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On some separation and mapping theorems

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Introduction

The problems treated here were discussed in some of my course lectures 1952—1953 on mapping theory. Publication of the results was originally intended for a book on Fixed Points, but it appears desirable to give them earlier circulation.

The stimulus for the first part of this paper comes from a homotopy view of perturbation theory. Thus if $h_t: X \to X$ where X is a compactum, and $0 \leq t \leq 1$, the fixed point set, $X(t)$, for each t is a compactum. The natural question is then whether when t changes slightly the fixed points change very little. Since $UX(t) \times t = C$ is easily verified to be a compactum, our question is essentially whether C contains a continuum joining $X \times 0$ and $X \times 1$. The remaining sections are concerned with sphere mappings. A theorem of Borsuk's, [B], asserts *a real valued map of an n sphere assigns the same value to some antipodal pair* $(z, -z)$. Dyson, [D], has proved *a real valued map of the 2 sphere assigns the same value to the four end points of some pair of orthogonal diameters.* (Livesay [Li] has shown any preassigned angle between the diameters can replace orthogonality). Dyson's proof is of set theoretic type. The present paper brings the methods of algebraic topology to bear on these seemingly metric problems. The key tool is the lemma that a *closed carrier of an n dimensional mod 2 cycle, non bounding over a product of a projective space P and a segment I carries non bounding cycles of all lower dimensions.* Let Z be an n sphere or more generally an n dimensional symmetric homologically sphere like set. Let f map Z into the j dimensional Euclidean space R^j. Our generalization of the Borsuk-Ulam theorem states the *symmetric sub set of Z for which* $f(z) = f(-z)$ *carries an* $n - j$ *dimensional cycle mod 2 which maps by identification of antipodal pairs into a non bounding cycle in* $P^n \times I$. Our generalization of the Dyson theorem states *there are* $n - j + 1$ *orthogonal lines through the origin whose end points lie in Z and are transformed by f into some* $j - 1$ *dimensional sphere about the origin of* R^j. Continuity of f can be weakened to upper semi continuity of $f(z) - f(-z)$ for both theorems. The separation idea is central in such aspects of the various proofs as (1), (3.04) and (4.00).

Throughout the paper we shall use the same letter for the inclusion map of spaces for the induced chain maps and for the induced homomorphisms of the homology groups. Thus $i: X \to Y$ induces $i: C(X) \to C(Y)$ and also $i: H(X, G) \to H(Y, G)$. The support of a chain on a geometrical complex is understood to be the carrier defined by the union of all the geometrical simplexes entering the chain. In dealing with the chain groups we shall often omit the i however, and write simply $C(X)$ in place of $iC(X)$. If D is a chain then $||D||$ is a point set attached to D which is either the support or the carrier. However, when no confusion is possible, the same symbol D will denote the associated point set. The field I_2 is that of integers mod 2. R^n is the Euclidean n dimensional space and S_n is the n sphere with center at the origin. I is the unit segment $0 \leq t \leq 1$. By $\underline{X}$ we mean the set of inner points of X.

1. Separation. The techniques involved and the arguments recur throughout the paper even for somewhat changed situations. Unless otherwise understood the cycles and carriers [W, p. 204] are Cech.

Theorem 1A. *Let M be compact Hausdorff and let A_n be a non bounding Cech cycle in M with the coefficient group G either compact or a field. Suppose C^0 and C^1 are disjunct compact sets in $M \times I$ and suppose C^1 does not meet $M^0 = M \times 0$ while C^0 does not meet $M^1 = M \times 1$. Then there is an n cycle B_n on $M \times I$, whose carrier does not meet $M^0 \cup M^1 \cup C^0 \cup C^1$ and $B_n \sim A_n(0)$ where $A_n(0)$ on M^0 corresponds to A_n on M.*

Since $M \times I$ is compact $M^0 \cup C^0$ and $M^1 \cup C^1$ can be covered by a finite collection of open sets in $M \times I$ whose union is N^0 and N^1 respectively with $N^0 \cap N^1 = 0$. Let $X^1 = \bar{N}^1$. Write $X^0 = \overline{M \times I - X^1}$, and $Q = X^0 \cap X^1$. Thus Q is the frontier of X^1 and is disjunct from both $M^0 \cup C^0$ and $M^1 \cup C^1$. Consider

$$\begin{tikzcd}
 & H_n(M^0) \arrow[d, "l"] & & \\
\arrow[r] H_n(Q) & H_n(X^0) \arrow[r, "j"] \arrow[d, "r"] \arrow[dr, phantom, "\mathrm{I}"] & H_n(X^0, Q) \arrow[r, "\partial"] \arrow[d, "l"] & H_{n-1}(Q) \\
 & H_n(M \times I) \arrow[r, "s"] & H_n(M \times I, X^1) &
\end{tikzcd}$$

with $H_n(Q) \xrightarrow{i} H_n(X^0)$.

Here i, r, s, l, are induced by the obvious inclusion maps and e is induced by an excision map for $X^1 - \underline{X}^1 = Q$, $M \times I - \underline{X}^1 = X^0$.

