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# Motions with Maximal Displacements

by HERBERT BUSEMANN, Los Angeles

*To Paul Finsler on his sixtieth birthday*

This note deals with the surprisingly strong implications of a nearly trivial remark. We consider an abstract Finsler space, that is, a space in which geodesics with the usual geometric properties, apart from differentiability, exist. The exact requirements are found below. The remark is this: *If for a motion  $\Phi$  of a Finsler space a point  $z$  exists at which the displacement, or the distance  $xx\Phi$  from a point  $x$  to its image  $x\Phi$  under  $\Phi$ , attains a maximum which is not too large, then the shortest geodesic arc from  $z$  to  $z\Phi^2$  passes through  $z\Phi$ .*

Among the facts which we deduce from this observation we mention the following: A closed group of motions of a compact space (without any differentiability conditions) is a Lie group. In a compact space without conjugate points and with an abelian fundamental group no geodesic has multiple points, and the closed geodesics in a given free homotopy class have the same length and cover the space simply.

## 1. The axioms. Proof of the remark.

The space is assumed to be a  $G$ -space, see [1] or [2]. The axioms for a  $G$ -space  $R$  are:

I  $R$  is metric. The distance of  $x$  and  $y$  is denoted by  $xy$ .

II  $R$  is finitely compact, i. e. the Theorem of Bolzano-Weierstrass holds.

III  $R$  is convex in Menger's sense, see [3]. If we introduce the notation  $(xyz)$  to indicate that  $x, y, z$  are distinct and  $xy + yz = xz$ , the last condition means: if  $x \neq z$  then  $y$  with  $(xyz)$  exists.  $S(p, \rho)$  will denote the set of points  $x$  satisfying  $px < \rho$ .

IV *Prolongation is locally possible*: every point  $p$  has a neighborhood  $S(p, \rho_p)$ ,  $\rho_p > 0$ , such that for any two distinct points  $x, y$  in  $S(p, \rho_p)$  a point  $z$  with  $(xyz)$  exists.

V *Prolongation is unique*: If  $(xyz_1)$ ,  $(xyz_2)$  and  $yz_1 = yz_2$  then  $z_1 = z_2$ .

It follows from I, II, III that any two points  $y, z$  can be connected by a segment  $T(y, z)$ , i. e. a curve  $x(t)$ ,  $\alpha \leq t \leq \beta = \alpha + yz$  such that  $x(\alpha) = y$ ,  $x(\beta) = z$  and  $x(t_1)x(t_2) = |t_1 - t_2|$ , see [3] or [7, p. 12]. A geodesic is a curve  $x(t)$ ,  $-\infty < t < \infty$ , with the property that for every real  $t_0$  a positive  $\varepsilon(t_0)$  exists such that  $x(t_1)x(t_2) = |t_1 - t_2|$  for  $|t_i - t_0| \leq \varepsilon(t_0)$   $i = 1, 2$ . Axioms I to IV imply the existence of geodesics: a representation  $x(t)$ ,  $\alpha \leq t \leq \beta$ ,  $\alpha < \beta$ , of a segment can be extended to all real  $t$  to represent a geodesic. This extension is unique if V holds.

The function  $\varrho_p$  in IV may be erratic but it can be replaced by a continuous function: if  $\varrho(p) = \sup \varrho_p$ , where  $\varrho_p$  satisfies IV at  $p$ , then  $S(p, \varrho(p))$  also satisfies IV. If  $\varrho(p) = \infty$ , then for any two distinct points  $x, y$  a point  $z$  with  $(xyz)$  exists. There fore  $\varrho(q) = \infty$  for any other point  $q$  and if  $x(t)$  represents a geodesic, then

$$x(t_1)x(t_2) = |t_1 - t_2|$$

for any  $t_1, t_2$ . We call a geodesic with this property a *straight line*. Thus for  $\varrho(p) = \infty$  all geodesics are straight lines, and the  $G$ -space is called *straight*. In the terminology of the calculus of variations the straight spaces are the simply connected spaces without conjugate points.

