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A generalization of Tauber's theorem and some Tauberian constants (III)

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1. Introduction. In a previous paper [6] in this journal, I extended in a particular direction Tauber's well-known conditional converses of Abel's theorem for power series, following H. Hadwiger and R. P. Agnew. My extensions concern transformations of the kind

$$\Phi(t) \equiv \int_0^\infty K(ut)d\{A(u)\} , \quad t>0 , \tag{1}$$

with suitable $K(u)$, applied to functions $A(u)$ which are assumed to be of bounded variation in every finite interval of $u \geqslant 0$ and (for simplicity) subject to the condition $A(0) = 0$. The results obtained by me include inequalities of the type:

$$\overline{\lim_{t\to+0}} \mid A(\delta/t) - \Phi(t) \mid , \quad \delta>0 ,$$

$$\leqslant \begin{cases} T(\delta) \overline{\lim_{u\to\infty}} \ \overline{\underset{u\leqslant u'\leqslant\lambda u}{\text{bound}}} \dfrac{\mid A(u') - A(u) \mid}{\log \lambda} , \\ T^*(\delta) \overline{\lim_{u\to\infty}} \mid u^{-1} \int_0^u x d\{A(x)\} \mid , \end{cases}$$

where the upper limits are supposed to be finite, and $T(\delta)$, $T^*(\delta)$ are functions of the parameter δ, involving $K(u)$ but not $A(u)$. My results thus overlap in part certain theorems of Delange ([2], Théorèmes 3, 5), a fact of which I was unfortunately unaware when I wrote my paper [6]. However, in two later papers bearing the same title as the present one, I discuss results which supplement the theorems of Delange. In the first of these papers [7], I treat a general method of obtaining the Tauberian constants $T(\delta)$ for Riesz, Laplace-Abel, Lambert and Stieltjes transforms of $A(u)$, simultaneously with a similar absolute constant for the Borel transform of a sequence; while, in the second paper [8], I introduce a constant analogous to $T(\delta)$ useful in dealing with $A(u)$ which are λ_n-step functions defined in relation to a sequence

$$0<\lambda_1<\lambda_2<\cdots, \lambda_n \to\infty ,$$

with "wide steps", i. e. with $\liminf (\lambda_{n+1}/\lambda_n) > 1$. In the present note I modify slightly a lemma of Agnew's ([1], § 4) and reach with ease Delange's $T^*(\delta)$ in Theorem A and the more general constant $T^*(\delta, \lambda)$ in Theorem B for the special $K(u)$ of (17), revealing these constants at the same time as the best possible in the context of our inquiry[1]).

On the lines of my last-mentioned paper [8], the kernel $K(u)$ of the transform (1) is defined in terms of a function $N(x)$ which is bounded in every finite interval of $x > 0$ and such that

$$\left.\begin{array}{ll} N(x) \in L(0, \infty)\,, & N(x) \log x \in L(0, \infty)\,, \\ K(u) \equiv \int\limits_u^\infty N(x)dx\,, & K(0) = \int\limits_0^\infty N(x)dx = 1\,. \end{array}\right\} \tag{2}$$

Thus the $\varphi(u)$, $\psi(u)$ of my previous paper [6] in this journal are replaced by the more general $K(u)$, $N(u)$ respectively. Otherwise the notation of that paper is retained.

2. Lemmas. Two modifications of Agnew's lemma already referred to, required for the purpose of this note, will now be established.

Lemma 1. *If $f(x,t)$ is a real function of $x > 0$, $t > 0$, integrable in every finite x-interval and such that*

$$\int\limits_0^\infty |f(x,t)|\,dx < \infty\,, \qquad \overline{\lim_{t\to+0}} \int\limits_0^\infty |f(x,t)|\,dx = M\,, \tag{3}$$

$$\lim_{t\to+0} f(x,t) = 0 \text{ uniformly with respect to } x \text{ in } (0, X) \tag{4}$$

for any fixed $X > x_0 > 0$, then each real bounded function $g(x)$ of $x > 0$, for which

$$\underline{\lim_{x\to\infty}}\, g(x) = -L\,, \qquad \overline{\lim_{x\to\infty}}\, g(x) = L\,, \qquad 0 \leqslant L < \infty\,, \tag{5}$$

has plainly a transform

$$F(t) = \int\limits_0^\infty f(u,t)g(u)du\,. \tag{6}$$

And this transform is such that, for any given $\delta > 0$,

$$-(M+1)L \leqslant \overline{\underline{\lim_{t\to+0}}}\,[F(t) + g(\delta/t)] \leqslant (M+1)L\,. \tag{7}$$