Actually e is an excision isomorphism. [E. S.; Theorem 5.4, p. 266.] Since all the homomorphisms in the square I are induced by either inclusions or excisions commutativity obtains. Indicate the coset corresponding to a cycle by curly brackets. Thus $A_n(0)$ is a representative of $\{A_n(0)\} \in H_n(M^0)$. Since $A_n(0) \sim A_n(1)$ over $M \times 1$, $A_n(0) \sim 0$ mod M^1 and hence $A_n(0) \sim 0$ mod X^1. In the notation of (1) we have

$$s\, r\, l\, \{A_n(0)\} = 0\ , \tag{1.01}$$

whence

$$e^{-1} s\, r\, l\, \{A_n(0)\} = 0\ , \tag{1.02}$$

or

$$j\, l\, \{A_n(0)\} = 0\ . \tag{1.03}$$

The upper horizontal sequence is either exact or partially exact depending on G. In either case the kernel of j includes the image of i. Since according to (1.03), $l\{A_n(0)\}$ is in the kernel of j, there must exist an element $\{B_n\} \in H_n(Q)$ such that $i\{B_n\} = l\{A_n(0)\}$ or

$$i\, B_n \sim l\, A_n(0)\ . \tag{1.04}$$

Interesting special cases arise when M is taken as a closed n dimensional orientable manifold[2]) or orientable pseudo manifold or orientable circuit imbedded in Euclidean space with A_n the fundamental integral cycle. In such cases we have a partial converse. We first state a useful lemma.

Lemma 1B. *Let K^0 and K^1 be compact Hausdorff spaces with union K and common part Q. Suppose L is a compact subset of K^0. Let A_n be an n cycle of L with $A_n \sim 0$ on K. Then there is a cycle B_n in Q homologous to A_n over K^0.*

The triad K, K^0, K^1 is proper since the sets are compact [E. S., p. 257]. The Mayer Vietoris sequence is

$$\begin{array}{c} H_n(L) \\ \downarrow l \\ \leftarrow H_{n-1}(Q) \overset{\Delta}{\leftarrow} H_n(K) \overset{\Phi}{\leftarrow} H_n(K^0) + H_n(K') \overset{\Psi}{\leftarrow} H_n(Q) \leftarrow \end{array}$$

Square brackets indicate the Bibliography.

[1]) C. T. Yang attended some of my lectures and independently obtained results like those in sections 3—5. His methods though different in appearance are basically like mine. In the long interval since submission of this manuscript he has obtained interesting variants.

[2]) A theorem for this case somewhat like Theorem IA was independently found by Livesay.

We have $\Phi\, l\{A_n\} = 0$ according to the hypothesis. Thus $l\{A_n\}$ is in the kernel of Φ whence by exactness some $\{B_n\} \in N_n(Q)$ satisfies $\{B_n\} = l\{A_n\}$. This implies the assertion of the theorem.

Theorem 1C. *Suppose M is a closed orientable n dimensional manifold with $M \times I$ in R_{n+1} and with fundamental cycle A_n. C is a continuum in $M \times I$ meeting both M^0 and M'. Suppose N is a closed n dimensional orientable manifold with base cycle E_n, where $E_n \sim A_n(0)$ over $M \times I$, and $\| N \|$ is a carrier of E_n. Then $\| N \|$ meets C.*

Compactify R^{n+1} by adding the point ∞ to get S_{n+1}. The coefficient group below is that of the integers. Suppose N separates S_{n+1} into the domains $N(1)$ and $N(2)$, M^j separates S_{n+1} into the domains $M^j(1)$ and $M^j(2)$, $j = 0, 1$. Suppose $N(1)$ contains M^0 and M'. By suitable labelling we can require that $M^0(1) \supset M'$ and $M'(1) \supset M^0$. Then

$$\overline{M^0(1)} \cap \overline{M'(1)} \supset M \times 1 \; .$$

Indeed if some point (m, τ) of $M \times I$ were not in $M^0(1)$ the line

$$\{(m, t) \mid \tau \leq t \leq 1\}$$

would cut M^0. Consider the sets

$$K^0 = L = X^0 = M \times I - N(2), \quad K' = \overline{N(2)}, \quad X' = \overline{M^0(2) \cup M'(2)} \; .$$

The compact sets X^0, X^1, $X = X^0 \cup X^1$ constitute a proper triad. Note $E_n - A_n(0) \sim 0$ over $K = K^0 \cup K'$. Evidently also $E_n - A_n(1) \sim 0$ over K. Recourse to Lemma 1B establishes there are cycles $C^0 = l_0 E_n$ and $C' = l_1 E_n$ on N, such that (a) $C^0 \sim E_n - A_n(0)$ and (b) $C^1 \sim E_n - A_n(1)$, both over K^0. — Since neither $A_n(0)$ nor $A_n(1)$ bounds on $M \times 1$, $m_i = 1 - l_i \neq 0$, and $C_n = m_0 A_n(1) - m_1 A_n(0)$ is a cycle on $X^0 \cap X^1 = M^0 \cup M'$ whose homology class in $H_n(X^0 \cap X^1)$, denoted by $\{C_n\}$, *is not* 0. On the other hand C_n is evidently a bounding cycle on both X^0 and X^1. Thus $\{C_n\}$ is in the kernel of the Mayer Vietoris map (ψ) into $H_n(X^0) + H_n(X^1)$ and therefore is the image (under Δ) of $H_{n+1}(X)$. Since X is a proper subset of S_{n+1}, $H_{n+1}(X) = 0$ and so $\{C_n\} = 0$. In short M^0 and M^1 cannot both be in domain $N(1)$ (or in $N(2)$). Also M^0 (or M') cannot meet both $N(1)$ and $N(2)$ for then so does $M^0(2)$ whence

$$0 \neq M^0(2) \cap N \subset \widetilde{M \times I} \cap N = 0 \; .$$

Suppose $\| N \| \cap C = 0$, then since the common boundary of $N(1)$ and $N(2)$ is N it would follow that C is contained entirely in one or the other of $N(1)$ or $N(2)$. This would stand at variance with our requirement that C meet both M^0 and M^1.

