

**Zeitschrift:** Commentarii Mathematici Helvetici  
**Herausgeber:** Schweizerische Mathematische Gesellschaft  
**Band:** 29 (1955)

**Artikel:** On fibre spaces in which the fibre is contractible.  
**Autor:** Spanier, E.H. / Whitehead, J.H.C.  
**DOI:** <https://doi.org/10.5169/seals-23274>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 25.04.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

# On fibre spaces in which the fibre is contractible

E. H. SPANIER<sup>1)</sup> and J. H. C. WHITEHEAD

*Dedicated to H. Hopf on his 60<sup>th</sup> birthday*

1. Let  $f : X \rightarrow Y$  be a map of a space  $X$  in a space  $Y$  and let  $A = f^{-1}y_0$ , for some point  $y_0 \in Y$ . Let  $A$  be either a locally finite  $CW$ -complex, as defined in [16], or a compactum which is an  $ANR$ . Then so are  $A \times A$  and<sup>2)</sup>

$$A \vee A = (A \times a_0) \cup (a_0 \times A) \subset A \times A \quad (a_0 \in A) .$$

We assume that the covering homotopy theorem is valid with respect to  $f$  for any map  $g : A \times A \rightarrow X$  and any homotopy of  $f \circ g$ . This will be so if  $f$  determines a fibering of  $X$  with a local product representation ([13], § 11.7, and [7], § 5). Subject to the latter condition we describe  $f$  as a *bundle mapping*. In § 3 below we prove :

**Theorem (1.1).** *If  $A$  is contractible in  $X$  it is an  $H$ -space<sup>3)</sup>.*

The method of proof is suggested by the following observation and an (unpublished) construction, due to M. G. Barratt, for defining the “generalized Whitehead product”,  $[\alpha, \beta] \in \pi_{m+n-1}(X, A)$ , of given elements  $\alpha \in \pi_m(X, A)$ ,  $\beta \in \pi_n(A)$ . Let  $f$  be a fibre mapping. Since  $f_* \beta = 0$  it follows that  $f_*[\alpha, \beta] = [f_* \alpha, 0] = 0$ , where each  $f_*$  denotes the appropriate homomorphism induced by  $f$ . Therefore  $[\alpha, \beta] = 0$ . We have  $\partial[\alpha, \beta] = \pm [\partial\alpha, \beta]$ , where

$$\partial : \pi_{q+1}(X, A) \rightarrow \pi_q(A)$$

is the boundary homomorphism and  $[\beta', \beta] \in \pi_{m+n-2}(A)$  is the ordinary Whitehead product of  $\beta' \in \pi_{m-1}(A)$  and  $\beta$ . Hence it follows that

<sup>1)</sup> This note arose out of consultations during the tenure of a John Simon Guggenheim Memorial Fellowship by Spanier.

<sup>2)</sup> The fact that, if  $A$  is an  $ANR$  compactum, so is  $A \vee A$ , follows from Theorem 1 in [15].

<sup>3)</sup> i. e. there is a map  $h : A \times A \rightarrow A$  such that  $h(a, a_1) = h(a_1, a) = a$  for some  $a_1$  and every  $a \in A$ .

$[\beta', \beta] = 0$  if  $i_*\beta' = 0$ , where  $i_* : \pi_{m-1}(A) \rightarrow \pi_{m-1}(X)$  is the injection. In particular  $\pi_1(A)$  is Abelian if  $X$  is simply connected (cf. [14], p. 289).

