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The associated form of a variety over a field of prime characteristic p

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Introduction

Wei-Liang Chow and van der Waerden in a publication [1] have introduced the associated form of an irreducible variety V . If d is the dimension of V , the associated form $F(u)$ is defined as an irreducible form in u_0, u_1, \dots, u_n , depending on d generic hyperplanes $u^{(1)}, \dots, u^{(d)}$, such that $F(u)$ becomes zero as soon as the hyperplane u is specialised so as to contain one of the points of intersection of $u^{(1)}, \dots, u^{(d)}$ with V . The form $F(u)$ is symmetric or antisymmetric in the $d + 1$ sets of variables $u, u^{(1)}, \dots, u^{(d)}$.

André Weil in his "Foundations of Algebraic Geometry" [2] gave new definitions of the fundamental notions of algebraic geometry. In particular, he introduced the notions of algebraically disjoint and of linearly disjoint fields and he proved the theorem ([2], Th. 5, p. 18): An extension $k(x)$ of a field k and the algebraic closure \bar{k} of k are linearly disjoint if and only if k is algebraically closed in $k(x)$, and $k(x)$ separably generated over k .

W.-L. Chow used the characteristic form in his investigation of "Algebraic systems of positive cycles in an algebraic variety" [3]. In the introduction of his paper he mentioned, without proof, the following property of the characteristic form: If the variety is separably generated then the associated form has no multiple factors.

We shall investigate quite generally, how the characteristic form, which is irreducible in K , factorises in an extension field L of K , and how this factorisation is related to the splitting of V into varieties V_1, V_2, \dots irreducible over L . In particular Chow's assertion mentioned above will be proved.

1. Definitions and notations

Let us take an arbitrary field k as *ground field*. We shall assume k to be of characteristic p . The *universal extension field* Ω is obtained from k by

adjunction of a countable number of indeterminates and algebraic closure. All coordinates of points and all coefficients of equations are always taken from Ω .

Let K, L, \dots stand for intermediate fields which contain k and are contained in Ω . These intermediate fields are always supposed to be generated by the adjunction of a finite number of elements to k .

An intermediate field L is said to be *separably generated* over K , if L is generated from K by adjunction of algebraically independent elements and separable algebraic functions of these elements.

A series of n coordinates p_1, p_2, \dots, p_n from Ω is called a *point of the affine space* R_n , and a *point of the projective space* S_n is a ray of the affine space R_{n+1} consisting of all points $(\omega p_0, \omega p_1, \dots, \omega p_n)$, where $(p_0, \dots, p_n) \neq (0, 0, \dots, 0)$ is a fixed point of R_{n+1} and ω runs over all the elements of Ω .

A *variety* is the set of all points of R_n or S_n which satisfy a finite system of algebraic equations,

$$f_k(p_1, p_2, \dots, p_n) = 0 \quad \text{or} \quad f_k(p_0, p_1, \dots, p_n) = 0$$

where f_k shall be polynomials in the first case, forms in the second case with coefficients from Ω . We shall suppose that the set is non-empty.

If a variety can be represented as a union of two proper parts (sub-varieties), it is said to be *divisible*. The variety is *indivisible* if such a representation is not possible.

If the equations that define the variety have their coefficients in K , the variety is called a *variety over* K . It is *irreducible over* K if it does not split into proper parts which are again varieties over K . By definition an indivisible variety remains irreducible over any extension field, i. e., it is *absolutely irreducible*.

A point P is said to be a *specialisation* of a point X with respect to a field K , if all equations $f(x_1, \dots, x_n) = 0$ with coefficients from K , or in the projective case all homogenous equations $f(x_0, x_1, \dots, x_n) = 0$, which are valid for the point X , remain valid if X is replaced by P .

An irreducible variety V over K has always a *generic point* X such that all points of V can be obtained by specialisation (with respect to K) of X . The generic point is uniquely determined by V except for isomorphisms. That is, in the affine case the coordinates x_1, \dots, x_n are uniquely determined except for a field isomorphism applied to all x_k , which leaves the elements of K unaltered. In the projective case the x_k are uniquely determined only up to a common factor ω . We may number the coordinates so that $x_0 \neq 0$ and then normalise ω so that $x_0 = 1$. The non-

homogeneous coordinates x_1, \dots, x_n of the point X are then uniquely determined but for an isomorphism. The number of the algebraically independent coordinates among the so normalised x_k is called the *dimension of V* .

