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A property of bounded analytic functions

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A well known theorem of Fatou [4] asserts that if $f(z)$ is a bounded regular function defined in $|z|<1$, then the radial limit $\lim_{r\to 1} f(re^{i\theta})$ where r tends to unity through real values less than one exists for all θ except for a set of at most zero measure, $0 \leqslant \theta < 2\pi$. We denote the set of all points at which this limit exists by $F(f)$, see [3].

For any point ζ of $|z|=1$ the cluster set of $f(z)$ at ζ, $C(f;\zeta)$, is defined to be the set of all complex numbers w such that there exists a sequence $\{z_n\}$ with the three properties

$$|z_n|<1 \quad n=1,2,\ldots; \qquad z_n \to \zeta; \qquad f(z_n)\to w .$$

The set of points ζ for which $C(f;\zeta)$ is a single point will be denoted by $G(f)$.

It is trivial that $F(f)\supset G(f)$. The object of this note is to show that the set of points in $F(f)$ and not in $G(f)$ is a set of the first category. This result has a similar appearance to the theorem of E. F. Collingwood [3], concerning the sets of points $S(f)$ and $I(f)$.[1] But both the content and method of proof are different from those of Collingwood.

The connection of this result with Fatou's theorem is that it enables us to disprove a natural conjecture. Since sets of zero measure and sets of first category can to some extent be regarded as interchangeable[2] one might suppose that the set $F(f)$ would contain all of $|z|=1$ except a set of first category. In fact not only is this false but it is even true that $F(f)$ itself need only be a set of first category. As concrete examples of such functions $f(z)$ there are,

(A) a Blaschke product

$$\mathrm{f}(z)=\prod_{\nu=1}^{\infty}\frac{ze^{-i\theta_\nu}-|z_\nu|}{1-z\bar{z}_\nu},$$

where $|z_\nu|<1$, amp $z_\nu=\theta_\nu$, $\sum_{\nu=1}^{\infty}(1-|z_\nu|)<\infty$ and the points z_ν are

[1]) See [3], p. 177, for definitions.

[2]) See [8], p. 77.

so distributed that the closure of their union contains the whole circumference $|z| = 1$.

(B) a certain function $g(z)$ which maps $|z| < 1$ onto a bounded simply connected domain whose prime-ends are all of diameter greater than some positive number δ. The existence of such functions has been established by F. Frankl [5]. $g(z)$, unlike the Blaschke product of (A), is univalent.

For both the function $f(z)$ of (A) and $g(z)$ of (B) and for every ζ of $|z| = 1$, the cluster set at ζ does not reduce to a single point. This is obvious for the function $g(z)$ since the cluster set of $g(z)$ at ζ is a prime end of the domain onto which $g(z)$ maps $|z| < 1$. For the function $f(z)$ of (A), the radial limit exists for almost all θ and is of modulus unity (see [7], p. 196). But for any ζ there is a subsequence of zeros of $f(z)$ that tends to ζ. Thus for almost all ζ of $|z| = 1$, $C(f; \zeta)$ has diameter greater than or equal to unity. But the set of ζ for which the diameter of $C(f; \zeta)$ is greater than or equal to unity is a closed set, and thus contains the whole of $|z| = 1$.

We shall refer to a point that belongs to the set $F(f)$ as a Fatou point and a point that belongs to $G(f)$ as a convergence point. We prove next the theorem :

Theorem. *If a non-constant, regular and bounded function* $f(z)$ *defined in* $|z| < 1$ *is such that for a certain arc* γ *of* $|z| = 1$, *the set* $F(f)$ *is of second category on every subarc of* γ, *then all points of* γ *except at most those of a set of first category belong to* $G(f)$.

It is sufficient to show that, given any positive number ε, and any subarc γ_1 of γ, there is a subarc of γ_1 say γ_2 such that if ζ belongs to γ_2, then the diameter of $C(f; \zeta)$ is less than ε. For suppose that this had been established then it would follow that the set of points ζ on $|z| = 1$ for which the diameter of $C(f; \zeta)$ is greater than or equal to ε would not contain any interval of $|z| = 1$ in γ and, since this set is clearly closed, it would be non-dense in γ. We allow ε to assume a sequence of positive values decreasing to zero and we conclude that the set of points ζ for which the diameter of $C(f; \zeta)$ is positive is a set of first category in γ. This is the statement of the theorem.