The above conclusion is the best possible in the sense that there are two real functions $g(x)$ satisfying (5) *and such that each of the signs $\leqslant$ in* (7) *is in turn reduced to $=$ by one of the functions.*

1) My procedure simplifies Delange's treatment of $T^*(\delta)$ in [2], §§ 3.6–3.63, and so dispenses with the separate discussions of *Hadwiger* [4] and *Hartman* [5] which deal with case $\delta = 1$, $N(u) = e^{-u}$.

Proof. (5) implies that we can choose $X > x_0 > 0$, corresponding to any small $\varepsilon > 0$, so that $|g(x)| < L + \varepsilon$ for $x > X$. Hence (6) gives

$$F(t) + g(\delta/t) \left\{ \begin{aligned} &\leqslant \int_0^X |f(x,t)| \cdot |g(x)|\, dx + (L+\varepsilon) \int_X^\infty |f(x,t)|\, dx + g(\delta/t), \\ &\geqslant -\int_0^X |f(x,t)| \cdot |g(x)|\, dx - (L+\varepsilon) \int_X^\infty |f(x,t)|\, dx + g(\delta/t). \end{aligned} \right.$$

The first part of the lemma follows at once from the above step when we let $t \to +0$ and use (4), (5).

To prove the second part of the lemma we argue with $M > 0$ and $L > 0$, say $L = 1$, the case of either $M = 0$ or $L = 0$ being trivial. By (3) we can choose $t = t_1$ and then $x_1 > \max(x_0, \delta/t_1)$ so that

$$\int_0^\infty |f(x,t_1)|\, dx > M - \varepsilon\ , \qquad \int_{x_1}^\infty |f(x,t_1)|\, dx < \varepsilon\ .$$

In fact, we can determine inductively a null sequence $\{t_r\}$ and a divergent sequence $\{x_r\}$, $r = 1, 2, 3, \ldots$, as follows. After t_{r-1} and x_{r-1} have been chosen. $t_r < \min(t_{r-1}, \delta/x_{r-1})$ is chosen subject to the condition

$$\int_0^{x_{r-1}} |f(x,t_r)|\, dx < \varepsilon^r\ , \qquad \int_0^\infty |f(x,t_r)|\, dx > M - \varepsilon^r\ , \tag{8'}$$

and then $x_r > \max(x_{r-1}, \delta/t_r)$ is chosen so that

$$\int_{x_r}^\infty |f(x,t_r)|\, dx < \varepsilon^r\ , \tag{8''}$$

the choices of t_r and x_r in (8′) and (8″) being possible by (4) and (3). Now let

$$\left. \begin{aligned} g(x) &= \operatorname{sgn} f(x,t_r)\ . \qquad x_{r-1} < x \neq \delta/t_r < x_r\ , \\ g(\delta/t_r) &= 1\ , \qquad g(x_r) = -1\ , \end{aligned} \right\} \ r = 1, 2, \ldots, \tag{9}$$

where as usual $\operatorname{sgn} f = 0$ when $f = 0$ and $\operatorname{sgn} f = |f|/f$ when $f \neq 0$. Then $g(x)$ satisfies (5) with $L = 1$, and we obtain from (6):

$$\begin{aligned} F(t_r) &= \int_0^{x_{r-1}} f(x,t_r) g(x) dx + \int_{x_{r-1}}^{x_r} \cdots + \int_{x_r}^\infty \cdots \\ &\geqslant -\int_0^{x_{r-1}} |f(x,t_r)|\, dx + \int_{x_{r-1}}^{x_r} f(x,t_r) \operatorname{sgn} f(x,t_r) dx - \int_{x_r}^\infty |f(x,t_r)|\, dx \\ &\geqslant -2\int_0^{x_{r-1}} |f(x,t_r)|\, dx + \int_0^\infty |f(x,t_r)|\, dx - 2\int_{x_r}^\infty |f(x,t_r)|\, dx \\ &> -2\varepsilon^r + M - \varepsilon^r - 2\varepsilon^r = M - 5\varepsilon^r \end{aligned}$$