The above terminology is in accordance with the suggestions of van der Waerden in one of his recent papers [4].

If $V = V_1 + V_2 + \dots + V_r$, and all the imbedded V_i are left out and the rest have the same dimension then the variety is said to be *unmixed* or *pure*.

We shall call with André Weil [2] an extension $K(X)$ of a field K *regular over K* or a *regular extension of K* if \overline{K} (the algebraic closure of K) and $K(X)$ are linearly disjoint over K .

2. The associated form of a variety

Let V be an irreducible variety of dimension d over a field K in the projective space S_n .

Let $u^{(1)}, \dots, u^{(d)}$ be hyperplanes with indeterminate coordinates $u_k^{(v)}$. The indeterminates $u_k^{(v)}$ shall be algebraically independent over K . The hyperplanes intersect V in a finite number of points $X^{(1)}, \dots, X^{(g)}$, conjugate over K .

Now we take in addition a further series of indeterminates,

$$u_k \quad (k = 0, 1, \dots, n) .$$

The product,

$$P = \prod_1^g (u_0 x_0^{(v)} + u_1 x_1^{(v)} + \dots + u_n x_n^{(v)}) \quad (1)$$

is a symmetric function in $X^{(1)}, \dots, X^{(g)}$.

In case of characteristic zero the product is rational in

$$K(u, u^{(1)}, \dots, u^{(d)}) :$$

In this case we write $P = Q(u, u^{(1)}, \dots, u^{(d)})$.

In case of characteristic p a p^e th power of the product P is rational and we write, taking e to be the lowest possible exponent,

$$P^q = Q(u, u^{(1)}, \dots, u^{(d)}), \quad (q = p^e) . \quad (2)$$

Q is integral in u and rational in $u^{(1)}, \dots, u^{(d)}$. We can, therefore, write

$$Q = \frac{A}{B} F(u, u^{(1)}, \dots, u^{(d)}) \quad (3)$$

where A and B depend only on $u^{(1)}, \dots, u^{(d)}$, while F is integral in $u, u^{(1)}, \dots, u^{(d)}$, and contains no more factors depending only on $u^{(1)}, \dots, u^{(d)}$.

Q is irreducible in $K(u^{(1)}, \dots, u^{(d)})[u]$ and hence F is irreducible in $K[u^{(1)}, \dots, u^{(d)}, u]$.

For, if F is reducible in $K[u^{(1)}, \dots, u^{(d)}, u]$, let $F = GH$, where G and H both contain u . Consequently, $Q = \frac{A}{B} GH = \left(\frac{A}{B} G\right)H$, contrary to hypothesis.

The irreducible form F is called the *associated form of V* .

We shall now show that a permutation of the variable series $u, u^{(1)}, u^{(2)}, \dots, u^{(d)}$ leaves F unaltered up to a factor ± 1 .

The condition,

$$F(v, v^{(1)}, \dots, v^{(d)}) = 0$$

is necessary and sufficient in order that the hyperplanes $v, v^{(1)}, \dots, v^{(d)}$ have a point in common with V ([5] § 36, p. 157).

In the same way the condition,

$$F(v^{(1)}, v, \dots, v^{(d)}) = 0$$

(with v and $v^{(1)}$ interchanged) is necessary and sufficient in order that $v, v^{(1)}, \dots, v^{(d)}$ have a point in common with V . The two conditions being equivalent, and both forms $F(u, u^{(1)}, \dots, u^{(d)})$ and

$$F(u^{(1)}, u, u^{(2)}, \dots, u^{(d)})$$

being irreducible, they must be proportional :

$$F(u^{(1)}, u, \dots, u^{(d)}) = \gamma F(u, u^{(1)}, \dots, u^{(d)})$$

where γ is a constant. The square of a transposition being identity, γ^2 must be equal to 1, so γ can only be $+1$ or -1 . The same is true for all transpositions of two of the $d+1$ series $u, u^{(1)}, \dots, u^{(d)}$.

Since every permutation is a product of two transpositions, it follows that every permutation leaves F invariant but for a factor ± 1 .

In the following we shall be concerned only with the associated forms of varieties over a field K of characteristic p , where p is a prime number.

3. The behaviour of the associated form over an extended field

Let V be irreducible over a field K and d be the dimension of V . Then over any extension L of K , V is an unmixed variety of dimension d .

