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On fibre spaces in which the fibre is contractible

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Dedicated to H. Hopf on his 60th birthday

1. Let $f: X \to Y$ be a map of a space X in a space Y and let $A = f^{-1}y_0$, for some point $y_0 \in Y$. Let A be either a locally finite CW-complex, as defined in [16], or a compactum which is an ANR. Then so are $A \times A$ and A

$$A \smile A = (A \times a_0) \cup (a_0 \times A) \subset A \times A \qquad (a_0 \in A) .$$

We assume that the covering homotopy theorem is valid with respect to f for any map $g: A \times A \to X$ and any homotopy of $f \circ g$. This will be so if f determines a fibering of X with a local product representation ([13], § 11.7, and [7], § 5). Subject to the latter condition we describe f as a bundle mapping. In § 3 below we prove:

Theorem (1.1). If A is contractible in X it is an H-space³).

The method of proof is suggested by the following observation and an (unpublished) construction, due to M. G. Barratt, for defining the "generalized Whitehead product", $[\alpha, \beta] \in \pi_{m+n-1}(X, A)$, of given elements $\alpha \in \pi_m(X, A)$, $\beta \in \pi_n(A)$. Let f be a fibre mapping. Since $f_* \beta = 0$ it follows that $f_* [\alpha, \beta] = [f_* \alpha, 0] = 0$, where each f_* denotes the appropriate homomorphism induced by f. Therefore $[\alpha, \beta] = 0$. We have $\partial [\alpha, \beta] = \pm [\partial \alpha, \beta]$, where

$$\partial: \pi_{q+1}(X, A) \to \pi_q(A)$$

is the boundary homomorphism and $[\beta', \beta] \in \pi_{m+n-2}(A)$ is the ordinary Whitehead product of $\beta' \in \pi_{m-1}(A)$ and β . Hence it follows that

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¹) This note arose out of consultations during the tenure of a John Simon Guggenheim Memorial Fellowship by Spanier.

²) The fact that, if A is an ANR compactum, so is $A \searrow A$, follows from Theorem 1 in [15].

³) i. e. there is a map $h: A \times A \to A$ such that $h(a, a_1) = h(a_1, a) = a$ for some a_1 and every $a \in A$.

 $[\beta', \beta] = 0$ if $i_*\beta' = 0$, where $i_* : \pi_{m-1}(A) \to \pi_{m-1}(X)$ is the injection. In particular $\pi_1(A)$ is Abelian if X is simply connected (cf. [14], p. 289).

Before considering the consequences of (1.1), in its full generality, we draw a corollary from the preceding observation. Let X be a finite dimensional, locally compact, separable metric space, which is an AR (absolute retract). Let $f: X \to Y$ be a bundle mapping with a connected fibre A. Since X is an AR it follows from an argument on p. 467 of [12] that A is acyclic and from the above observation that $\pi_1(A)$ is Abelian. Therefore $\pi_1(A) \approx H_1'(A) = 0$, where $H_n'(A)$ is the n^{th} integral, singular homology group of A. Therefore $\pi_n(A) \approx H'_n(A) = 0$ for every $n \geqslant 1$. It follows from the local product representation that A is a neighbourhood retract of X and hence an ANR. Since X is locally compact so, obviously, is A and since dim $A \leq \dim X < \infty$ it follows that A may be imbedded as a closed 4) sub-set in some Euclidean space, E, of which it is a neighbourhood retract. Since A is connected and $\pi_n(A) = 0$ for every $n \ge 1$ it follows that A is a retract of E, and hence an AR. The map f is obviously open and it follows without difficulty that Y is a C_{σ} -space, as defined in § 11.3 of [13]. In particular Y is covered by a countable set of open sub-sets, U_1, U_2, \ldots such that \overline{U}_i , the closure of U_i , lies in a coordinate neighbourhood V_i (i. e. a neighbourhood such that $f^{-1}V_i$ is represented as $A \times V_i$). Let $Y_n = \overline{U}_1 \circ \cdots \circ \overline{U}_n$ $(n \geqslant 1)$ and assume that there is a map $g_n: Y_n \to X$ such that $fg_n y = y$ for every $y \in Y_n$. It follows from the local product representation that this is so if n=1. Let $T_{n+1}=\overline{U}_{n+1} \cap Y_n$. Then T_{n+1} is a closed sub-set of \overline{U}_{n+1} and the latter is a separable metric space, since it is homeomorphic to a sub-set of X. Since A is an AR it follows that every map $T_{n+1} \to A$ has an extension $\overline{U}_{n+1} \to A$. Therefore it follows from the local product representation that g_n has an extension $g_{n+1}:Y_{n+1}\to X$ such that $fg_{n+1}y = y$ $(y \in Y_{n+1})$. Hence it follows by induction on n that f has a right inverse, $g: Y \to X$, by means of which Y may be imbedded in X in such a way that f becomes a retraction. Therefore Yis an AR and it follows from § 11.6 in [13] that X, as a bundle over Y, is equivalent to the product $A \times Y$. That is to say there is a homeomorphism $h: A \times Y \to X$, onto X, such that $h(A \times y) = f^{-1}y$ for every $y \in Y$. Thus we have:

⁴) By the addition of a single point, c, we can imbed A in a compactum, C, such that dim $C \leq \dim A + 1$. We imbed C in a p-sphere, S^p , for some large value of p, and $E = S^p - c$.