Before considering the consequences of (1.1), in its full generality, we draw a corollary from the preceding observation. Let  $X$  be a finite dimensional, locally compact, separable metric space, which is an  $AR$  (absolute retract). Let  $f : X \rightarrow Y$  be a bundle mapping with a connected fibre  $A$ . Since  $X$  is an  $AR$  it follows from an argument on p. 467 of [12] that  $A$  is acyclic and from the above observation that  $\pi_1(A)$  is Abelian. Therefore  $\pi_1(A) \approx H'_1(A) = 0$ , where  $H'_n(A)$  is the  $n^{\text{th}}$  integral, singular homology group of  $A$ . Therefore  $\pi_n(A) \approx H'_n(A) = 0$  for every  $n \geq 1$ . It follows from the local product representation that  $A$  is a neighbourhood retract of  $X$  and hence an  $ANR$ . Since  $X$  is locally compact so, obviously, is  $A$  and since  $\dim A \leq \dim X < \infty$  it follows that  $A$  may be imbedded as a closed<sup>4)</sup> sub-set in some Euclidean space,  $E$ , of which it is a neighbourhood retract. Since  $A$  is connected and  $\pi_n(A) = 0$  for every  $n \geq 1$  it follows that  $A$  is a retract of  $E$ , and hence an  $AR$ . The map  $f$  is obviously open and it follows without difficulty that  $Y$  is a  $C_\sigma$ -space, as defined in § 11.3 of [13]. In particular  $Y$  is covered by a countable set of open sub-sets,  $U_1, U_2, \dots$  such that  $\bar{U}_i$ , the closure of  $U_i$ , lies in a coordinate neighbourhood  $V_i$  (i. e. a neighbourhood such that  $f^{-1}V_i$  is represented as  $A \times V_i$ ). Let  $Y_n = \bar{U}_1 \cup \dots \cup \bar{U}_n$  ( $n \geq 1$ ) and assume that there is a map  $g_n : Y_n \rightarrow X$  such that  $fg_n y = y$  for every  $y \in Y_n$ . It follows from the local product representation that this is so if  $n = 1$ . Let  $T_{n+1} = \bar{U}_{n+1} \cap Y_n$ . Then  $T_{n+1}$  is a closed sub-set of  $\bar{U}_{n+1}$  and the latter is a separable metric space, since it is homeomorphic to a sub-set of  $X$ . Since  $A$  is an  $AR$  it follows that every map  $T_{n+1} \rightarrow A$  has an extension  $\bar{U}_{n+1} \rightarrow A$ . Therefore it follows from the local product representation that  $g_n$  has an extension  $g_{n+1} : Y_{n+1} \rightarrow X$  such that  $fg_{n+1} y = y$  ( $y \in Y_{n+1}$ ). Hence it follows by induction on  $n$  that  $f$  has a right inverse,  $g : Y \rightarrow X$ , by means of which  $Y$  may be imbedded in  $X$  in such a way that  $f$  becomes a retraction. Therefore  $Y$  is an  $AR$  and it follows from § 11.6 in [13] that  $X$ , as a bundle over  $Y$ , is equivalent to the product  $A \times Y$ . That is to say there is a homeomorphism  $h : A \times Y \rightarrow X$ , onto  $X$ , such that  $h(A \times y) = f^{-1}y$  for every  $y \in Y$ . Thus we have :

---

<sup>4)</sup> By the addition of a single point,  $c$ , we can imbed  $A$  in a compactum,  $C$ , such that  $\dim C \leq \dim A + 1$ . We imbed  $C$  in a  $p$ -sphere,  $S^p$ , for some large value of  $p$ , and  $E = S^p - c$ .

**Theorem (1.2).** *Any fibre bundle with a connected fibre, which, as a space, is a finite dimensional, locally compact, separable metric ANR, is equivalent to a product bundle.*

It follows from the arguments in [13] that, if  $G$  is a topological transformation group of  $A$  and if  $X$  is a bundle with  $G$  as its group, then (1.2) is valid if equivalence is interpreted as equivalence with respect to  $G$ .

We now turn to the consequences of (1.1). Let  $f : X \rightarrow Y$  be a bundle mapping of a compactum  $X$ . Since  $f$  is an open mapping onto  $Y$  it follows that  $Y, A$  are compacta ( $A = f^{-1}y_0$ ). Let  $A$  be connected, contractible in  $X$  and an ANR. Since  $A$  is a neighbourhood retract of  $X$  it will certainly be an ANR if  $X$  is an ANR. Let  $\dim X < \infty$ . Then  $\dim A, \dim Y \leq \dim X < \infty$  in consequence of the local product representation. Let  $H^*(P, G), H^n(P, G)$  denote the (discrete) Čech cohomology ring and the  $n^{\text{th}}$  Čech cohomology group of a given compactum  $P$ , with coefficients in a given ring  $G$ . We assume that  $Y$  is locally and globally pathwise connected and that  $\pi_1(Y)$ , operating on  $H^*(A, G)$  as in [9], operates simply for every  $G$ . Since  $A$  is connected, and hence pathwise connected, this will certainly be the case if  $X$  is simply connected. For then  $\pi_1(Y) = 1$ . Let  $I_0, R$  and  $S^n$  denote respectively the ring of integers, the ring of rational numbers and an  $n$ -sphere. We write  $H^*(P, I_0) = H^*(P), H^i(P, I_0) = H^i(P)$ . In § 3 we prove :