by (8′) and (8″). Therefore, for the $g(x)$ in (9),

$$\overline{\lim_{r\to\infty}}\,[F(t_r)+g(\delta/t_r)] \geqslant M+1=(M+1)L\,;$$

while, by the first part of the lemma, the above relation is also true with $\leqslant$ instead of $\geqslant$. Hence, for the $g(x)$ defined by (9),

$$\overline{\lim_{r\to\infty}}\,[F(t_r)+g(\delta/t_r)] = (M+1)L\ .$$

For the $g(x)$ which is the negative of the $g(x)$ in (9), we have

$$\underline{\lim_{r\to\infty}}\,[F(t_r)+g(\delta/t_r)] = -\,(M+1)L\ ,$$

and so the proof is complete.

Lemma 2. *This is a restatement of Lemma* 1 *for complex-valued* $g(x)$ *with*

$$(5)\ \textit{replaced by:}\quad \overline{\lim_{x\to\infty}}\,|\,g(x)\,| = L\ , \tag{5a}$$

$$(7)\ \textit{replaced by:}\quad \overline{\lim_{t\to+0}}\,|\,F(t)+g(\delta/t)\,| \leqslant (M+1)L\ , \tag{7a}$$

where the equality signs cannot be omitted.

3. Theorems. The theorems which follow are implicit in Lemmas 1, 2.

Theorem A. *In* (1), *let* $A(u)$ *be real- or complex-valued, in the latter case the real and the imaginary parts of* $A(u)$ *satisfying the condition already imposed on real* $A(u)$. *Also let*

$$\overline{\lim_{u\to\infty}}\,|\,u^{-1}B(u)\,| \equiv \overline{\lim_{u\to\infty}}\,\Big|\,A(u)-u^{-1}\int_0^u A(x)dx\,\Big| < \infty\ . \tag{10}$$

Then, for any $\delta>0$,

$$\overline{\lim_{t\to+0}}\,|\,A(\delta/t)-\Phi(t)\,| \leqslant T^*(\delta)\,\overline{\lim_{u\to\infty}}\,|\,u^{-1}B(u)\,| \tag{11}$$

where the equality sign is indispensable and

$$T^*(\delta)=1+\int_0^\delta\left|\frac{1-K(x)}{x}-N(x)\right|dx+\int_0^\infty\left|\frac{K(x)}{x}+N(x)\right|dx\,.^{2)} \tag{12}$$

---

[2]) Hypotheses (2) ensure the existence of the integrals composing $T^*(\delta)$ since it can be proved that they ensure the existence of

$$\int_0 \frac{|\,1-K(x)\,|}{x}\,dx\ ,\quad \int^\infty \frac{|\,K(x)\,|}{x}\,dx\ .$$

Proof. It is easy to show that (10) ensures first

$$\overline{\text{bound}}_{u\leqslant u'\leqslant\lambda u} |A(u') - A(u)| = O(1)[1 + \log\lambda] , \qquad u \to\infty ,$$

and thence $A(u) = O(\log u)$. The last relation, in conjunction with the manner of our defining $K(u)$ in (2), gives us, as $u\to\infty$,

$$K(ut)A(u) = K(ut)O(\log u) = o(1)$$

for every $t>0$. Hence we get, by an integration of (1) by parts,

$$\Phi(t) \equiv \int_0^\infty K(ut)d\{A(u)\} = t\int_0^\infty N(ut)A(u)du \equiv \Psi(t) .$$

(10) also ensures the existence of

$$\Psi_1(t) \equiv t\int_0^\infty N(ut)\frac{A_1(u)}{u}du , \qquad t>0 , \qquad A_1(u) \equiv \int_0^u A(x)dx ,$$

through the existence of $\Psi(t)$. In the above step we can express first $A_1(u)$ and then $\Psi_1(t)$ as follows:

$$\frac{A_1(u)}{u} = \int_0^u \frac{B(x)}{x^2}dx , \text{[3]} \tag{13}$$

$$\Psi_1(t) = t\int_0^\infty N(ut)du\int_0^u \frac{B(x)}{x^2}dx$$

$$= t\int_0^\infty \frac{B(x)}{x^2}dx\int_x^\infty N(ut)du = \int_0^\infty \frac{B(x)}{x^2}K(xt)dx , \tag{14}$$

justifying the inversion of integration by an appeal to Fubini's theorem with the help of (10). Hence the identity