Theorem (1.2). Any fibre bundle with a connected fibre, which, as a space, is a finite dimensional, locally compact, separable metric AR, is equivalent to a product bundle.

It follows from the arguments in [13] that, if G is a topological transformation group of A and if X is a bundle with G as its group, then (1.2) is valid if equivalence is interpreted as equivalence with respect to G.

We now turn to the consequences of (1.1). Let $f:X\to Y$ be a bundle mapping of a compactum X. Since f is an open mapping onto Y it follows that Y, A are compacta $(A=f^{-1}y_0)$. Let A be connected, contractible in X and an ANR. Since A is a neighbourhood retract of X it will certainly be an ANR if X is an ANR. Let $\dim X<\infty$. Then $\dim A$, $\dim Y\leqslant \dim X<\infty$ in consequence of the local product representation. Let $H^*(P,G)$, $H^n(P,G)$ denote the (discrete) Čech cohomology ring and the n^{th} Čech cohomology group of a given compactum P, with coefficients in a given ring G. We assume that Y is locally and globally pathwise connected and that $\pi_1(Y)$, operating on $H^*(A,G)$ as in [9], operates simply for every G. Since A is connected, and hence pathwise connected, this will certainly be the case if X is simply connected. For then $\pi_1(Y)=1$. Let I_0 , R and S^n denote respectively the ring of integers, the ring of rational numbers and an n-sphere. We write $H^*(P,I_0)=H^*(P)$, $H^i(P,I_0)=H^i(P)$. In § 3 we prove:

Theorem (1.3). Let $H^*(X) \approx H^*(S^n)$ $(n \geqslant 1)$. Then:

- a) either A is an⁵) AR or $H^*(A, R) \approx H^*(S^q, R)$, for some odd value of q,
- b) if A is homeomorphic to a topological product, $A_1 \times A_2$, then one of A_1 , A_2 is an AR.

In consequence of the second alternative in (1.3a) we have the exact sequence of Gysin ([3], [9], Ch. III)

$$f^*$$
 $\cdots \to H^{j-1}(X, R) \to H^{j-q-1}(Y, R) \to H^j(Y, R) \to \cdots$

in which $\theta v = v \circ \Omega$ for some $\Omega \in H^{q+1}(Y,R)$. Since dim $Y < \infty$ there is a $k \ge 0$ such that $\Omega^k \ne 0$, $\Omega^{k+1} = 0$, where $\Omega^0 = 1 \in H^0(Y,R)$, $\Omega^r = \Omega \circ \cdots \circ \Omega \in H^{r(q+1)}(Y,R)$. It may be verified that k > 0, since A is contractible in X, that n = k(q+1) + q and that

$$H^{i}(Y, R) \approx R$$
 for $i = 0, q + 1, ..., k(q + 1)$
 $H^{i}(Y, R) = 0$ for all other values of i . (1.4)

⁵) This will be the case, for example, if X = Y and f = 1.

Let $S^n \to Y$ be a bundle mapping with a connected fibre F. Then F is an ANR, which is contractible in S^n except in the trivial case $F = S^n$. Therefore we have:

Corollary (1.5). If F is homeomorphic to $A_1 \times A_2$, then one of A_1 , A_2 is an AR.

The results (1.2), (1.5) above extend the two theorems concerning the fibering of Euclidean spaces and spheres by tori which are proved in [4]. Also (1.5) extends a theorem due to A. Borel ([1], [2], p. 165). It will be seen that our (1.3) is an easy corollary of (1.1) together with this theorem of Borel's.