2. Basic Notions. We add the following conventions: All homology and chain groups are over I_2. Nevertheless in the interests of naturalness we shall use both $+$ and $-$. Indicate the metric norm in R^{n+1} by $|y|$. Let Y be the closed shell (in R^{n+1}), $\{y \mid 1 \leq |y| \leq 2\}$ or any other positive bounds where necessary. $S_m(s)$, $m \leq n$ is a sphere in Y of radius s about 0. (If the radius is arbitrary we write S_m.) This is also the basic chain of a symmetric triangulation of the sphere. By X we shall invariably mean a symmetric set (with respect to the origin) in Y. The projective n dimensional space is indicated by P^n. Let px be the reflection of x in the origin. Let Tx denote the identification of x and px, i. e. $T(x \cup px) = x' = (x, px)$. The next few remarks are essentially special cases of known results for periodic transformations [S]. Use the same symbol T for the chain transformation which identifies σ and $p\sigma$, i. e. $T(1+p)\,\sigma = \sigma'$. *Throughout a prime on a set or chain indicates the identification under T or under the corresponding simplex identification $T(1+p)$.*

Let σ be given by the vertex scheme $[y_0, \ldots, y_l]$. Indicate this by $[y]$. Then $p\sigma = [py_0, \ldots, py_l]$ or $[py]$ and $T(1+p)[y] = \sigma' = [y']$. Observe $T^{-1}\sigma' = (1+p)\sigma$. This is a unique correspondence though σ is not unique since $p\sigma$ serves as well. Write $[z]_i$ for

$$[z_0, \ldots, z_{i-1}, z_{i+1}, \ldots, z_l] \;.$$

The choice $z = y$, py or y' is that of interest below. Thus $\partial z = \Sigma\,[z]_i$. We make use of the relations

$$\begin{aligned} \partial p\sigma &= p\partial\sigma \\ \partial\sigma' &= (\partial(1+p)\,\sigma)' \\ \partial T^{-1}\sigma' &= T^{-1}\partial\sigma' \end{aligned} \tag{2.00}$$

For instance,

$$\begin{aligned} \partial p\sigma &= \partial[py] = \Sigma_i[py]_i = p\,\Sigma\,[y]_i = p\partial\sigma \;, \\ \partial\sigma' &= \partial[y'] = \Sigma\,T(1+p)[y]_i = T(1+p)\partial\sigma \;. \end{aligned}$$

This shows incidentally that T is a chain map (on symmetric chains). A chain, C_m is symmetric if and only if

$$(1+p)\,C_m = 0 \qquad \text{or} \qquad C_m = (1+p)\,{}_1C_m \;. \tag{2.01}$$

In applications we always assume that ${}_1C_m$ contains no antipodal pair of m simplexes. The closed half spaces to one side or other of an n dimensional hyperplane, R^n, containing the origin are indicated by $R^n(+)$ and $R^n(-)$ respectively. The intersections with Y are written $Y(+)$ and $Y(-)$. $Q^{n+1} = P^n \times I = Y'$.

The following lemma and its direct proof are central in the developments of this paper. X and Y are here considered simplicial complexes.

Lemma 2A. *Let k denote the inclusion map $X \to Y$ where X is a simplicial subcomplex of Y. If A_m is a symmetric simplicial m cycle on X, $m \leq n$, and kA'_m is non bounding (on Y') then $|| A_m ||$ carries a symmetric cycle A_{m-j} where A'_{m-j} is non bounding (on Y') for all $j \leq m$.*

It is sufficient to establish the lemma for $j = 1$. A trivial application of the Künneth relations shows the m dimensional homology groups over Q^{n+1} and over P^n are isomorphic for $m \leq n$ and are therefore isomorphic to I_2. Plainly the chain $S'_m = P^n$ is non bounding. The hypotheses imply

$$A'_m - S'_m = \partial(C'_{n+1}) \ . \tag{2.02}$$

The symmetric chain $C_{m+1} = T^{-1}(C'_{n+1})$ may be represented as $(1 + p)\,{}_1C_{m+1}$. Thus $\partial(C'_{m+1}) = (\partial C_{m+1})' = ((1 + p)(\partial_1 C_{m+1}))'$ by virtue of (2.00). Then applying T^{-1} to (2.02) there results

$$(1 + p)({}_1A_m - {}_1S_m - \partial_1 C_{m+1}) = 0 \tag{2.03}$$

where ${}_1A_m$ may be chosen in a variety of ways conditioned merely by the requirement that $(1 + p)\,{}_1A_m = T^{-1}(A_m)'$ and that ${}_1A_n$ and p_1A_m have no m simplexes in common. Similar statements apply to ${}_1S_n$. Accordingly

$${}_1A_m - {}_1S_m - \partial_1 C_{m+1} = D_n \ , \tag{2.04}$$

where $(1 + p)\,D_n = 0$. Then

$$\partial_1 A_n - \partial_1 S_n = \partial D_m \ . \tag{2.05}$$

All chains in (2.05) are symmetric. Indeed since A_n is a symmetric cycle,

$$0 = \partial A_n = \partial(1 + p)\,{}_1A_m = (1 + p)\partial_1 A_m \ ,$$

and so (2.01) establishes our assertion here. Similarly $\partial_1 S_m$ is symmetric. Since D_m is symmetric so is ∂D_m. We therefore derive from (2.05),

$$(\partial_1 A_m)' - (\partial_1 S_n)' = (\partial D_m)' = \partial(D_m) \ . \tag{2.06}$$

Write B_{m-1} for $\partial_1 A_m$. Then B_{m-1} is evidently an $m - 1$ symmetric cycle. After suitable subdivision of the simplexes of the triangulation of Y, if necessary, we can choose ${}_1S_m$ as the upper cap of the section of S_m by a suitable hyperplane. Then $\partial_1 S_n$ is the equatorial $m-1$ sphere and so $(\partial_1 S_m)'$ is a non bounding $m-1$ cycle in Y'. Thus (2.06) guarantees B_{m-1} is a non bounding cycle in Y'.