$$A(u) - \Phi(t) = \left[A(u) - \frac{A_1(u)}{u}\right] - [\Psi(t) - \Psi_1(t)] + \left[\frac{A_1(u)}{u} - \Psi_1(t)\right]$$

yields, when we use (13) and (14) in the last term $[\cdots]$ of the right-hand member, and put $u = \delta/t$, the following relations:

$$A(\delta/t) - \Phi(t) = \frac{B(\delta/t)}{\delta/t} - t\int_0^\infty N(xt)\frac{B(x)}{x}dx + \left[\int_0^{\delta/t}\frac{B(x)}{x^2}dx - \int_0^\infty\frac{B(x)}{x^2}K(xt)dx\right]$$

$$= \frac{B(\delta/t)}{\delta/t} + \int_0^{\delta/t}\left[\frac{1-K(xt)}{x} - tN(xt)\right]\frac{B(x)}{x}dx$$

$$+ \int_{\delta/t}^\infty\left[-\frac{K(xt)}{x} - tN(xt)\right]\frac{B(x)}{x}dx . \tag{15}$$

[3]) To avoid useless complications we may suppose that $A(u) = O(u)$ as $u\to+0$ and thus ensure the existence of the integral in (13).

Now, in Lemma 2, we can choose

$$
\left.\begin{aligned}
g(x) &= B(x)/x \ , \\
f(x,t) &= \begin{cases} [1 - K(xt)]/x - tN(xt) & \text{for} \quad 0 < x < \delta/t \ , \\ -K(xt)/x - tN(xt) & \text{for} \quad x \geqslant \delta/t \ . \end{cases}
\end{aligned}\right\} \tag{16}
$$

The above choice of $g(x)$ is justified by the fact that (5a) holds in the form (10). And the choice of $f(x,t)$ is justified by the following facts. (a) If $t \leqslant 1$ (such a restriction on t being permissible as we are going to let $t \to +0$), and x is in any finite interval $(0, X)$ where $X > x_0 = \delta$, then

$$
|f(x,t)| \leqslant \begin{cases} t \,|\int\limits_0^{xt} N(u)du\,|/xt + t\,|N(xt)| < C_1 t + C_2 t \quad (0 < xt < \delta) & , \\ t\,|K(xt)|/xt + t\,|N(xt)| \quad < C_3 t + C_4 t \quad (\delta \leqslant xt \leqslant X) \ , \end{cases}
$$

where the C's are constants depending only on δ and X, and therefore (4) holds. (b) Furthermore (3) holds since, defining $T^*(\delta)$ by (12), we have

$$
\int\limits_0^{\infty} |f(x,t)|\,dx = \int\limits_0^{\delta/t} \cdots + \int\limits_{\delta/t}^{\infty} \cdots = T^*(\delta) - 1 \ .
$$

Thus, finally, an appeal to Lemma 2, with the choices of f and g in (16), enables us to pass from (15) to the conclusion (11). That the equality sign in (11) is indispensable is established by choosing the particular $g(x)$ of (16), in terms of the particular $f(x,t)$ of (16), exactly as in the general case where we establish the indispensability of the equality sign in (7a). Of course the specification of g involves the following specification of $A(u)$ in consequence of (13):

$$
\left.\begin{aligned} A(u) &\equiv g(u) + u^{-1}A_1(u) \\ &= g(u) + \int\limits_0^u x^{-1} g(x)dx \end{aligned}\right\} g(x) = \frac{B(x)}{x} \ .
$$

A generalization of the proof of Theorem A brings to light a constant $T_k^*(\delta)$ which is featured in the corollary that follows.

Corollary A. *In the integral transform defined by*

$$
\Psi_k(t) \equiv t \int\limits_0^{\infty} N(ut)\,\sigma_k(u)du \ , \qquad t > 0, \qquad k \geqslant 0 \ ,
$$

where

$$
\sigma_r(u) \equiv \frac{r}{u^r} \int\limits_0^u (u - x)^{r-1} A(x)dx \ , \qquad r > 0 \ , \qquad \sigma_0(u) \equiv A(u) \ ,
$$

let

$$
\varlimsup_{u \to +\infty} |u^{-k-1} B_k(u)| \equiv \varlimsup_{u \to \infty} (k+1)\,|\sigma_k(u) - \sigma_{k+1}(u)| < \infty \ .
$$

Then, for any $\delta > 0$,

$$\overline{\lim_{t\to+0}} \mid \sigma_k(\delta/t) - \Psi_k(t) \mid \leqslant T_k^*(\delta) \overline{\lim_{u\to\infty}} \mid u^{-k-1} B_k(u) \mid {}^{4)}$$

where $T_k^*(\delta)$ *is obtained from* $T^*(\delta)$ *by changing* $N(x)$ *to* $N(x)/(k+1)$ *in the two integrals composing* $T^*(\delta)$ *in* (12).