If the space is not straight, then  $0 < \varrho(p) < \infty$  and

$$|\varrho(p) - \varrho(q)| \leq pq.$$

For if  $\varrho(p) > \varrho(q)$  and  $\varrho(p) > pq$  then the triangle inequality yields  $S(q, \varrho(p) - pq) \subset S(p, \varrho(p))$  hence  $\varrho(p) - \varrho(q) \leq pq$ . The number  $\varepsilon(t_0)$  occurring in the definition of a geodesic  $x(t)$  may be chosen as  $\varrho(x(t_0))$ .

Axiom V implies that the segment  $T(x, y)$  is unique if a point  $z$  with  $(xyz)$  exists, see [1, p. 216]. In particular  $T(x, y)$  is always unique for  $x, y \in S(p, \varrho(p))$ .

The following is now an exact formulation of the remark mentioned in the introduction:

(1) If  $\Phi$  is a motion of the  $G$ -space  $R$  which is not the identity  $E$  and if  $zz\Phi = \sup_{x \in R} xx\Phi < \varrho(z)/2$ , then  $(zz\Phi z\Phi^2)$ .<sup>1)</sup>

*Proof.* Because  $zz\Phi < \varrho(z)/2$  there is a point  $u$  such that  $(zz\Phi u)$  and  $zz\Phi = z\Phi u$ , briefly  $z\Phi$  is a midpoint of  $z$  and  $u$ . Then  $z\Phi^2$  is a midpoint of  $z\Phi$  and  $u\Phi$  and the only one, because  $\varrho(z\Phi) = \varrho(z)$ . The relation

$$zz\Phi \geq uu\Phi \geq z\Phi u\Phi - z\Phi u = zu - z\Phi u = zz\Phi$$

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<sup>1)</sup> It is also true, and has many applications(see [2]), that  $0 < zz\Phi = \inf_{x \in R} xx\Phi < \varrho(z)/2$  implies  $(zz\Phi z\Phi^2)$ .

shows that  $u$  is a midpoint of  $z\Phi$  and  $u\Phi$ , hence  $u = z\Phi^2$ , which proves the assertion.

## 2. Applications to compact spaces.

In the compact case the following additional statements can be made :

(2) For any motion  $\Phi \neq E$  of a compact  $G$ -space  $R$  a point  $z$  of maximal displacement  $\alpha$  (i. e.  $\alpha = zz\Phi = \sup_{x \in R} xx\Phi$ ) exists. If  $k$  is the first integer for which  $k\alpha \geq \varrho(z)/2$ , then  $zz\Phi^k = k\alpha$ . If  $k > 1$  then a geodesic  $x(t)$  exists such that  $x(i\alpha) = z\Phi^i$ ,  $i = 0, \pm 1, \pm 2, \dots$  and  $x(t)$  represents a segment for  $i\alpha \leq t \leq (i+k)\alpha$ .

*Proof.* The existence of  $z$  is obvious and for  $k = 1$  there is nothing to prove. If  $k > 1$  then  $k\alpha < \varrho(z)$  and with  $z_i = z\Phi^i$  it follows from (1) that  $(zz_1z_2)$ , hence  $(z_{i-1}z_i z_{i+1})$  for all  $i$ . Since  $\varrho(z_i) = \varrho(z)$  the segment  $T(z_{i-1}, z_{i+1})$  is unique and passes through  $z_i$ . The existence of  $x(t)$  follows, and  $x(t)$  represents a segment for  $i\alpha \leq t \leq (i+k)\alpha$  because  $k\alpha < \varrho(z_i)$ . In particular  $x(0)x(k\alpha) = zz\Phi^k = k\alpha$ .