Suppose the l cycle ${}_1D_l$, $l<1$, contains no pair σ_l, $p\sigma_l$. Since the mod 2 Betti numbers of Y vanish if the dimension is inferior to n, we must have ${}_1D_l = \partial C_{l+1}$.

Then $((1+p)\,D_l)' = \partial\,((1+p)\,C_{l+1})'$ or $((1+p)_1D_l)' \sim 0$. Hence A_m and B_{n-1} can be replaced by *subcycles* E_m and E_{m-1} on components for $m>1$ [L, p. 91] and

$$E_l' \sim S_l', \ l = m-1, m \ . \tag{2.07}$$

Theorem 2B. *Let X be a closed symmetric carrier of a symmetric Cech cycle A_m with $A_m' \nsim 0$ on Y'. Suppose $X = Z \cup pZ$ where Z is compact. Denote frontiers in X of subsets of X by $Bd(\ )$. Suppose $Z \cap pZ = BdZ = W$. Then W carries a symmetric Cech cycle E_{m-1} with $E_{m-1}' \sim S_{m-1}'$ in Y'.*

For a symmetric cover $\mathfrak{U}$, $U \in \mathfrak{U}$ implies $pU \in \mathfrak{U}$. We may assume below $U \cap pU = 0$. Then $\mathfrak{U}' = \{U_i' \mid U_i' = T(1+p)\,U_i,, \ \ U_i \in \mathfrak{U}\}$. We remark there is a cofinal sequence of finite symmetric open covers $\{\mathfrak{U}(r) \mid r = 1, 2, \ldots\}$ with the following properties: (a) $\mathfrak{U}(s)$ is the star of a symmetric triangulation of $Y, \Delta(s)$, and refines $\mathfrak{U}(r)$ for $r<s$.

(b) if $Y(r)$ is the nerve of $U(r)$ and $Y'(r)$ that of $\mathfrak{U}'(r)$ then

$$H_i(Y(r) \approx H_i(Y)\} \ \text{ and } \ H_i\{Y'(r)) \approx H_i(Y')$$

for all i and (c) if a kernel [L, p. 245] meets both Z and pZ then it meets W.

The only assertion not immediately obvious is (c). Appeal may be made to the proof of an analogous assertion when no symmetry restriction is imposed on the sets or covers [W, p. 202] and the result required here may be established by similar arguments. An alternative derivation (indeed the original one of the writer's) starts with a symmetric triangulation, δ, of Y. A prescription can be given for the introduction of new vertices to give δ' whose zero and one dimensional kernels satisfy (c). Next new vertices are introduced to give δ^2 such that the kernels of dimension 2 or less satisfy (c). This inductive construction yields $\Delta = \delta^{n+1}$ satisfying (c).

If Q is closed in Y then $Q(r)$ is the subcomplex of $Y(r)$ consisting of simplexes whose kernels meet Q. We write $||\,Q(r)\,||$, here for the point set closure of the union of the kernels of simplexes in $Q(r)$. Evidently $Q(r)$ may also be viewed as the closure of the Euclidean subcomplex of $\Delta(r)$ consisting of simplexes (of $\Delta(r)$) meeting Q with $\| Q(r) \|$ the associated point set. The m dimensional skeleton of $Q(r)$ is written $Q_m(r)$.

We show first that $W \neq 0$. Assume the contrary. Thus for r, $r_0 < r$, $X(r) = Z(r) \cup pZ(r)$ where $Z(r)$ and $pZ(r)$ are disjunct. We proceed as in a similar situation occuring in the proof of Theorem 2A. The symmetric Cech cycle A_m has the representation $\{A_m(s)\}$, where $A_m(s)$ is a symmetric simplicial cycle and the hypotheses of Theorem 2B require that for $r_1 < r$, $A'_m(r) \nsim 0$ on $Y'(r)$. Choose r larger than either r_0 or r_1. Let ${}_1A_m(r)$ consist of those simplexes of $A_m(r)$ which are in $Z(r)$. Then no pair $\sigma_m(r)$, $p\sigma_m(r)$ occurs in ${}_1A_m(r)$ and, since

$$\| {}_1A_m(r) \| \cap \| p\, {}_1A_m(r) \| = 0, \qquad \partial\, {}_1A_m(r) = 0 \ .$$

Hence, recalling property (b) of $U(r)$, ${}_1A_m(r) = \partial C_{m+1}(r)$ where $C_{m+1}(r)$ is a chain on $Y(r)$ and then

$$A'_m(r) = \big((1+p)\,{}_1A_n(r)\big)' = \partial\big((1+p)C'_{m+1}(r)\big)' \quad \text{or} \quad A'_m(r) \sim 0$$

in violation of our requirement.

The complexes $Z_m(r)$, $(pZ)_m(r)$ share a symmetric complex

$$\dot{M}_m(r) \cup (pM)_m(r) \ .$$

$M_m(r)$ and $(pM)_m(r)$ are closed complexes with no common m simplexes. Let

$$K_m(r) = M_n(r) \cup {}'Z_m(r)$$

where ${}'Z_m(r)$ is the maximal closed subcomplex of $Z_m(r)$ which contains no m simplexes occurring in $M_m(r)$ or $(pM)_m(r)$.