This theorem, which is proved by Hodge and Pedoe ([6], § 11, Th. 1,

p. 69) for the case of a field of characteristic zero, is also true for the case of a field of characteristic $p > 0$, since the conditions mentioned in the proof of the above theorem are independent of the characteristic of the field.

Let the field K be of characteristic p . The associated form F defined in § 1 is irreducible over K .

Let $L = K(t_1, \dots, t_s)$ be a purely transcendental extension. That is, let t_1, \dots, t_s be algebraically independent over K . Now we shall prove

Theorem 1. *A purely transcendental extension $L = K(t_1, \dots, t_s)$ leaves F and V irreducible.*

Proof: Suppose F could be factorised in $K(t)[u]$, e. g.

$$F(u) = g(t, u) \cdot h(t, u) .$$

By a well known theorem of Gauss ([7] I, § 23) this factorisation would imply a factorisation in $K[t, u] = K[t][u]$, say

$$F(u) = G(t, u) \cdot H(t, u)$$

where G and H are polynomials in t and u . Putting all $t_i = 0$, we would obtain a factorisation of $F(u)$ in K , which is impossible, i. e. $F(u)$ cannot be factorised in $K(t_1, \dots, t_s) = L$.

If V were reducible, the points of intersection $X^{(v)}$ would split up into the generic points of V_1 , generic points of V_2 and so on. This implies a factorisation of $F(u)$, as will be shown in the proof of theorem 4.

Theorem 2. *A transcendental extension L of K , in which the form F can be factorised into h factors,*

$$F(u) = G_1(u) G_2(u) \dots G_h(u), \quad (\text{in } L[u])$$

always contains an algebraic extension A , in which $F(u)$ can be factorised in the same way:

$$F(u) = C F_1(u) F_2(u) \dots F_h(u), \quad (\text{in } A[u])$$

so that the factors F_j are not essentially different from G_j .

Proof: For the sake of convenience, the u_j and $u_j^{(i)}$ of our earlier notation will be replaced by $u_j^{(0)}$ and $u_j^{(i)}$. Let F be of order g and let k be any integer greater than g which we can choose once and for all. Let us fix $(d + 1)(n + 1)$ integers r_{ij} such that

$$0 \leq r_{00} < r_{01} < \dots < r_{0n} < r_{10} < \dots < r_{1n} < \dots < r_{d0} < \dots < r_{dn} .$$

Let $\Phi(u_j^{(i)})$ be any polynomial in the $u_j^{(i)}$ such that no $u_j^{(i)}$ appears to a power greater than g and let $\varphi(t)$ be the polynomial in t obtained by replacing $u_j^{(i)}$ in $\Phi(u_j^{(i)})$ by t to the power $k^{r_{ij}}$ ($i = 0, \dots, d; j = 0, \dots, n$). Consider now a term in $\Phi(u_j^{(i)})$ in which $u_j^{(i)}$ has exponent ρ_{ij} . From this we get a term in $\varphi(t)$ with the exponent $\sum \rho_{ij} k^{r_{ij}}$. Another term in $\Phi(u_j^{(i)})$ in which $u_j^{(i)}$ has exponent σ_{ij} leads to a term in t with exponent $\sum \sigma_{ij} k^{r_{ij}}$ and since $\rho_{ij} \leq g < k, \sigma_{ij} \leq g < k$ we have $\sum \rho_{ij} k^{r_{ij}} = \sum \sigma_{ij} k^{r_{ij}}$ if and only if $\sigma_{ij} = \rho_{ij}$ for $i = 0, \dots, d; j = 0, 1, \dots, n$. Therefore, the set of coefficients of $\Phi(u_j^{(i)})$ must exactly be the same as the set of coefficients of $\varphi(t)$.

Now let L be any extension of K over which the associated form $F(u)$ becomes reducible,

$$F(u) = F(u^{(0)}, u^{(1)}, \dots, u^{(d)}) = \prod_{j=1}^h G_j(u^{(0)}, u^{(1)}, \dots, u^{(d)}) = \prod_{j=1}^h G_j(u) .$$

Let the corresponding polynomials in t be

$$f(t) = \prod_{j=1}^h g_j(t) .$$

If C_j is the leading coefficient of $g_j(t)$, i. e., the coefficient of the highest power of t , we may write $g_j(t) = C_j f_j(t)$, where $f_j(t)$ have leading coefficient 1. Hence $f(t) = \prod_{j=1}^h C_j f_j(t)$. The set of coefficients of $g_j(t)$ is the same as the set of coefficients of $G_j(u)$. Hence we can write $G_j(u) = C_j F_j(u)$ and $F(u) = \prod_{j=1}^h C_j F_j(u)$ corresponding to the above equation in t .