The proof of Corollary A is like that of Theorem A but makes use of the following easily proved relations in place of (13) and (14):

$$\sigma_{k+1}(u) = \int_0^u \frac{B_k(x)}{x^{k+2}} dx \ , \quad \Psi_{k+1}(t) = \int_0^\infty \frac{B_k(x)}{x^{k+2}} K(xt) dx \ .$$

Next follows a theorem which supplements Theorem A in the following cases of $K(u)$ considered in my previous paper [6].

$$\left.\begin{array}{l} \text{(i)}\ \ K(u) = (1-u)^k,\ k \geqslant 1,\ \text{for}\ u \leqslant 1;\ K(u) = 0\ \text{for}\ u > 1. \\ \text{(ii)}\ \ K(u) = e^{-u}. \quad \text{(iii)}\ \ K(u) = (1+u)^{-\varrho},\ \varrho < 0. \\ \text{(iv)}\ \ K(u) = u/(e^u - 1)\ \text{for}\ u \neq 0,\ K(0) = 1. \end{array}\right\} \quad (17)$$

Theorem B. *Suppose that, in Theorem* A, $A(u)$ *is real and* $N(u)$ *is additionally assumed to be positive and monotonic decreasing for* $u > 0$. *Suppose further that the hypothesis* (10) *is replaced by*

$$\underline{\lim_{u\to\infty}}\, u^{-1} B(u) = -L/p \ , \quad \overline{\lim_{u\to\infty}}\, u^{-1} B(u) = L/q \ ,$$

$$L \geqslant 0 \ , \quad p > 0 \ , \quad q > 0 \ , \quad p^{-1} + q^{-1} = 1 \ .$$

Then the conclusion (11) *will be replaced by*

$$-\frac{L}{p} T^*(\delta, p) \leqslant \overline{\lim_{t\to+0}} [A(\delta/t) - \Phi(t)] \leqslant \frac{L}{q} T^*(\delta, q)$$

where the equality signs are indispensable and

$$T^*(\delta, \lambda) = 1 + \int_0^\delta \left[\frac{1-K(x)}{x} - N(x)\right] dx + (\lambda - 1) \int_\delta^\infty \left[\frac{K(x)}{x} + N(x)\right] dx \ , \ \lambda > 1 .$$

[4]) On condition that (i) $\Phi(t)$ in (1) exists as a Lebesgue-Stieltjes integral (and not merely as a Riemann-Stieltjes integral which sufficies for the results of this note), (ii) $N(u)$ is positive and monotonic decreasing, as in the cases of the $K(u)$ of (17), it can be shown that $\Psi_k(t)$ exists for $t > 0$ and

$$\liminf \Phi(t) \leq \begin{array}{c}\limsup \\ \liminf\end{array} \Psi_k(t) \leq \limsup \Phi(t) \ , \qquad t \to +0 \ ,$$

provided that the extreme members of the above inequalities are finite. Thus, assuming (i), (ii) and the last-stated proviso, we can connect $\sigma_k(\delta/t)$ with $\Phi(t)$ as well as with $\Psi_k(t)$, as, for instance, in the special relations (29), (30) of my paper [6].

Proof. Theorem B is easily deduced from Lemma 1, exactly as Theorem A from Lemma 2, with the same choice of $f(x, t)$ as in (16) but with

$$g(x) = 2\frac{B(x)}{x} + L\left(\frac{1}{p} - \frac{1}{q}\right).$$

In the particular cases of the $K(u)$ of (17), there is a result which includes Theorem A and may be proved like that theorem. This result, stated below as Theorem C, is similar to Agnew's ([1], Theorem 3.1) where the Tauberian condition on $A(u)$, instead of being in the Kronecker form (10), is in the simpler Hardy form [5]).