**Theorem (1.3).** *Let  $H^*(X) \approx H^*(S^n)$  ( $n \geq 1$ ). Then :*

- a) *either  $A$  is an<sup>5)</sup> ANR or  $H^*(A, R) \approx H^*(S^q, R)$ , for some odd value of  $q$ ,*
- b) *if  $A$  is homeomorphic to a topological product,  $A_1 \times A_2$ , then one of  $A_1, A_2$  is an ANR.*

In consequence of the second alternative in (1.3a) we have the exact sequence of Gysin ([3], [9], Ch. III)

$$\dots \xrightarrow{f^*} H^{j-1}(X, R) \xrightarrow{\theta} H^{j-q-1}(Y, R) \xrightarrow{f^*} H^j(Y, R) \rightarrow \dots,$$

in which  $\theta v = v \cup \Omega$  for some  $\Omega \in H^{q+1}(Y, R)$ . Since  $\dim Y < \infty$  there is a  $k \geq 0$  such that  $\Omega^k \neq 0, \Omega^{k+1} = 0$ , where  $\Omega^0 = 1 \in H^0(Y, R), \Omega^r = \Omega \cup \dots \cup \Omega \in H^{r(q+1)}(Y, R)$ . It may be verified that  $k > 0$ , since  $A$  is contractible in  $X$ , that  $n = k(q+1) + q$  and that

$$\left. \begin{array}{l} H^i(Y, R) \approx R \quad \text{for } i = 0, q+1, \dots, k(q+1) \\ H^i(Y, R) = 0 \quad \text{for all other values of } i. \end{array} \right\} \quad (1.4)$$

---

<sup>5)</sup> This will be the case, for example, if  $X = Y$  and  $f = 1$ .

Let  $S^n \rightarrow Y$  be a bundle mapping with a connected fibre  $F$ . Then  $F$  is an *ANR*, which is contractible in  $S^n$  except in the trivial case  $F = S^n$ . Therefore we have :

**Corollary (1.5).** *If  $F$  is homeomorphic to  $A_1 \times A_2$ , then one of  $A_1, A_2$  is an *AR*.*

The results (1.2), (1.5) above extend the two theorems concerning the fibering of Euclidean spaces and spheres by tori which are proved in [4]. Also (1.5) extends a theorem due to A. Borel ([1], [2], p. 165). It will be seen that our (1.3) is an easy corollary of (1.1) together with this theorem of Borel's.

2. Let  $P, Q$  be topological spaces which are either locally finite *CW*-complexes or *ANR* compacta. Then so are  $P \times Q$  and

$$P \vee Q = (P \times q_0) \cup (p_0 \times Q) \subset P \times Q ,$$

where  $p_0, q_0$  are points in  $P, Q$ , which are 0-cells if  $P, Q$  are *CW*-complexes. Let  $P, Q$  be imbedded in  $P \vee Q$  so that  $p = (p, q_0)$ ,  $q = (p_0, q)$  for each point  $p \in P$  and each  $q \in Q$ . Let

$$P \xrightarrow{u} A \xleftarrow{v} Q$$

be given maps such that  $u p_0 = v q_0 = a_0$ , say, and  $u$  is homotopic in  $X$  (and hence homotopic rel.  $p_0$ ) to the constant map  $P \rightarrow a_0$ . Let

$$g_0 : P \times Q \rightarrow X$$

be defined by  $g_0(p, q) = vq$ . Then  $g_0 P = a_0$  and there is a homotopy  $u_t : P \rightarrow X$ , rel.  $p_0$ , such that  $u_0 p = g_0 p = a_0$ ,  $u_1 p = up$  ( $p \in P$ ). This can be extended, first to a homotopy  $u'_t : P \vee Q \rightarrow X$  such that  $u'_t q = vq$  if  $q \in Q$ , and then ([16], p. 228, [15]) to a homotopy  $g_t : P \times Q \rightarrow X$ . Then  $g_1 p = up$ ,  $g_1 q = vq$ . Let

$$h : (P \times Q, P \vee Q) \rightarrow (X, A)$$

be the map determined by  $g_1$ . We describe  $h$  as *inessential* if, and only if, it is related by a homotopy of the form  $(P \times Q, P \vee Q) \rightarrow (X, A)$  to a map with values in  $A$ . We describe  $v$  as *inessential* if, and only if, it is homotopic, and hence homotopic rel.  $q_0$ , to the constant map  $Q \rightarrow a_0$ .