The hypotheses of the theorem require that the symmetric Cech cycle $A_m = \{A_m(r)\}$ satisfies $A'_m(r) \sim S'_m(r)$. Write $A_m(r) = (1+p)\,{}_1A_m(r)$ where ${}_1A_m(r)$ consists of the m simplexes of $A_m(r)$ occurring in $K_n(r)$. Evidently $BdK_m(r)$ is the symmetric complex consisting of all $m-1$ dimensional simplexes common to $K_m(r)$ and $pK_m(r)$ and is shown to be non vacuous by the same type of argument used to establish $W \neq 0$. Moreover

$$BdK_m(r) \subset W_{m-1}(r) \ . \tag{2.08}$$

In fact if $\sigma_{m-1}(r)$ is a face of a simplex of $K_m(r)$ as well as of one of $pK_m(r)$ then the kernel of $\sigma_{m-1}(r)$ meets both Z and pZ and hence meets W. Write $B_{m-1}(r) = \partial\,{}_1A_m(r)$. We assert

$$B_{m-1}(r) \subset W_{m-1}(r) \ . \tag{2.09}$$

Indeed ${}_1A_m(r)$ is on $K_m(r)$ and therefore so is $B_{m-1}(r)$. This last chain is symmetric and therefore is on $pK_m(r)$ also. Accordingly $B_{m-1}(r)$ is on $BdK_m(r)$ and so appeal to (2.08) establishes (2.09).

Since $W(r)$ may be considered a symmetric Euclidean complex, $H_{m-1}(W'(r)) \approx H_{m-1}(\|W'(r)\|)$. Let $p^r s$ be the projection homomorphism induced by the inclusion map of $\| W'(r) \|$ into $\| W'(s) \|$ while $i(r)$ is induced by the inclusion map $\| W'(r) \| \to Y'$. We have commutativity in the squares below for $s<r$

$$\begin{array}{lllr} H_{m-1}(\| W'(r) \|) & \xrightarrow{i(r)} H_{m-1}(Y') & \to 0 & \\ p^r s \downarrow & \quad\quad \downarrow & \quad \downarrow & \quad\quad (2.10) \\ H_{m-1}(\| W'(s) \|) & \xrightarrow{i(s)} H_{m-1}(Y') & \to 0\ . & \end{array}$$

The justification for asserting $i(r)$ is onto in (2.10) is that $i(r)\, B'_{m-1}(r) \not\sim 0$ in $Y'(r)$. We note $W' = \cap \| W'(r) \|$. Since the groups occuring in (2.10) are compact and $\| W(r) \| \subset \| W(s) \|$, $s < r$ we may take the inverse limits and invoke the continuity property of the Cech groups to get the exact sequence [E. S., p. 226],

$$H_{m-1}(W') = \underset{\leftarrow}{L} H_{m-1}(\| W'(r) \|) \xrightarrow{i} H_{m-1}(Y') \to 0\ .$$

Since i is onto, $iH_{m-1}(W')$ is not trivial. Accordingly some cycle B'_{m-1}, non bounding in Y', is carried by W'.

3. Sets Circumscribing a Frame of Orthogonal Diameters.

The main result here, Theorem 3A, is half of the Generalized Dyson theorem.

Theorem 3A. *If X carries a symmetric m cycle A_n of Y and A'_m does not bound in Y' there exist $m+1$ mutually orthogonal diameters of some n dimensional sphere about 0, $m \leq n$, whose termini lie in X.*

Clearly a standard compactness argument serves to establish the assertion once it is verified for neighborhoods (symmetric) of X. We may therefore assume that X is a finite complex with symmetric triangulation. We tacitly assume throughout that the triangulations are always *so chosen that the simplexes or faces are in the required sub spaces.* By (2.07) *X can be supposed a component for $m>0$.*

The proof is by induction. The assertion of the theorem is patently valid for $m=0$ and arbitrary n, $n \geq m$. Suppose then that for fixed n and all $j \leq m-1$ the assertion of the theorem is valid.

Let $a \cup b \in X$ where a is a nearest point to 0 and b a furthest point from 0. Let $L : L(t \mid 0 \leq t \leq 1)$ be a polygonal line in X joining a and b. Denote by $r(t)$ the length of the line $l(t)$ from 0 to the point $L(t)$. Let $\{e_i \mid 1 \leq i \leq n+1\}$ be a fixed orthonormal frame in R^{n+1}. We require that e_{n+1} lie along $l(0)$. Designate the orthogonal complement of $l(t)$ by $R^n(t)$. Then the linear extension of $\{e_i \mid i \leq n\}$ is $R^n(0)$ which we write R_n. The rotation of e_{n+1} as $l(t)$ describes L determines a linear isometry of R_n onto $R^n(t)$. Denote this map by $f : y \times t = y(t)$ where $y \in R_n$ and $y(t) \in R^n(t)$. Thus f is on $R_n \times I$ onto $U_I R^n(t) \subset R^{n+1}$. Introduce $Y_n = Y^n(0)$ where $Y^n(0) = R^n(0) \cap Y$. Similarly $Y^n(t) = R^n(t) \cap Y$. Then f induces a map of $Y_n \times I$ onto $U_I Y^n(t) \subset Y$. We introduce also the homeomorphic map, g^{-1} of Y_n into a "funnel" in $Y_n \times I$. Specifically let $t = |y| - 1$ for $y \in Y_n$. Then $S_{n-1}(1+t)$ is the linear map of $S_{n-1}\big(r(t)\big) \times t$ determined by a dilatation in $Y_n \times t$ followed by a projection onto Y_n. Thus $g^{-1} : (y \mid Y_n) \to z \times (|y| - 1) \in Y_n \times I$ where $|z| = r(|y| - 1) \in Y$. The construction and maps introduced in this paragraph are suggested by the work of Yamabe and Yujobo [YY] on the Kakutani problem.