Now each coefficient of $f_j(t)$ is a symmetric function of the roots and hence lies in the root field B of the polynomial $f(t)$ over K . The coefficients of $f_j(t)$ also lie in L , because they are quotients of coefficients of $g_j(t)$. Hence they lie in the intersection field A of B and L . Thus the theorem is proved.

Theorem 3. F can be split into absolutely irreducible factors $F = C F_1^q \cdot F_2^q \dots F_h^q$ with coefficients in an algebraic extension field of K .

Proof: If F can be factorised, let us write $F = F_1 \cdot F_2$. If F_1 or F_2 can be factorised we shall continue the factorisation until we arrive at absolutely irreducible factors: $F = G_1 G_2 \dots G_h$.

By theorem 2, the G_j may be replaced by F_j with coefficients from an algebraic extension A . Thus we get:

$$F = C F_1 F_2 \dots F_h .$$

The F_j are absolutely irreducible, because they are proportional to the G_j .

Some of the factors may be repeated. In this case we shall write

$$F = C F_1^{q_1} \cdot F_2^{q_2} \dots F_h^{q_h} .$$

Later on we shall see that F can have repeated factors only if F is the q th power of a form F_0 without repeated factors, q being a power of the characteristic p . So the decomposition of F into absolutely irreducible factors must have the form,

$$F = C F_1^q F_2^q \dots F_h^q .$$

Theorem 4. *Let L be any extension of K . Let $V = V_1 + V_2 + \dots + V_h$ be the decomposition of V in L . Let F_1, \dots, F_h be the associated forms of V_1, \dots, V_h . Then the decomposition of F in $L[u]$ is*

$$F = C F_1^{a_1} \cdot F_2^{a_2} \dots F_h^{a_h} .$$

Proof: We have, $V = V_1 + V_2 + \dots + V_h$, where V_1, V_2, \dots, V_h are irreducible over L and they are of the same dimension. The points of intersection $X^{(\nu)}$ ($\nu = 1, 2, \dots, g$) are split up into generic points of V_1 , generic points of V_2 and so on.

So if F_1 and F_2 are the associated forms of V_1 and V_2 the linear factors of F are partly contained in F_1 and partly in F_2 and so on.

Hence F can only be

$$F = C F_1^{a_1} \cdot F_2^{a_2} \dots F_h^{a_h} .$$

Corollary 1. *If V is absolutely irreducible then F is a power of a prime form.*

Proof: Suppose F can be expressed in some extension L of K as a product of different factors, say, $F = F_1 \cdot F_2$ having no prime factor in common. If F_1 is factorised into linear factors as in (1), it must contain with every factor all conjugate linear factors as well. Now all points of intersection of V with the hyperplanes $u^{(1)}, \dots, u^{(d)}$ are conjugate, because V is irreducible over L . Hence F_1 contains all prime factors of (1), each once at least. The same holds for F_2 . Hence F_1 and F_2 have factors in common, against hypothesis. Thus, F can only be a power of a prime form in L .

In the special case when F has no multiple factors, $F = F_1 \cdot F_2 \dots F_h$. By Theorem 4, each of the prime factors F_1, \dots, F_h defines a separate variety. These sub-varieties cannot be further subdivided, since the associated forms are irreducible.

Conversely, to every irreducible part of V corresponds a prime factor of F . For, if to an irreducible part of V corresponds a factor of F which is again factorisable into separate factors we arrive at a contradiction.

To each factor of F corresponds exactly one irreducible part of V . Hence the number of factors is the same. Therefore, we have :

Corollary 2. *If F has no repeated factors, the decomposition of F is $F = F_1 \cdot F_2 \cdot \dots \cdot F_n$. In this case to every prime factor of F corresponds an irreducible part of V and conversely. The number of factors is equal to the number of irreducible parts.*

Corollary 3. *If V is absolutely irreducible and F has no repeated factors, F is absolutely irreducible.*

Corollary 4. *If F is absolutely irreducible or a power of an absolutely irreducible factor, then V is absolutely irreducible.*

Proof: Suppose V is reducible over some extension L of K , say into V_1 and V_2 .