**Lemma (2.1).** *If  $v$  is inessential, so is  $h$ .*

Let  $v_t : Q \rightarrow A$ , rel.  $q_0$ , be a homotopy such that  $v_0 = v$ ,  $v_1 Q = a_0$ . Let  $g' : P \times Q \times I \rightarrow X$  be defined by

$$\begin{aligned}
g'(p, q, t) &= g_{1-3t}(p, q) & \text{if } 0 \leq t \leq 1/3 \\
&= v_{3t-1}(q) & \text{if } 1/3 \leq t \leq 2/3 \\
&= u_{3t-2}(p) & \text{if } 2/3 \leq t \leq 1 .
\end{aligned}$$

Then  $g'(p, q, 0) = g_1(p, q) = h(p, q)$ ,  $g'(Q \times I) \subset A$  and  $g'(P \times Q \times 1) \subset A$ . Also  $g'(p_0 \times q_0 \times I) = a_0$  and since  $g_s(p, q_0) = u_s(p)$  it follows that  $g' | (P \vee Q) \times I$  is homotopic, rel.  $(P \times 0) \cup (Q \times I) \cup (P \times 1)$ , to a map in which  $(p, t) \rightarrow up$ . Therefore (2.1) follows from the homotopy extension theorem, applied to the pair  $(P \times Q \times I, K)$ , where

$$K = (P \times Q \times 0) \cup (P \vee Q) \times I \cup (P \times Q \times 1) .$$

Notice that we have used the form of the homotopy extension theorem in which the argument spaces are of a special sort and the image space is arbitrary. The definition of  $h$  and the proof of (2.1) apply unchanged if  $X$  is an ANR, of the sort appropriate to some general category of spaces to which  $P, Q, P \times Q$  etc. belong (cf. [6]).

3. *Proof of (1.1).* Let  $f' : (X, A) \rightarrow (Y, y_0)$  be the map determined by  $f$ . Then

$$f' \circ h : (P \times Q, P \vee Q) \rightarrow (Y, y_0)$$

is defined in the same way as  $h$ , in § 2, with  $g_t, v$  replaced by  $f \circ g_t$  and the constant map  $Q \rightarrow y_0$ . Therefore it follows from (2.1) that  $f' \circ h$  is homotopic, rel.  $P \vee Q$ , to the constant map  $c$ , where  $c(P \times Q) = y_0$ . Assuming that a homotopy  $f' \circ h \simeq c$  can be lifted it follows that  $h$  is inessential. Therefore  $h | P \vee Q$  has an extension  $P \times Q \rightarrow A$  and (1.1) follows on taking  $P = Q = A$  and  $u = v =$  the identical map.

Let  $f$  be a bundle mapping and let  $X$  be a locally compact, separable metric space. Then  $X$  and likewise  $A$  and  $A \times A$  are obviously  $C_\sigma$ -spaces. Therefore we have, in consequence of the concluding remarks in § 2 and § 11.3 in [13]:

**Theorem (3.1).** *If  $f : X \rightarrow Y$  is a bundle mapping, if  $X$  is a locally compact, separable metric ANR and if a fibre,  $A$ , is contractible in  $X$ , then  $A$  is an  $H$ -space.*

4. *Proof of (1.3).* Let  $g : E \rightarrow B$  be a fibre mapping, with fibre  $F$ , where  $E, B$ , and hence also  $F$ , are compacta. Let  $H^i(P) = 0$  for  $P = B, E, F$  and all sufficiently large values of  $i$ . This will be the case, for example, if  $\dim P < \infty$ . Also let  $H^i(P)$  be finitely generated for all values of  $i$ . It follows from the theory of the spectral sequence associated with the mapping  $g$  that this will be the case if any two of  $H^i(B), H^i(E),$

