii) $\alpha_\nu = \pm 1$,

(iii) $\varepsilon'_n = \alpha_\nu$ for $\lambda_n \in \{I_\nu\}$,
$\varepsilon'_n = 1$ for $\lambda_n \bar{\in} \{I_\nu\}$.

Then, by lemma 6 (B), (3.1) has the simple convergence-abscissa $\sigma = 0$.

For $\{\delta_n\} \not\equiv \{\delta'_n\} (\delta_n, \delta'_n = \pm 1)$, we have

$$\left.\begin{aligned} f(s : \{\delta_n\}) - f(s : \{\delta'_n\}) &= \sum_{n=1}^{+\infty} \delta''_n f_n(s) \\ &= \sum_{\nu=1}^{+\infty} \pm 2 \cdot u^*_\nu(s) \\ &= \sum_{n=1}^{+\infty} \varepsilon_n a_n \exp(-\lambda_n s) , \end{aligned}\right\} \qquad 3.2$$

where

(i) $\delta''_n = \delta_n - \delta'_n$ (Since $\{\delta_n\} \not\equiv \{\delta'_n\}$, there exists at least one n such that $\delta''_n = \pm 2$),

(ii) $u^*_\nu(r) = \sum_{[x'_\nu]+f(x'_\nu) \leqq \lambda_n < x'_\nu} a_n \exp(-\lambda_n s) \quad (\{x'_\nu\} :$ subsequence of $\{x_\nu\})$,

(iii) $\varepsilon_n = \pm 2$ for $\lambda_n \in \{I'_\nu\} \quad (I'_\nu : \ [x'_\nu] + f(x'_\nu) \leqq x < x'_\nu)$
$\varepsilon_n = 0$ for $\lambda_n \bar{\in} \{I_\nu\}$.

On account of lemma 4 (b) and lemma 6 (A), (3.2) has also the simple convergence-abscissa $\sigma = 0$.

(3.2) has $\sigma = 0$ as the natural boundary. To prove this, by lemma 1, it is sufficient to show that

$$\overline{\lim_{m\to+\infty}} |O_m(1,\omega;t)|^{1/m} \geqq 1 \quad \text{for arbitrary } t \quad (-\infty<t<+\infty) , \tag{3.3)}$$

where

$$O_m(1,\omega;t) = \sum_{m(1-\omega)\leqq\lambda_n\leqq m(1+\omega)} \varepsilon_n a_n (\lambda_n e/m)^m \cdot \exp(-\lambda_n(1+it)) .$$

Since $O_{[x'_\nu]}(1,\omega;t) = \pm 2O(x'_\nu,t)$ by lemma 4 (a), we have by lemma 4 (c)

$$\begin{aligned}\overline{\lim_{m\to\infty}} |O_m(1,\omega;t)|^{1/m} &\geqq \overline{\lim_{\nu\to+\infty}} |O_{[x'_\nu]}(1,\omega;t)^{1/[x'_\nu]} \\ &= \overline{\lim_{\nu\to+\infty}} |O(x'_\nu,t)|^{1/[x'_\nu]} \\ &\geqq \underline{\lim_{\nu\to+\infty}} |O(x_\nu,t)|^{1/[x_\nu]} \geqq 1 ,\end{aligned}$$

which proves (3.3).

Let us denote by $E(\{\delta_n\})$ the set of regular points on $\sigma = 0$ of (3.1), which is evidently an open set. Then we have

$$E(\{\delta_n\}) \frown E(\{\delta'_n\}) = 0 \quad \text{for} \quad \{\delta_n\} \not\equiv \{\delta'_n\} . \tag{3.4}$$

In fact, if $s_0 \in E(\{\delta_n\}) \frown E(\{\delta'_n\}) \neq 0$, (3.2), would be regular at $s = s_0$, which is impossible by (3.3). If $E(\{\delta_n\}) \neq 0$ for all $\{\delta_n\}$, by (3.4) the set of all functions $\{f(s:\{\delta_n\})\}$ is at most enumerable, which contradicts the power of continuum of the set of all $\{\delta_n\}$. Hence $E(\{\delta_n\})=0$ for at least one $\{\delta_n\}$, which shows that $f(s:\{\delta_n\})$ has $\sigma = 0$ as the natural boundary. (3.1) is evidently of the form $\sum_{n=1}^{+\infty} \varepsilon'_n a_n \exp(-\lambda_n s)$ $(\varepsilon'_n = \pm 1)$. q. e. d.