Suppose then that γ_1 is a given subarc of γ which we shall assume without real loss of generality to be a small arc of $|z| = 1$. Denote by D_1 the part of $|z| < 1$ bounded by γ_1 and by the linear segment joining the endpoints of γ, and denote by $E_1 = f(D_1)$ the set of values assumed by $f(z)$ as z varies in D_1.

If E_1 is of diameter less than ε then any subarc of γ_1 (strictly interior to γ_1) will serve as the arc γ_2.

Otherwise select a point p_1 of E_1 and let K_1 and L_1 denote respectively the subsets of D_1 for which

$$| f(z) - p_1 | = \varepsilon/2 \ , \qquad | f(z) - p_1 | = \varepsilon/4 \ .$$

Since the values taken by $f(z)$ at the set of points z where $f'(z) = 0$ form an enumerable set, we can select p_1 such that for every point of K_1 and L_1, $f'(z) \neq 0$. We also choose p_1 so that its distance from the component of the complement of E_1 which contains the point at infinity, is less than $\varepsilon/8$.

Under the above circumstances each of K_1 and L_1 consists of the intersection of $|z| < 1$ with an at most enumerable set of arcs whose end points belong to $|z| = 1$.[3])

Let $k_1(r)$ denote the union of all the arcs of K_1 that are completely contained in the annulus, $1 - r \leqslant |z| \leqslant 1$, and define $l_1(r)$ similarly. For any arc α of K_1 let $p(\alpha)$ be the projection of α in $0 < |z| < 1$ from the point $z = 0$ onto the circumference $|z| = 1$ by means of the radii $\arg z =$ constant. $p(\alpha)$ is a subinterval of γ which may be open or closed or semi-open. Further write

$$h_1(r) = \bigcup_{\alpha \in k_1(r)} p(\alpha) \ ; \qquad m_1(r) = \bigcup_{\alpha \in l_1(r)} p(\alpha) \ .$$

Let r_i be a sequence of positive numbers decreasing to zero and write

$$H_1 = \bigcap_{i=1}^{\infty} h_1(r_i) \ ; \qquad M_1 = \bigcap_{i=1}^{\infty} m_1(r_i) \ .$$

Now if $\zeta \in H_1$ the radius vector joining $z = 0$ to $z = \zeta$ intersects K_1 in a sequence of points whose closure contains the point ζ; for if this were not the case, there would be a segment of the radius say $\zeta_1 \zeta$ disjoint from the set K_1. Choose the integer i so large that r_i is less than the distance of ζ from ζ_1. Since $\zeta \in H_1 \subset h_1(r_i)$ there is an arc α of $k_1(r_i)$ that intersects the radius joining $z = 0$ to $z = \zeta$. Moreover by the definition of $k_1(r_i)$, α must meet segment $\zeta \zeta_1$ and this is a contradiction which establishes the statement above.

It follows that if ζ belongs to $H_1 \cap M_1$ it does not belong to $F(f)$ i. e.

$$H_1 \cap M_1 \cap F(f) = 0 \ . \tag{1}$$

Next consider the set $h_1(r_i)$. This is the union of an at most enumerable

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[3]) This follows because $f'(z) \neq 0$ for $z \in K_1$ or $z \in L_1$ and by applications of Koebe's lemma; see [4], [2], p. 96. A similar argument is used in [9], p. 247.

sequence of intervals $p(\alpha)$ of γ_1. If $h_1(r_i)$ is dense in γ_1, so is the set obtained from $h_1(r_i)$ by replacing $p(\alpha)$ by the same arc without its end points; i. e. if $h_1(r_i)$ is dense in γ_1 it contains an open subset of γ_1 also dense in γ_1. Thus finally if each $h_1(r_i)$, $i = 1, 2, \ldots$ is dense in γ_1, then

$$H_1 = \bigcap_{i=1}^{\infty} h_1(r_i)$$

contains a G_δ set dense in γ_1. By a similar argument applied to M_1 we see that either one of the sets $h_1(r_i)$, $m_1(r_i)$ is not dense in the whole of γ or the set N, the union of the complements of H_1 and M_1 in $|z| = 1$ is of first category on γ_1. But (1) implies $F(f) \subset N$ and $F(f)$ is by hypothesis of second category; thus this situation cannot arise and at least one of the sets $h_1(r_i)$, $m_1(r_i)$, $i = 1, 2, \ldots$ is not dense in the whole of γ_1.