Let F_1, F_2 be the corresponding associated forms ; then by Theorem 4,

$$F = F_1^{a_1} \cdot F_2^{a_2} \quad \text{contrary to hypothesis.}$$

Theorem 5. *If $L = \Omega$ is chosen so that F factors into absolutely irreducible factors $F = F_1^{a_1} \cdot \dots \cdot F_n^{a_n}$, then V decomposes into absolutely irreducible varieties in Ω .*

Proof: To each absolutely irreducible factor F_j , or to a power of an absolutely irreducible factor F_j^q , corresponds a part V_j of V according to Theorem 4.

Now, by corollary 4 these V_j are indivisible (i. e., absolutely irreducible) parts of V .

This concludes the proof of theorem 5.

4. The case of a purely inseparable extension field

Now we shall consider the case of a purely inseparable extension of a field K . A purely inseparable extension of K of characteristic p is defined as an extension L in which every element is a p^e th root of an element of K .

Theorem 6. *The variety V remains irreducible in a purely inseparable extension of K .*

Proof: Let p be the characteristic of K and let the algebraic extension

L be purely inseparable. Then L consists only of p^e th roots (which are unique) of elements of K .

If V were reducible over L , there would be a product of forms G and H with coefficients in L , such that GH contains V but neither G nor H contains V . Now $q = p^e$ can be so chosen as a power of p such that the q th powers of all coefficients of G and H are in L . By the well known rule, $(a + b + \dots)^q = a^q + b^q + \dots$ it follows that G^q and H^q are forms with coefficients in K . Now the form

$$(GH)^q = G^q H^q$$

contains V , but neither G^q nor H^q contains V . This is impossible since V is irreducible over K .

Now let $q = p^e$ have the same meaning as in formula (2), § 1. We shall prove

Theorem 7. *In a suitable, purely inseparable extension K_0 of K the form F becomes equal to F_0^q , where F_0 has no multiple factors any more.*

Proof: The formula (2) in § 2 implies that Q contains the indeterminates u_0, \dots, u_n only in the q th power.

The same holds good for F on account of (3) § 1. Now on account of the possibility of interchanging it follows, that F also contains the $u_k^{(v)}$ only in the q th power.

Therefore, F is a q th power of a form in u_k and $u_k^{(v)}$ with coefficients from a field K_0 , which arises out of K by the adjunction of the q th roots of all coefficients of F . Thus we have

$$F = F_0^q . \tag{4}$$

Formula (3) now becomes

$$P^q = \frac{A}{B} F_0^q . \tag{5}$$

By (1), § 1, the product P has no multiple factors. Hence the left side of (5) and therefore, also the right side contains every factor exactly q times; it follows that F_0 contains every linear factor of P only once, i. e., F_0 does not contain multiple factors. This concludes the proof of Theorem 7.

Theorem 8. *If $q = 1$, the variety V is separably generated, i. e., all X are separable algebraic functions of d independent elements.*

In the proof 2 cases will be distinguished.

Case 1. We suppose K to be an infinite field. In the case of a field of characteristic p an irreducible polynomial $f(t)$ of one variable t is inseparable if and only if it may be written as a polynomial in t^p .

Suppose $e = 0$, i. e., $q = p^e = 1$. By (1) § 1 and (5), F_0 is a product of different linear factors :

$$u_0 x_0^{(\nu)} + u_1 x_1^{(\nu)} + \cdots + u_n x_n^{(\nu)} .$$

Now if we normalise $x_0 = 1$, we obtain

$$u_0 + u_1 x_1^{(\nu)} + u_2 x_2^{(\nu)} + \cdots + u_n x_n^{(\nu)} \quad \text{as factors.}$$

Now consider F_0 as a polynomial in one variable u_0 . This polynomial is a product of linear factors

$$(u_0 - \vartheta) (u_0 - \vartheta') \dots$$

all different. Consequently $\vartheta = -(u_1 x_1^{(\nu)} + u_2 x_2^{(\nu)} + \cdots + u_n x_n^{(\nu)})$ is separable with respect to the field, $K(u_1, \dots, u_n; u^{(1)}, \dots, u^{(d)})$.

Let V be defined over a field K . We shall enlarge the field K by the adjunction of n^2 indeterminates t_{ik} , where i and k take all values from 1 to n . Let the enlarged field $K(t_{ik})$ be denoted by K' . By Theorem 1, V is still irreducible with respect to K' . We shall first prove our theorem with respect to K' .