Let the parameter range for a typical line segment of L be $t_0 \leq t \leq t_1$. Denote this interval by J. Remark that $Y_n \times J$ is deformable to a homeomorph of $U_J Y^n(t)$ as follows easily from the fact that $Y_n \times J - (E \times J)$ is the homeomorph of $U_J Y^n(t) - E$ where $E = Y^n(t_0) \cap Y^n(t_1)$. Let $X(t) = Y^n(t) \cap X$. Write $X(J)$ for $U_J\big(X(t)\big)$. Write $\big(X(J), J\big)$ for $U_J X(t) \times t$ and $E \cap \big(X(J), J\big)$ for $U_J\big(E \cap X(t) \times t\big)$. As usual, primes will indicate identification under T.

We require the following lemmas.

Lemma 3(B). $H_i\big(X'(J)\big) \approx H_i\big(X'(J), J\big)$.

Observe f yields a homeomorphism of $X(J) - E \cap X(J)$ and $\big(X(J), J\big) - E \cap \big(X(J), J\big)$ and hence of $X'(J) - E' \cap X'(J)$ and $\big(X'(J), J\big) - E' \cap \big(X'(J), J\big)$. If $y \in X(t_0) \cap E$ for some $t_0 \in J$ then y may be considered in $X(t) \cap E$ for each $t \in J$ and

$$f^{-1}(y) = y \times J \subset E \cap \big(X(J), J\big) .$$

Furthermore this relation is valid for the corresponding primed sets. Thus $f^{-1}\big(y' \mid X'(t_0) \cap E'\big) = y' \times J \subset E' \cap \big(X'(J), J\big)$. Since the augmented homology groups of $y' \times J$ are certainly trivial the assertion of the lemma is then a consequence of the generalized Vietoris theorem, [*Be*], [*Bo*].

Lemma 3C. $H_i(Y'_n \times J) \approx H_i\big(U_J(Y'_n(t))\big)$.

The demonstration is clear from that for Lemma 3B.

We proceed with the argument for the theorem. Let $Y(t_0, +)$ be the half section of Y containing $l(t_0)$. Recall the notation of Lemma 2A. Let ${}_1A_m$ and ${}_1S_m$ be chosen in $Y(t_0, +)$. Thus if $Y^n(t_0)$ is transverse then $A_m(t_0, +) = {}_1A_m = A_m \frown Y(t_0, +)$. Accordingly B_{m-1} and S_{m-1} have their supports in $Y^n(t_0)$ and are therefore written $B_{m-1}(t_0)$, $S_{m-1}(t_0)$ and hence also $B'_{m-1}(t_0)$ and $S'_{m-1}(t_0)$. Let J be the interval $(0{=}t_0, t_1)$. We impose a consistency requirement on $A_m(t_1, +)$. Let $C_m \mid Z$ be the section of the chain C_m consisting of simplexes in Z. Observe $X(t_1, +) = \big(X(t_0, +) \frown X(t_1, +)\big) \smile \big(pX(t_0, +) \frown X(t_1, +)\big)$. We may therefore define $A_m(t_1, +)$ as

$$A_m(t_1, +) = A_m(t_0, +) \mid X(t_1, +) + pA_m(t_0, +) \mid X(t_1, +) \ . \tag{3.00}$$

Starting with $A_m(0, +)$ we use (3.0) to give the determination of $A_m(t; +)$ at the end points of each sub interval...

We have then for any interval $J:(t_0, t_1)$,

$$B_{m-1}(t_0) - B_{m-1}(t_1) = \partial K \ , \tag{3.01}$$

where $K = A_m(t_0, +) - A_m(t_1, +)$ and is symmetric in view of the consistency condition imposed in (3.0). Hence

$$B'_{m-1}(t_0) - B'_{m-1}(t_1) = (\partial K)' = \partial(K') \ . \tag{3.02}$$

Thus $B'_{m-1}(t_0)$ and $B'_{m-1}(t_1)$, considered in $X'(J)$, are homologous over $X'(J)$.

Let $D(x, t) = |\,x\,| - r(t)$, $x \in X(t)$. We transfer attention to the space $Y_n \times I$. Under the map f^{-1} restricted to $X(t_i)$ we can consider $B_{m-1}(t_i)$ as on $X(t_i) \times t_i$ and then on $X'' = U_J\big(X(J), J\big) = U_I X(t) \times t$ under an inclusion map. Similarly $B'_{m-1}(t_i)$ on $X'(t_i) \times t_i$ can be supposed mapped by inclusion on $X''' = U_I X'(t) \times t$. Application of Lemma 3B to (3.02) shows that

$$B'_{m-1}(0) \sim B'_{m-1}(1) \tag{3.03}$$

over X'''. We can assume $D(x, t)$ defined on X'' to R^1. Denote by F (in X'') the point set for which $D(x, t)$ vanishes, i.e.
$F = U_I\big(S_{n-1}(r(t)\big) \frown X(t) \times t$. Define a set $H \subset X''$ as line symmetric if $x(t) \times t \in H$ implies $px(t) \times t \in H$. F is line symmetric. We assert F contains a line symmetric cycle homologous to $B_{m-1}(0)$ over X''. Let

$$U = \{(x, t) \mid (x, t) \in X'', D(x, t) \leq 0\} \ ,$$
$$V = \{(x, t) \mid (x, t) \in X'', D(x, t) \geq 0\} \ .$$

Thus $F = U \cap V = U - \underline{U}$. Observe $\| B_{m-1}(0) \| \subset X(0) \times 0 \subset U$ and $\| B_{m-1}(1) \| \subset X(1) \times 1 \subset V$. Accordingly $\| B'_{m-1}(1) \| \subset X'(1) \times 1$ and $F' = U' \cap V' = U' - \underline{U'}$. $\underline{U'}$ is the open part of TU where T is defined on line symmetric sets by $T(x(t) \times t \cup px(t) \times t) = x'(t) \times t,\ x'(t) \in X'(t)$.