2. Let P, Q be topological spaces which are either locally finite CWcomplexes or ANR compacta. Then so are $P \times Q$ and

$$P \smile Q = (P \times q_0) \smile (p_0 \times Q) \subset P \times Q$$
,

where p_0 , q_0 are points in P, Q, which are 0-cells if P, Q are CW-complexes. Let P, Q be imbedded in $P \searrow Q$ so that $p = (p, q_0)$, $q = (p_0, q)$ for each point $p \in P$ and each $q \in Q$. Let

$$P \xrightarrow{u} A \xleftarrow{v} Q$$

be given maps such that $up_0 = vq_0 = a_0$, say, and u is homotopic in X (and hence homotopic rel. p_0) to the constant map $P \to a_0$. Let

$$g_0: P \times Q \to X$$

be defined by $g_0(p,q) = vq$. Then $g_0P = a_0$ and there is a homotopy $u_t: P \to X$, rel. p_0 , such that $u_0 \ p = g_0 \ p = a_0$, $u_1 \ p = up \ (p \in P)$. This can be extended, first to a homotopy $u_t': P \lor Q \to X$ such that $u_t'q = vq$ if $q \in Q$, and then ([16], p. 228, [15]) to a homotopy $g_t: P \times Q \to X$. Then $g_1 \ p = up$, $g_1 \ q = vq$. Let

$$h:(P\times Q,\,P\smile Q)\to (X,\,A)$$

be the map determined by g_1 . We describe h as inessential if, and only if, it is related by a homotopy of the form $(P \times Q, P \vee Q) \to (X, A)$ to a map with values in A. We describe v as inessential if, and only if, it is homotopic, and hence homotopic rel. q_0 , to the constant map $Q \to a_0$.

Lemma (2.1). If v is inessential, so is h.

Let $v_t: Q \to A$, rel. q_0 , be a homotopy such that $v_0 = v$, $v_1Q = a_0$. Let $g': P \times Q \times I \to X$ be defined by

$$g'(p, q, t) = g_{1-3t}(p, q)$$
 if $0 \le t \le 1/3$
= $v_{3t-1}(q)$ if $1/3 \le t \le 2/3$
= $u_{3t-2}(p)$ if $2/3 \le t \le 1$.

Then $g'(p,q,0) = g_1(p,q) = h(p,q)$, $g'(Q \times I) \subset A$ and $g'(P \times Q \times 1) \subset A$. Also $g'(p_0 \times q_0 \times I) = a_0$ and since $g_s(p,q_0) = u_s(p)$ it follows that $g' \mid (P \vee Q) \times I$ is homotopic, rel. $(P \times 0) \vee (Q \times I) \vee (P \times 1)$, to a map in which $(p,t) \to up$. Therefore (2.1) follows from the homotopy extension theorem, applied to the pair $(P \times Q \times I, K)$, where

$$K = (P \times Q \times 0) \circ (P \smile Q) \times I \circ (P \times Q \times 1) .$$

Notice that we have used the form of the homotopy extension theorem in which the argument spaces are of a special sort and the image space is arbitrary. The definition of h and the proof of (2.1) apply unchanged if X is an ANR, of the sort appropriate to some general category of spaces to which $P, Q, P \times Q$ etc. belong (cf. [6]).

3. Proof of (1.1). Let $f':(X,A)\to (Y,y_0)$ be the map determined by f. Then

$$f' \circ h : (P \times Q, P \vee Q) \rightarrow (Y, y_0)$$

is defined in the same way as h, in § 2, with g_t , v replaced by $f \circ g_t$ and the constant map $Q \to y_0$. Therefore it follows from (2.1) that $f' \circ h$ is homotopic, rel. $P \searrow Q$, to the constant map c, where $c(P \times Q) = y_0$. Assuming that a homotopy $f' \circ h \cong c$ can be lifted it follows that h is inessential. Therefore $h \mid P \searrow Q$ has an extension $P \times Q \to A$ and (1.1) follows on taking P = Q = A and u = v = the identical map.

Let f be a bundle mapping and let X be a locally compact, separable metric space. Then X and likewise A and $A \times A$ are obviously C_{σ} -spaces. Therefore we have, in consequence of the concluding remarks in § 2 and § 11.3 in [13]:

Theorem (3.1). If $f: X \to Y$ is a bundle mapping, if X is a locally compact, separable metric ANR and if a fibre, A, is contractible in X, then A is an H-space.

4. Proof of (1.3). Let $g: E \to B$ be a fibre mapping, with fibre F, where E, B, and hence also F, are compacta. Let $H^i(P) = 0$ for P = B, E, F and all sufficiently large values of i. This will be the case, for example, if dim $P < \infty$. Also let $H^i(P)$ be finitely generated for all values of i. It follows from the theory of the spectral sequence associated with the mapping g that this will be the case if any two of $H^i(B)$, $H^i(E)$,

 $H^{i}(F)$ are finitely generated for every 6) i. Therefore it will be the case if $H^{*}(E) \approx H^{*}(S^{n})$ and F is an ANR. We quote the universal coefficient theorem 7)

$$H^r(Q,G) \approx H^r(Q) \otimes G + H^{r+1}(Q) *G$$
 (3.1)

for the (discrete) Čech cohomology groups of a compactum Q, with coefficients in G. It follows from (3.1) that, if

$$H^m(Q) \approx I_0 \; , \quad H^i(Q) = 0 \qquad for \quad i > m \; , \qquad (3.2)$$

then

$$H^m(Q,G) \approx G$$
 , $H^i(Q,G) = 0$ for $i > m$. (3.3)

Let $H^{j}(Q)$ be finitely generated for every $j \ge m$. Then (3.2) is true if (3.3) holds for every field, G, as group of coefficients.