We have proved that

$$-\vartheta = u_1 x_1^{(\nu)} + u_2 x_2^{(\nu)} + \cdots + u_n x_n^{(\nu)}$$

is separable with respect to the field $K(u_1, \dots, u_n; u^{(1)}, \dots, u^{(d)})$. In this enunciation, the indeterminates u_k and $u_k^{(i)}$ may be replaced by any other set of indeterminates. Now replace,

$$\begin{aligned} u_k & \text{ by } t_{ek} (k = 1, \dots, n; \quad e = d + 1) , \\ u_k^{(i)} & \text{ by } t_{ik} (k = 1, \dots, n) , \\ u_0^{(i)} & \text{ by new indeterminates } z_i (i = 1, \dots, d). \end{aligned}$$

It follows that,

$$-\vartheta_e = t_{e1} x_1 + t_{e2} x_2 + \cdots + t_{en} x_n \tag{6}$$

is separable with respect to the field $K'(z_1, \dots, z_d)$, where X is any one of the points of intersection of V with the hyperplanes

$$z_i + t_{i1} x_1 + t_{i2} x_2 + \cdots + t_{in} x_n = 0 . \tag{7}$$

Now the problem may be simplified by a linear transformation of the coordinates x_1, \dots, x_n :

$$y_i = \sum t_{ik} x_k; \quad (i = 1, \dots, n) . \quad (8)$$

Equations (6) and (7) now simplify to

$$\begin{aligned} z_i + y_i &= 0 . \\ -\vartheta_e &= y_e . \end{aligned}$$

Hence y_1, \dots, y_d are equal to $-z_1, \dots, -z_d$, and $y_{d+1} = y_e = -\vartheta_e$ is a separable function of the indeterminates z_1, \dots, z_d .

The same holds, if $d + 1$ is replaced by any one of the numbers $d + 2, d + 3, \dots, n$. Hence y_{d+1}, \dots, y_n are separable functions of z_1, \dots, z_d . Also y_1, \dots, y_d are separable functions of z_1, \dots, z_d , for they are equal to $-z_1, \dots, -z_d$. So all y_i are separable functions of z_1, \dots, z_d . Solving (8) with respect to the x_k , it is seen that also x_1, \dots, x_n are separable functions of the indeterminates z_1, \dots, z_d .

Thus the theorem 8 is true provided K' [equal to $K(t_{ik})$] is taken as a field of constants instead of K . Now we have to pass from K' to K .

Let e be anyone of the numbers, $d + 1, \dots, n$. We have an algebraic equation defining y_e as an algebraic function of y_1, \dots, y_d :

$$f_e(y_1, \dots, y_d, y_e) = 0 . \quad (9)$$

The coefficients of this equation are rational functions of the t_{ik} , but they may be made integral rational. To express this, we shall write

$$f_e(t_{ik}, y_1, \dots, y_d, y_e) = 0 . \quad (10)$$

Now we can show that X is a generic point of V over $K(t_{ik})$:

y_1, \dots, y_d are algebraically dependent on x_1, \dots, x_n by (8); and y_1, \dots, y_n are algebraically dependent on y_1, \dots, y_d by (9). By solving (8) we see that x_1, \dots, x_n are dependent on y_1, \dots, y_n . Hence x_1, \dots, x_n are algebraically dependent on y_1, \dots, y_d . Therefore x_1, x_2, \dots, x_n are equivalent to y_1, \dots, y_d .

That is, the degree of transcendency of X over $K(t_{ik})$ is d . Hence X is a generic point of V over $K(t_{ik})$.

The equations (8) and (9) or (10) may be interpreted in another way. We have considered z_1, \dots, z_d as indeterminates and x_1, \dots, x_n as algebraic functions of z_1, \dots, z_d . We may also start with a generic point X of V , define y_1, \dots, y_n by (8) and define z_1, \dots, z_d by $z_i = -y_i$. The equations (9) remain valid in this interpretation, because all algebraic equations, valid for one generic point of V , remain valid for any other generic point. This means: if y_1, \dots, y_d and y_e are substituted from equation (8) into (10), we get an identity in the t_{ik} :

$$f_e(t_{ik}, \Sigma t_{ik} x_k) = 0 . \quad (11)$$

Such an identity remains valid, if the t_{ik} are specialised to t'_{ik} , and the y_i accordingly to $y'_i = \Sigma t'_{ik} x_k$.