We use an argument akin to that involved in (1.0). Consider

$$\begin{array}{ccccccccc} & & H_{m-1}(X'(1)\times 1) & & & & & & \\ & & \downarrow l & & & & & & \\ \longrightarrow H_{m-1}(F') & \xrightarrow{i} & H_{m-1}(V') & \xrightarrow{j} & H_{m-1}(V', W') & \xrightarrow{\partial} & H_{m-2}(F') & & (3.04) \\ & & \downarrow r & \text{II} & \downarrow e & & & & \\ & & H_{m-1}(X''') & \xrightarrow{s} & H_{m-1}(X''', U') & & & & \end{array}$$

where i, r, s, l are inclusions. Again, [E. S., p. 266], e is an excision isomorphism, with $X''' - \underline{U'}, U' - \underline{U'} = V', F'$.

We derive from $B'_{m-1}(0) \sim B'_{m-1}(1)$ on X''' that

$$B'_{m-1}(1) \sim 0 \bmod X'(0) \times 0$$

and therefore $B'_m(1) \sim 0 \bmod U'$. We make these remarks more precise by writing

$$srl\{B'_{m-1}(1)\} = 0$$

whence

$$e^{-1}srl\{B'_{m-1}(1)\} = 0\ .$$

Since all our homomorphisms in the square II are either inclusions or excisions, commutativity holds and we have

$l\{B_{m-1}(1)\}$ is in kernel j.

Since I_2 is a field, the upper horizontal sequence in (3.04) is exact and so $l\{B'_{m-1}(1)\}$ is in image i. Thus there is a cycle D'_{m-1} in F' with $i\{D'_{m-1}\} = l\{B'_{m-1}(1)\}$.

If we wish we can carry out the arguments in terms of simplicial complexes. Thus $D(x, t)$ can be replaced by simplicial approximations and $U_I(X(t) \times t)$ by a sequence of simplicial neighborhoods. Appeal to compactness gains the final conclusions required (say those arising from the existence of D'_{m-1}). It is more convenient now, however, to interpret all the groups as Čech groups.

We again indicate by the context the inclusion space in which D_{m-1} and its transforms are considered. Thus $D'_{m-1} \sim B'_{m-1}(1)$ over X'''.

Recalling the interpretation of T and T^{-1} for line symmetric chains we have $T^{-1}D'_{m-1} = D_{m-1} \subset F$. Consider, in Y_n, $A_{m-1} = gD_{m-1}$. We assert A'_{m-1} is *non bounding on* Y'_n. Suppose this were untrue. Then since g^{-1} is a homeomorphism on Y_n onto $U_I S_{n-1}(r(t)) \times t$ we should infer $D'_{m-1} \sim 0$ on $U_I S_{n-1}(r(t)) \times t$ and hence on $Y'_n \times I$. In view of Lemma 3C, $S'_{m-1}(1)$ is non bounding not only on $Y^{n'}(1)$ and Y' but also on $Y'_n \times I$. Since $D'_{m-1} \sim B'_{m-1}(1) \sim S'_{m-1}(1)$, D'_{m-1} cannot be homologous to 0 on $Y'_n \times I$. This contradiction establishes our assertion about A'_{m-1}. By replacing A_{m-1} or some sub cycle by its homologue S_{m-1} in Y_n we immediately establish A'_{m-1} is non bounding when considered on Y'.

The induction hypothesis guarantees the existence of m orthogonal diameters of some sphere whose termini $\{a_i, pa_i \mid i = 1, \ldots, m\}$ lie in $||A_{m-1}||$. We define t_0 by $|pa_i| = |a_i| = 1 + t_0$. Moreover $fg^{-1}||A_{m-1}|| \subset X$. Thus $\{x_i \mid fg^{-1}(a_i) = x_i\}$ satisfy $|x_i| = |px_i| = r(t_0)$. Let $x_{m+1} = L(t_0)$. Then $|x_{m+1}| = |px_{m+1}| = r(t_0)$ also and the assertion of the theorem is established.

By a diameter of the symmetric set X we shall mean a segment bisected by the origin with end points x, $px \in X$.

Corollary 3D. *Suppose X is a compact symmetric set exterior to 0 in R^{n+1} with X' a carrier of an m cycle on Y'. Let F be a continuous map of Y to the reals, satisfying $F(x) = F(px)$. Then there are $m+1$ orthogonal diameters to X, whose end points lie in X and map into a common point under F.*

The non trivial case is that when $F(x) \not\equiv 0$. One proof consists in the observation that any non negative function can replace the distance function from the origin in the proof of Theorem 3A and so with $s(x) = 2 \sup |F(x)| + F(x)$ we have the preceeding argument valid in all details. An alternative proof for the case $X = S^n$ proceeds by replacing $F(x)$ by $w(x) = F(x)/2 \sup |F(x)|$. Consider the points $x \in X$ as vectors from 0. Replace $x \in X$ by the vector $x(1 + w(x))$. This gives a new symmetric set X_1 homeomorphic to X, $x = h(x_1)$, whence $khH_m(X') = kH_m(X') \neq 0$. Thus Theorem 3A applied to the set X_1 yields the existence of $m+1$ orthogonal diameters with end points $x_i(1 + w(x_i))$, $px_i(1 + w(px_i))$ on a common sphere, i. e. $w(x_i) = w(x_{m+1})$ and this implies the assertion of the corollary.

4. Generalizations of a Borsuk Theorem. The following theorem for the special case $Z = S_n$, $j = n$ reduces to a classic result of Borsuk's [B].

Theorem 4A. *Let Z be a compact symmetric set which separates* 0 *and* ∞. *Let $\{f_i \mid i = 1, \ldots, j\}$ be* j *continuous real valued functions on Z and suppose $X = \{x \mid f_i(x) = f_i(px),\ 1 \leq i \leq j \leq n,\ x \in Z\}$. Then X' carries a non bounding n —* j *dimensional cycle over Y'.*

The case $Z = S_n$, j $= n - 1$ is already new. Let $F_i(z) = f_i(z) - f_i(pz)$.