Let E satisfy (3.2) for some $m \ge 0$ and let K be a given field. Then it follows from Theorem (9.1) on p. 189 of [9] that there are integers $r = r_K$, $s = s_K$ such that r + s = m and

$$H^r(B,K) \approx K$$
, $H^i(B,K) = 0$ if $i > r$, $H^s(F,K) \approx K$, $H^j(F,K) = 0$ if $j > s$, (3.4)

in which \approx indicates isomorphism between vector spaces over K. Let $k=r_R$, $l=s_R$. Since $H^i(B)$, $H^i(F)$ are finitely generated it follows from (3.1) and (3.4), with K=R, that

$$H^k(B) pprox I_0 + T$$
 , $H^l(F) pprox I_0 + T'$,

where T, T' are finite groups. Hence, and from (3.1), it follows that $H^k(B,K)$ and $H^l(F,K)$ each contains a summand which is isomorphic to K. Therefore $k \leq r$, $l \leq s$ and since k+l=m=r+s we have k=r, l=s. Thus r, s are independent of the choice of K. Therefore B, F satisfy (3.2) with m replaced by r or s according as Q=B or F.

Let $f: X \to Y$ and A be as in (1.3). Then $H^*(X) = H^*(S^n)$ and it follows from the preceding paragraph that

a)
$$H^{p}(Y) \approx I_{0}$$
, $H^{i}(Y) = 0$ if $i > p$
b) $H^{q}(A) \approx I_{0}$, $H^{j}(A) = 0$ if $j > q$, (3.5)

⁶) The argument is essentially the same as the one on p. 465 of [12]. See also § 9 of [9]

⁷⁾ See Theorem 44.2 on p. 823 of [5], in which the term $H^{r+1}(Q) * G$ is expressed differently. The only property of this "product" which we need is that H * G = 0 if either H = 0 or if G has no (non-zero) element of finte order. We use + to indicate direct summation.

for some pair of integers p, q such that p + q = n. Moreover A, being contractible in X, is an H-space, according to (1.1).

First assume that q=0. Then it follows from (5.1) on p. 346 of [10] that $H_i(A)=0$ for every i>0, where $H_i(A)$ is the i^{th} discrete, integral Čech homology group of A. Since A is an ANR it follows that $H_i'(A)=0$ if i>0 where $H_i'(A)$ is the i^{th} singular homology group of A ([11], p. 107). Hence it follows, as in the proof of (1.2), that A is an AR. If A is homeomorphic to $A_1\times A_2$, then A_1 is homeomorphic to a retract of A. Therefore it follows that A_1 is an AR. This proves (1.3) if q=0 and we proceed on the assumption that q>0.

Since A is an H-space and $H^{j}(A) = 0$ if j > q we have ([8], No. 24)

$$H^*(A, R) \approx H^*(S^{i_1} \times \cdots \times S^{i_p}, R)$$

for certain odd values of i_1, \ldots, i_p . Since $H^*(X) \approx H^*(S^n)$ it follows from [1] and [2], p. 165, that p = 1. Thus

$$H^*(A, R) \approx H^*(S^q, R)$$
 , (3.6)

where q is odd. This proves (1.3a).

Let A be homeomorphic to $A_1 \times A_2$. On taking $g: E \to B$ to be the projection $A_1 \times A_2 \to A_2$, with F homeomorphic to A_1 , it follows from (3.5b) and (3.4) that

$$H^{q_j}\left(A_j
ight) pprox I_0$$
 , $H^i(A_j) = 0$ if $i > q_j$,

for some pair of integers q_1 , q_2 such that $q_1 + q_2 = q$. The group $H^{q_j}(A_1 \times A_2, R)$ contains a summand which is isomorphic to $H^{q_j}(A_j, R)$. Hence it follows from (3.6) that $q_j = 0$ or q. Since $q_1 + q_2 = q$ it follows that either $q_1 = 0$ or $q_2 = 0$, say $q_1 = 0$. Since A_1 is homeomorphic to a retract of A it is an ANR. Since $\pi_1(A) \approx \pi_1(A_1) \times \pi_1(A_2)$ and $\pi_1(A)$ is Abelian, because A is contractible in X, it follows that $\pi_1(A_1)$ is Abelian. Therefore it follows from an argument similar to the one used above to dispose of the case q = 0 that A_1 is an AR. This completes the proof.

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