Thus we get,

$$f_e(t'_{ik}, y'_1, \dots, y'_d, y'_e) = 0 . \quad (12)$$

Let A_e be the coefficient of the highest power of y_e in (10) and D_e the discriminant of (10), considered as an equation for y_e . A_e does not vanish, nor does D_e , because the equation is separable. A_e and D_e are polynomials in t_{ik} and y_1, \dots, y_d , and upon substitution of (8) they become polynomials in t_{ik} and x_1, \dots, x_n . Further, let D be the determinant of the t_{ik} ($i = 1, \dots, n; k = 1, \dots, n$).

Now specialise t_{ik} into t'_{ik} so that $D \prod_{d+1}^n A_e D_e$ remains $\neq 0$, where t'_{ik} are elements of K . Equation (12) now shows that all y'_e and hence all x_1, \dots, x_n are separable algebraic functions of y'_1, \dots, y'_d . This completes the proof of theorem 8 for case 1.

Case 2. Now, let K be a finite field and hence perfect. In this case the theorem follows from the following¹⁾

Lemma: x_1, \dots, x_d can be numbered in such a way that x_{d+1}, \dots, x_n are separable algebraic functions of x_1, \dots, x_d .

Theorem 9. *If V is separably generated then $q = p^e = 1$ (i. e., $e = 0$, where e is the exponent).*

Proof: By Kronecker's substitution, $F(u)$ is replaced by $f(t)$, where $f(t) = t^n + a_1 t^{n-1} + a_2 t^{n-2} + \dots + a_n$.

Suppose it contains only t^q . Then we can write,

$$\begin{aligned} f(t) &= t^{mq} + a_1 t^{(m-1)q} + \dots + a_n = g(t^q) ; \\ g(v) &= v^m + a_1 v^{(m-1)} + \dots + a_n . \end{aligned}$$

Now $g(v)$ is separable, otherwise it could be written as a polynomial in t^p .

Hence there is a separable extension L in which $g(v)$ is a product of different linear factors :

$$g(v) = (v - v_1) (v - v_2) \dots (v - v_m) .$$

In L let the variety be $V = V_1 + V_2 + \dots + V_h$ where V_1, V_2, \dots, V_h

¹⁾ For a proof see [8], p. 620, § 1

are irreducible. Then,

$$F(u) = F_1 \cdot F_2 \dots F_n .$$

By Kronecker's substitution this is replaced by

$$f(t) = f_1(t) \cdot f_2(t) \dots f_n(t) .$$

i. e., $f(t) = g(t^a) = \prod_{\nu} (t^a - v_{\nu}) .$

In L every $f_k(t)$ is a product of some factors $(t^a - v_{\nu})$. Hence in $L^{1/a}$ every $f_k(t)$ is a product of some factors $(t - w_{\nu})^a$ where $v_{\nu} = w_{\nu}^a$. That is, in $L^{1/a}$, we have $f_k(t) = \{f'_k(t)\}^a$, where $f'_k(t)$ is a product of different linear factors.

Now suppose V_k were reducible in a larger field L^* ,

$$V_k = V_{k1}^* + V_{k2}^* .$$

Then, $F_k = F_{k1}^* \cdot F_{k2}^*$, where F_{k1}^* and F_{k2}^* have no factors in common. That is

$f_k = f_{k1}^* \cdot f_{k2}^*$, where f_{k1}^* and f_{k2}^* have no factors in common. We have then f_{k1}^* is a product of some factors $(t^a - v_{\nu})$, where v_{ν} is in L and f_{k1}^* is in L . Similarly, f_{k2}^* is also in L contrary to hypothesis.

Hence V_1, V_2, \dots, V_n are absolutely irreducible over L .

Now we shall prove the

Lemma: If V is absolutely irreducible and separably generated over L , then L is algebraically closed in $L(X)$.