Suppose, continuing the terminology of Section 2, that $Z \subset Y$. Assume f_i and hence F_i extended to Y by Tietze's theorem. Let K_2 be the component of $Y - Z$ containing $S_n(2)$, and K_1 the component of $Y - Z$ containing $S_n(1)$. Then $F_1 = Y - K_2$, $F_2 = Y - K_1$, are closed symmetric sets with $F_1 \cap F_2 = Z$, $F_1 \cup F_2 = Y$. We proceed with the analogue of (1.0) and (3.04), viz

$$\begin{tikzcd}
 & H_n(S'_n(1)) \arrow[d, "l"] & & \\
\longrightarrow H_m(Z') \arrow[r, "i"] & H_m(F'_1) \arrow[r, "j"] \arrow[d, "r"] \arrow[dr, phantom, "\mathrm{III}"] & H_m(F'_1, Z') \arrow[r, "\partial"] \arrow[d, "e"] & {} \\
 & H_m(Y') \arrow[r, "s"] & H_m(Y', F'_2) &
\end{tikzcd} \tag{4.00}$$

By the argument we have used earlier it follows that some element of $H_m(Z')$ maps by i into the non neutral element of $H_m(F'_1)$. Thus some cycle, A'_m, of Z' is homologous to $S'_m(1)$. Let $W' = \| A'_m \|$ and as usual let $W = T^{-1}W'$. Let $W_1 = \{z \mid F_1(z) \geq 0\} \cap W$. Then $BdW_1 = W_1 \cap pW_1 = \{z \mid F_1(z) = 0\} \cap W$. (That $BdW_1 \neq 0$ is established incidentally in the course of the proof of Theorem 2B). We use Theorem 2B to guarantee the existence of a symmetric $n - 1$ cycle A_{n-1} carried by BdW_1 with $A'_{n-1} \sim S'_{n-1}$ over Y'. If $j > 1$ let $W_2 = \{z \mid F_2(z) \geq 0\} \cap BdW_1$. We need only take points in $\| A_{n-1} \|$ really. Again $BdW_2 = \{z \mid F_1(z) = 0, F_2(z) = 0\} \cap W \neq 0$ and from Theorem 2B follows the existence of a symmetric $n - 2$ cycle A_{n-2} carried by BdW_2 with $A'_{n-2} \sim S'_{n-2}$ over Y'. On continuing the process if necessary we gain the conclusion: $X = \{z \mid F_i(z) = 0, i = 1, \ldots, j\}$ *carries a symmetric $n - j$ cycle A_{n-j} where $A'_{n-j} \sim S'_{n-j}$* on Y'. This is *the assertion of the theorem.*

Theorem 4B. *Let Z be a compact symmetric set in Y such that Z' carries an m cycle non bounding in Y'. Let $\{f_i \mid i = 1, \ldots, j\}$ be continuous real valued functions. Let $X = \{z \mid f_i(z) = f_i(pz), 1 \leq | \leq j, z \in Z\}$. Then X' is the carrier of an $m - j$ dimensional cycle A'_{m-j} which does not bound in Y'.*

Replace Z by a compact symmetric subset if necessary which carries A_m a connected symmetric m cycle with $A'_m \sim S'_m$ in Y'. The latter half of the proof of Theorem 4A applies verbatim.

The proofs of theorems 4A and 4B require merely that the sets $\{F_i(z) \mid z \geq 0\}$ be closed. Thus *these theorems remain in force if the requirements of continuity on $f_i(x)$ are weakened to, say, upper semi-continuity on $f_i(x) - f_i(px)$.* This strengthens even the classical Borsuk theorem.

5. The Generalized Dyson Theorem. We gather together some of our earlier results to give an extension of Dyson's result.

Theorem 5A. *Suppose Z separates 0 from ∞ in R^{n+1}.*

$$\{f_i, \mid i = i, \ldots, j \leq n\},$$

with $f_i(z) - f_i(pz)$ upper semicontinuous, are j real valued functions on Z. Let $f(z)$ be the point of R^j whose coordinates are $f_1(z)\ f_2(z), \ldots, f_j(z)$. Then there exist $n - j + 1$ orthogonal diameters to Z whose termini map into a single point under

$$\mid f(z) \mid = (\Sigma_{i=1}^{j} \mid f_i(z) \mid^2)^{1/2} .$$

We invoke Theorem 4A to obtain a subset of the set of common zeros of $f(z) - f(pz)$ which satisfies the hypothesis of Theorem 3A and Corollary 3D with $\mid fz \mid = F(z)$. Similarly using Theorem 4B we get

Theorem 5B. *Suppose Z is a compact symmetric set in Y such that Z' carries an m cycle non bounding in Y', $m \leq n$. Let $\{f_i \mid \leq i \leq j \leq m\}$, with $f(z) - f(pz)$ upper semicontinuous, be j real valued functions on Z and let $f(z)$ be the point of R^j whose coordinates are $f_1(z) \ldots f_i(z)$. Then there exist $m - j + 1$ orthogonal diameters for Z whose termini map into a single point under $\mid f \mid$.*

Remarks. The arguments require merely that p be a fixed point free continuous involution such that the identification space is homologically a projective space. Accordingly the results and demonstrations in Sections 3, 4, and 5 are formally valid in detail if p is interpreted as the reflection in an l dimensional hyperplane. Then $Y = pY$ and $X = pX$ are sets symmetric with respect to this hyperplane. The identification space is now $Y' = p^{n-l} \times I^{l+1}$ (so the dimension bound on X is now $n - l$ rather than n).

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