$H^i(F)$  are finitely generated for every <sup>6)</sup>  $i$ . Therefore it will be the case if  $H^*(E) \approx H^*(S^n)$  and  $F$  is an ANR. We quote the universal coefficient theorem <sup>7)</sup>

$$H^r(Q, G) \approx H^r(Q) \otimes G + H^{r+1}(Q) * G, \quad (3.1)$$

for the (discrete) Čech cohomology groups of a compactum  $Q$ , with coefficients in  $G$ . It follows from (3.1) that, if

$$H^m(Q) \approx I_0, \quad H^i(Q) = 0 \quad \text{for } i > m, \quad (3.2)$$

then

$$H^m(Q, G) \approx G, \quad H^i(Q, G) = 0 \quad \text{for } i > m. \quad (3.3)$$

Let  $H^j(Q)$  be finitely generated for every  $j \geq m$ . Then (3.2) is true if (3.3) holds for every field,  $G$ , as group of coefficients.

Let  $E$  satisfy (3.2) for some  $m \geq 0$  and let  $K$  be a given field. Then it follows from Theorem (9.1) on p. 189 of [9] that there are integers  $r = r_K, s = s_K$  such that  $r + s = m$  and

$$\left. \begin{aligned} H^r(B, K) \approx K, \quad H^i(B, K) = 0 \quad \text{if } i > r, \\ H^s(F, K) \approx K, \quad H^j(F, K) = 0 \quad \text{if } j > s, \end{aligned} \right\} \quad (3.4)$$

in which  $\approx$  indicates isomorphism between vector spaces over  $K$ . Let  $k = r_R, l = s_R$ . Since  $H^i(B), H^i(F)$  are finitely generated it follows from (3.1) and (3.4), with  $K = R$ , that

$$H^k(B) \approx I_0 + T, \quad H^l(F) \approx I_0 + T',$$

where  $T, T'$  are finite groups. Hence, and from (3.1), it follows that  $H^k(B, K)$  and  $H^l(F, K)$  each contains a summand which is isomorphic to  $K$ . Therefore  $k \leq r, l \leq s$  and since  $k + l = m = r + s$  we have  $k = r, l = s$ . Thus  $r, s$  are independent of the choice of  $K$ . Therefore  $B, F$  satisfy (3.2) with  $m$  replaced by  $r$  or  $s$  according as  $Q = B$  or  $F$ .

Let  $f: X \rightarrow Y$  and  $A$  be as in (1.3). Then  $H^*(X) = H^*(S^n)$  and it follows from the preceding paragraph that

$$\left. \begin{aligned} \text{a) } H^p(Y) \approx I_0, \quad H^i(Y) = 0 \quad \text{if } i > p \\ \text{b) } H^q(A) \approx I_0, \quad H^j(A) = 0 \quad \text{if } j > q \end{aligned} \right\} \quad (3.5)$$

<sup>6)</sup> The argument is essentially the same as the one on p. 465 of [12]. See also § 9 of [9]

<sup>7)</sup> See Theorem 44.2 on p. 823 of [5], in which the term  $H^{r+1}(Q) * G$  is expressed differently. The only property of this "product" which we need is that  $H * G = 0$  if either  $H = 0$  or if  $G$  has no (non-zero) element of finite order. We use  $+$  to indicate direct summation.

for some pair of integers  $p, q$  such that  $p + q = n$ . Moreover  $A$ , being contractible in  $X$ , is an  $H$ -space, according to (1.1).

First assume that  $q = 0$ . Then it follows from (5.1) on p. 346 of [10] that  $H_i(A) = 0$  for every  $i > 0$ , where  $H_i(A)$  is the  $i^{\text{th}}$  discrete, integral Čech homology group of  $A$ . Since  $A$  is an  $ANR$  it follows that  $H'_i(A) = 0$  if  $i > 0$  where  $H'_i(A)$  is the  $i^{\text{th}}$  singular homology group of  $A$  ([11], p. 107). Hence it follows, as in the proof of (1.2), that  $A$  is an  $AR$ . If  $A$  is homeomorphic to  $A_1 \times A_2$ , then  $A_1$  is homeomorphic to a retract of  $A$ . Therefore it follows that  $A_1$  is an  $AR$ . This proves (1.3) if  $q = 0$  and we proceed on the assumption that  $q > 0$ .