Theorem C. *Suppose that, in Theorem* A, *the* $T^*(\delta)$ *of* (12), *considered as a function of* $\delta > 0$, *has a unique minimum which is necessarily the least* $T^*(\delta)$, *a condition which is satisfied in the cases of the* $K(u)$ *of* (17). *Then we have, in addition to* (11),

$$\overline{\lim_{t \to +0}} \mid A(u) - \Phi(t) \mid \leqslant \max \{T^*(\alpha), T^*(\beta)\} \overline{\lim_{u \to \infty}} \mid u^{-1} B(u) \mid$$

where the equality sign is indispensable and u in the left-hand member is such that

$$0 < \alpha = \underline{\lim_{t \to +0}}\, ut \leqslant \overline{\lim_{t \to +0}}\, ut = \beta < \infty .$$

The various special types of elementary Tauberian theorems deducible from Theorems A, B, by known methods ([6], pp. 222–223) are collected here for convenience.

(I) *In Theorem* A, *the additional condition* $\lim \Phi(t) = \infty$ *as* $t \to +0$ *implies* $\lim A(u) = \infty$ *as* $u \to \infty$.

(II) *In the special case of Theorem* A *where the upper limit in* (10) *is* 0, *the limit points of* $\Phi(t)$ *and* $A(u)$ *are identical.*

(III) *In Theorem* A, *whenever* $T^*(\delta)$ *has an absolute minimum* τ^* *as in the cases of the* $K(u)$ *of* (17), *each limit point* z' *of* $A(u)$ *corresponds to a limit point* z'' *of* $\Phi(t)$, *and conversely, such that*

$$\mid z' - z'' \mid \leqslant \tau^* \overline{\lim_{u \to \infty}} \mid u^{-1} B(u) \mid . \text{[6])} \tag{18}$$

---

[5]) Theorems A and B, like Theorem C, have analogues, suggested by my earlier work ([6], § 2), in which the Tauberian condition is of the Hardy form and involves $ua(u)$ instead of $u^{-1}B(u)$, $a(u)$ being such that $A(u) = \int_0^u a(x)dx$. The treatment of these analogues is of course similar to, but simpler than, that of Theorems A, B.

[6]) As Garten has shown ([3], Satz 2), τ^* in the particular case of the $K(u)$ of 17(i) is also the constant figuring in the analogue of (18) which connects a limit point z' of a sequence s_n and a limit point z'' of the sequence of kth Cesàro means of s_n $(k = 1, 2, 3 \ldots)$.

It is not known whether τ^* is the best (or least) constant in the above inequality[7]).

(IV) *In the special case of Theorem* B *in which* $q \to 1$ (*or* $p \to \infty$), *we have the following result* (*which is best-possible since Theorem* B *is so*)*:*
if

$$\varliminf_{u\to\infty} u^{-1}B(u) = 0\,, \qquad \varlimsup_{u\to\infty} u^{-1}B(u) < \infty\,,$$

then

$$\varliminf_{u\to\infty} A(u) = \varliminf_{t\to+0} \Phi(t)\,, \qquad \varlimsup_{u\to\infty} A(u) \leqslant \varlimsup_{t\to+0} \Phi(t) + \varlimsup_{u\to\infty} u^{-1}B(u)\,.$$

4. Corrigenda. Vol. 24 (1950), pp. 219–231. I take this opportunity to list the corrections which should be made in my paper [6]:

P. 219. In relation (2), read '$\varphi(u) = \dots$' for '$\psi(u) = \dots$'.

P. 220. At the end of relation (9), replace [1]) by [4]).

P. 223. In line 10, read

$$\text{`}\varlimsup_{u\to\infty} S(u) = \varlimsup_{t\to+0} F(t)\text{'} \quad \text{for} \quad \text{`}\varliminf_{u\to\infty} S(u) = \varlimsup_{t\to+0} F(t)\,\text{'}.$$

P. 223. In the Note, read '$k > 1$' for '$k \geqslant 1$'.

P. 227. In line 2, read '$(r+1)\int_0^u A_r(u)\,du$' for '$\frac{1}{r+1}\int_0^u A_r(u)\,du$'.

P. 229. In line 7, read '$J_{k+1}(t)$' for '$J_{k+t}(t)$'.

P. 231. In line 5 read '$0 \leqslant u < \lambda_1$' for '$0 \leqslant u < \lambda$'.

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7) *Agnew* ([1], § 4) has partially answered the corresponding question for the absolute minimum of $T(\delta)$ defined in the Introduction, in certain cases which include the $K(u)$ of (17).

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