Proof of theorem 2. Without any loss of generality, we can assume $\sigma_s = 0$. $\sigma_s = 0$. Let us denote by $\{x_\nu\}$ the sequence of lemma 4. Let us put

$$f(s,\theta,\alpha) = f_0(s) + f_r(s,\theta,\alpha) = \sum_{n=1}^{\infty} \varepsilon'_n a_n \exp(-\lambda_n s) , \tag{3.5}$$

where

(i) $$f_0(s) = \sum_{\lambda_n \bar{\in} \{I_\nu\}} a_n \exp(-\lambda_n s) \quad (I_\nu:\ [x_\nu] + f(x_\nu) \leqq x < x_\nu) ,$$

(ii) $$f_1(s,\theta,\alpha) = \sum_{\nu=1}^{+\infty} \exp(\alpha\theta/[x_\nu]) \cdot u_\nu(s) ,$$

$$u_\nu(s) = \sum_{\lambda_n \in I_\nu} a_n \exp(-\lambda_n s) ,$$

θ: a real constant, α: a parameter $(= \sqrt{-1}$ or $1)$,

$$\text{(iii)} \qquad \varepsilon'_n = \exp(\alpha\theta/[x_\nu]) \quad \text{for} \quad \lambda_n \in \{I_\nu\} ,$$
$$\varepsilon'_n = 1 \quad \text{for} \quad \lambda_n \bar{\in} \{I_\nu\} .$$

Then, by lemma 6 (B), (3.5) has the simple convergence-abscissa $\sigma = 0$. Put

$$\begin{aligned} & f(s,\theta,\alpha) - f(s,\theta',\alpha) \\ &= \sum_{\nu=1}^{+\infty} \{\exp(\alpha\theta/[x_\nu]) - \exp(\alpha\theta'/[x_\nu])\} \cdot u_\nu(s) \\ &= \sum_{n=1}^{+\infty} \varepsilon_n a_n \exp(-\lambda_n s) \quad \text{for} \quad \theta \neq \theta' , \end{aligned} \tag{3.6}$$

where

$$\varepsilon_n = 0 \quad \text{for} \quad \lambda_n \bar{\in} \{I_\nu\} ,$$
$$\begin{aligned} \varepsilon_n &= \exp(\alpha\theta/[x_\nu]) - \exp(\alpha\theta'/[x_\nu]) \\ &= \alpha(\theta - \theta')/[x_\nu] + 0(1/x_\nu^2) \quad \text{for} \quad \lambda_n \in \{I_\nu\} . \end{aligned}$$

Then, by lemma 4 (b) and lemma 6 (A), (3.6) has also the simple convergence-abscissa $\sigma = 0$.

(3.6) has $\sigma = 0$ as the natural boundary. To establish this fact, by lemma 1, it is sufficient to show that

$$\overline{\lim_{m\to\infty}} |O_m(1,\omega,t)|^{1/m} \geqq 1 \quad \text{for arbitrary } t \quad (-\infty < t < +\infty) . \tag{3.7}$$

Since

$$O_{[x_\nu]}(1,\omega,t) = \{\alpha(\theta - \theta')/[x_\nu] + O(1/x_\nu^2)\} \cdot O(x_\nu,t) ,$$

by lemma 4 (c)

$$\begin{aligned} \overline{\lim_{m\to\infty}} |O_m(1,\omega,t)|^{1/m} &\geqq \overline{\lim_{\nu\to\infty}} |O_{[x_\nu]}(1,\omega,t)|^{1/[x_\nu]} \\ &= \overline{\lim_{\nu\to\infty}} |O(x_\nu,t)|^{1/[x_\nu]} \\ &\geqq \underline{\lim_{\nu\to\infty}} |O(x_\nu,t)|^{1/[x_\nu]} \geqq 1 , \end{aligned}$$

from which follows (3.7).

Let us denote by $E(\theta,\alpha)$ the set of regular points on $\sigma = 0$ of (3.5), which is an open set. Then we have

$$E(\theta,\alpha) \frown E(\theta',\alpha) = 0 \quad \text{for} \quad \theta \neq \theta' . \tag{3.8}$$

In fact, if $s_0 \in E(\theta,\alpha) \frown E(\theta',\alpha) \neq 0$ (3.6), would be regular at $s = s_0$, which is impossible by (3.7). If $E(\theta,\alpha) \neq 0$ for all θ $(0 < \theta < \gamma$, γ: a fixed constant), by (3.8) the set of all functions $\{f(s,\theta,\alpha)\}$ is at most enumerable, which

contradicts the power of continuum of the set of θ. Hence $E(\theta,\alpha) = 0$ for at least one θ, which shows that $f(s,\theta,\alpha)$ has $\sigma = 0$ as the natural boundary. If $\alpha = \sqrt{-1}$ (or $=1$), (i) (or (ii)) of theorem 2 is established. q. e. d.

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