Since  $A$  is an  $H$ -space and  $H^j(A) = 0$  if  $j > q$  we have ([8], No. 24)

$$H^*(A, R) \approx H^*(S^{i_1} \times \dots \times S^{i_p}, R)$$

for certain odd values of  $i_1, \dots, i_p$ . Since  $H^*(X) \approx H^*(S^n)$  it follows from [1] and [2], p. 165, that  $p = 1$ . Thus

$$H^*(A, R) \approx H^*(S^q, R), \quad (3.6)$$

where  $q$  is odd. This proves (1.3a).

Let  $A$  be homeomorphic to  $A_1 \times A_2$ . On taking  $g : E \rightarrow B$  to be the projection  $A_1 \times A_2 \rightarrow A_2$ , with  $F$  homeomorphic to  $A_1$ , it follows from (3.5b) and (3.4) that

$$H^{q_j}(A_j) \approx I_0, \quad H^i(A_j) = 0 \quad \text{if} \quad i > q_j,$$

for some pair of integers  $q_1, q_2$  such that  $q_1 + q_2 = q$ . The group  $H^{q_j}(A_1 \times A_2, R)$  contains a summand which is isomorphic to  $H^{q_j}(A_j, R)$ . Hence it follows from (3.6) that  $q_j = 0$  or  $q$ . Since  $q_1 + q_2 = q$  it follows that either  $q_1 = 0$  or  $q_2 = 0$ , say  $q_1 = 0$ . Since  $A_1$  is homeomorphic to a retract of  $A$  it is an  $ANR$ . Since  $\pi_1(A) \approx \pi_1(A_1) \times \pi_1(A_2)$  and  $\pi_1(A)$  is Abelian, because  $A$  is contractible in  $X$ , it follows that  $\pi_1(A_1)$  is Abelian. Therefore it follows from an argument similar to the one used above to dispose of the case  $q = 0$  that  $A_1$  is an  $AR$ . This completes the proof.

## REFERENCES

- [1] *Armand Borel*, Impossibilité de fibrer une sphère par un produit de sphères, *C. R. Acad. Sci. Paris* 231 (1950) 943–45.
- [2] *Armand Borel*, Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts, *Ann. Math.* 57 (1953) 115–207.
- [3] *Shiing-Shen Chern and E. Spanier*, The homology structure of sphere bundles, *Proc. Nat. Acad. Sci.* 36 (1950) 248–55.
- [4] *B. Eckmann, H. Samuelson and G. W. Whitehead*, On fibering spheres by toruses, *Bull. Amer. Math. Soc.* 55 (1949) 433–38.
- [5] *Samuel Eilenberg and Saunders MacLane*, Group extensions and homology, *Ann. Math.* 43 (1942) 757–831.
- [6] *Olof Hanner*, Retraction and extension of mappings of metric and non-metric spaces, *Ark. Mat.* 2 (1952) 315–60.
- [7] *I. M. James and J. H. C. Whitehead*, Note on fibre spaces, *Proc. London Math. Soc.* (3), 4 (1954), 129–137.
- [8] *Jean Leray*, Sur la forme des espaces topologiques et sur les points fixes des représentations, *J. Math. Pures Appl.* 24 (1945) 95–167.
- [9] *Jean Leray*, L'homologie d'un espace fibré dont la fibre est connexe, *Ibid* 29 (1950) 169–213.
- [10] *S. Lefschetz*, Algebraic Topology, New York 1942.
- [11] *S. Lefschetz*, Topics in Topology, *Ann. Math. Studies* No. 10, Princeton 1942.
- [12] *Jean-Pierre Serre*, Homologie singulière des espaces fibrés, *Ann. Math.* 54 (1951) 425–505.
- [13] *N. E. Steenrod*, The topology of fibre bundles, Princeton 1951.
- [14] *J. H. C. Whitehead*, On the groups  $\pi_r(V_{n,m})$  and sphere bundles, *Proc. London Math. Soc.* (2) 48 (1944) 243–91.
- [15] *J. H. C. Whitehead*, Note on a theorem due to Borsuk, *Bull. Amer. Math. Soc.* 54 (1948) 1125–32.
- [16] *J. H. C. Whitehead*, Combinatorial homotopy (I), *Ibid.* 55 (1949) 213–45.

(Received October 8, 1953.)