*Proof*²⁾: Suppose there were an element α in $L(X)$, algebraic over L and not in L . α being separable over L , the conjugate elements α, α', \dots are all different. That is $\alpha \neq \alpha'$ and

$$L(\alpha) \cong L(\alpha') . \tag{i}$$

Now extend the isomorphism of $L(\alpha)$ to $L(X)$, so as to obtain an isomorphism $L(X) \cong L(X')$ as follows:

Let x_1, \dots, x_d be algebraically independent and let x_{d+1}, \dots, x_n be algebraic functions of x_1, \dots, x_d . Define the isomorphism as follows:

$$\begin{array}{l} x_1 \longrightarrow x_1 \\ \dots \dots \dots \\ x_d \longrightarrow x_d \\ L(\alpha, x_1, \dots, x_d) \cong L(\alpha', x_1, \dots, x_d) . \end{array}$$

²⁾ I owe the proof of this Lemma to Prof. B. L. van der Waerden.

$L(X)$ is algebraic over $L(\alpha, x_1, \dots, x_d)$, hence this isomorphism can be extended to

$$L(X) \cong L(X') \text{ — (Proof in [7], I, § 35).} \quad (\text{ii})$$

X is a point of V and of degree of transcendency d . V remains irreducible over $L(\alpha)$. Hence X is a generic point of V with respect to $L(\alpha)$.

Because of the isomorphism (ii), X' too is a generic point of V . As before, we conclude: X' is a generic point with respect to $L(\alpha)$.

That is, X and X' are generic points of V with respect to $L(\alpha)$. Hence there is an isomorphism :

$$L(\alpha)(X) \longrightarrow L(\alpha)(X') \text{ .} \quad (\text{iii})$$

The elements of $L(\alpha)$ remain fixed

$$\alpha \longrightarrow \alpha$$

and

$$X \longrightarrow X'$$

α is in $L(X)$. Hence $\alpha = f(X)$. Applying (ii) we get $\alpha' = f(X')$.

Applying (iii) we have,

$$\alpha = f(X')$$

Hence $\alpha = \alpha'$ contrary to hypothesis.

Now we can complete the proof of theorem 9 that was interrupted by this Lemma.

It is given that V is separably generated over K , i. e., the coordinates of X are separable algebraic functions of d independent elements. They are also independent over the algebraic closure \bar{K} of K , and hence independent over L . It follows that V_1 , the absolutely irreducible part of V is also separably generated over L .

Now by the theorem ([2], Th. 5, p. 18) :

— An extension $L(X)$ of a field L is regular over L , if and only if L is algebraically closed in $L(X)$ and $L(X)$ is separably generated over L , — we have that $L(X) = L(x_0, \dots, x_n)$ is regular over L , i. e., $L(X)$ and \bar{L} are linearly disjoint over L . That is, every set of linearly independent elements in $L(X)$ over L is still linearly independent over \bar{L} . Hence also $L(t_{ik}, X)$ and $\bar{L}(t_{ik})$ are linearly disjoint over $L(t_{ik})$, where t_{ik} are defined as in the proof of theorem 8.

Now it can be proved that F_1 corresponding to V_1 is a product of different linear factors and hence q is equal to 1.

For, if not suppose,

$F_1 = F_0^p$. Then also, $f_1 = f_0^p$ and we should have,

$$f_0(y_1, \dots, y_d, y_{d+1})^p = 0, \quad \text{i. e.,} \quad f_0(y_1, \dots, y_d, y_{d+1}) = 0.$$

Putting $g' = g/p$, where $g' = \text{degree of } f_0$ and $g = \text{degree of } f_1$, this would mean a linear dependence between,

$$1, y_1, \dots, y_{d+1}, y_1 y_2, \dots, y_1^g, y_1^{g-1} y_2, \dots, y_{d+1}^{g'}$$

with respect to $\bar{L}(t_{ik})$. Hence there is also a linear dependence with coefficients from $L(t_{ik})$. This means y_{d+1} has degree $g' (< g)$ at most with respect to $L(t_{ik}, y_1, \dots, y_d)$, contrary to hypothesis.

Lastly, we shall show that $p^e = 1$ with respect to L leads to the result $p^e = 1$ with respect to K also. We have,

$$F = F_1 \cdot F_2 \dots F_h \text{ in } L \text{ (} F \text{ irreducible in } K\text{)}$$

F_1 cannot be written as $f(u^p, \dots)$; hence F_1 is a product of different linear factors :

$$F_1 = \Pi(u_0 x_0 + \dots + u_n x_n)$$

$$F_2 = \Pi(- \quad - \quad -)$$

.....

$$F_h = \Pi(- \quad - \quad -)$$

Hence F is a product of different linear factors. Hence $p^e = 1$ with respect to K .

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