Suppose then that the set $h_1(r_i)$ contains no point of the closed subarc β of γ_1. Let β_1 be the arc which is the middle third of the arc β. Denote the circle $|z| = 1 - r_i$ by T and suppose that β_1 is the arc $\{z \mid z = e^{i\theta}$, $0 \leqslant \theta_1 \leqslant \theta \leqslant \theta_2 < 2\pi$. Let X be the set of points

$$\{z \mid 1 - r_i \leqslant |z| \leqslant 1, \theta_1 \leqslant amp\, z \leqslant \theta_2\} \ .$$

By the definition of β_1 any arc of K_1 which meets X must also meet T, for if this were not so, the arc of K_1 concerned would belong to $k_1(r_i)$ and some point of β_1 would belong to $h_1(r_i)$. If there were infinitely many such arcs then there would be a point say z_0 of T such that every neighbourhood of z_0 contains points of infinitely many arcs of K_1. But K_1 is closed and thus $z_0 \epsilon K_1$. But this implies that $f'(z_0) = 0$ and thus there cannot be infinitely many arcs of K_1 with points in every neighbourhood of z_0. Thus only a finite number of arcs of K_1 meet X.

Hence there is a point of β_1, say ζ_1, and a positive number δ such that every point of K_1 is distant at least δ from ζ_1. Select two points ζ', ζ'' of $|z| = 1$ whose distances from ζ_1 are less than δ. Denote the small arc of $|z| = 1$ whose end points are ζ' ζ'' by $\gamma(1)$ and let $D(1)$ be the part of $|z| < 1$ bounded by $\gamma(1)$ and the linear segment joining ζ' to ζ''.

Since $D(1)$ does not meet K_1 there are two possibilities; (i) the set of values $E(1)$ taken by $F(z)$ in $D(1)$ is contained in the circle whose centre is p_1 and whose radius is $\varepsilon/2$, or (ii) the set of values $E(1)$ is exterior to this circle.

If it had been one of the sets $m_1(r_i)$ that was not dense in the whole of γ_1 then we should have been led to similar conclusions except that the $\varepsilon/2$ would be replaced by $\varepsilon/4$. We can cover both cases without needless repetition by retaining the $\varepsilon/2$ in (i) and replacing it by $\varepsilon/4$ in (ii).

If (i) holds we have an arc $\gamma(1)$ which is a subarc of γ_1 and has all the properties that we required of the arc γ_2.

If (ii) holds we repeat the argument. That is to say we find a point p_2 of $E(1)$, an arc $\gamma(2)$ contained in $\gamma(1)$ and corresponding sets $D(2)$ and $E(2)$ such that either

(i) $E(2)$ is contained in the circle centre p_2 and radius $\varepsilon/2$ or

(ii) $E(2)$ is contained in the exterior of the circle centre p_2 and radius $\varepsilon/4$.

If (i) holds $\gamma(2)$ will serve as the arc γ_2. Otherwise we repeat the argument.

Since every pair of points of the sequence $p_1, p_2, \ldots$ are at a distance of at least $\varepsilon/4$ apart and since all these points are contained in the bounded set of values taken by $F(z)$ in $|z|<1$, only a finite number of repetitions of the argument are possible. We are eventually led to case (i) and to an arc with the required properties.

This completes the proof of the theorem.

Remark. To see that the Blaschke product $f(z)$ of (A) and the function $g(z)$ of (B) are such that their Fatou sets $F(f)$ and $F(g)$ are of first category we have only to observe that to any set of second category on $|z| = 1$, there always corresponds an arc such that the set is of second category at each point of the arc[4]). Now if $F(f)$ (or $F(g)$) was of second category we could find an interval of $|z| = 1$ as described above. In this interval the set $G(f)$ (or $G(g)$) would, by the theorem, be of second category. However this is not so since we know that $G(f)$ and $G(g)$ are void.

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[4]) See [1]. See also Bagemihl and Seidel, Proc. Nat. Acad. Sci. 39 (1953) 1068–1075.