We use the standard metric  $\delta(\Phi, \Psi) = \sup_{x \in R} x\Phi x\Psi$  for motions  $\Phi, \Psi$  of a compact space. Since  $\varrho(x)$  is continuous and positive it has on a compact space  $R$  a positive minimum  $\varrho(R)$ . An immediate consequence of (2) is

(3) *A non-trivial group of motions of a compact  $G$ -space  $R$  has at least diameter  $\varrho(R)/2$ .*

„Non-trivial“ means that the group contains at least one motion  $\Phi \neq E$ , and (2) implies that  $\delta(E, \Phi^k) \geq \varrho(z)/2 \geq \varrho(R)/2$  for a suitable positive  $k$ . Well known theorems on topological groups yield the further result:

(4) **Theorem.** *A closed group of motions of a compact  $G$ -space  $R$  is a Lie group. If the group  $\Gamma$  of all motions which  $R$  possesses is transitive on  $R$ , then  $R$  is a topological manifold and  $\dim \Gamma \leq \dim R (\dim R + 1)/2$ .*

The first statement follows from [4, Theorem 53] and the second from [5, Corollary 3', Theorem 9 and Theorem 12]. In spite of the recent result of Gleason it is an open question whether (4) extends to non-compact spaces, since no analogue to (3) is known, even when  $\inf \varrho(x) > 0$ .

The rotations about the  $z$ -axis of the surface  $z = (x^2 + y^2)^{-1/2}$  in  $E^3$ , with the length of the shortest connection on the surface as distance, show that a one-parameter group of motions of a non-compact  $G$ -space

may not have any orbits which are geodesics. (1) and (2) imply the existence of such orbits on compact  $G$ -spaces:

(5) **Theorem.** *A one-parameter group of motions of a compact  $G$ -space possesses an orbit which is a geodesic.*

We assume that the one-parameter group is given in the form  $\Phi(s)$  with  $\Phi(s_1)\Phi(s_2) = \Phi(s_1 + s_2)$ , and prove that a geodesic  $x(t)$  and a positive  $\alpha$  exist such that  $x(t) = x(0)\Phi(\alpha t)$ .

Choose  $\varepsilon > 0$  such that  $\delta(E, \Phi(s)) < \rho(R)/2$  for  $|s| < \varepsilon$ . Let  $0 < u < \varepsilon$ . By (2) there are points  $z$  and  $z'$  of maximal displacement under  $\Phi(u)$  and  $\Phi(u/2)$  respectively. Then the choice of  $\varepsilon$  and (1) imply

$$\begin{aligned} z'z'\Phi(u) &= 2z'z'\Phi(u/2) \geq 2zz\Phi(u/2) \\ &= zz\Phi(u/2) + z\Phi(u/2)z\Phi(u) \geq zz\Phi(u) \geq z'z'\Phi(u). \end{aligned}$$

Hence  $z$  is also a point of maximal displacement for  $\Phi(u/2)$  and generally for  $\Phi(2^{-n}u)$ . Moreover  $(zz\Phi(u/2)z\Phi(u))$  and generally

$$(zz\Phi(2^{-n-1}u)z\Phi(2^{-n}u)) .$$

If  $x(t)$  is the geodesic with  $x(0) = z$  which represents for

$$0 \leq t \leq zz\Phi(u) = \beta$$

the (unique) segment  $T(z, z\Phi(u))$  then (2) yields

$$x(i 2^{-n} \beta) = z\Phi(i 2^{-n} u)$$

for all  $i$  and non-negative  $n$ . A trivial continuity argument shows that  $x(\beta t) = z\Phi(ut)$  or  $x(t) = x(0)\Phi(\alpha t)$  for all  $t$ , where  $\alpha = u/\beta$ .

### 3. Compact spaces without conjugate points and abelian fundamental groups.

For a  $G$ -space  $R'$  which satisfies the usual differentiability hypotheses of the calculus of variations the absence of conjugate points means that the universal covering space  $R$  of  $R'$  is straight.

The relation of the theorem mentioned in the introduction to motions with maximal displacements comes from:

(6) **Theorem.** *If  $R$  is straight and  $\Phi$  is a motion of  $R$  for which a point  $z$  with  $0 < zz\Phi = \sup_{x \in R} xx\Phi$  exists, then  $xx\Phi$  is independent of  $x$ . The points  $x\Phi^i$ ,  $i = 0, \pm 1, \pm 2, \dots$  lie for each  $x$  on a straight line  $\mathfrak{g}_x$ .*

For it follows from (1) that the points  $z_i = z\Phi^i$  satisfy

$$(z_{i-1}z_i z_{i+1})$$

hence lie on a straight line  $\mathfrak{g}_z$ . If  $x$  is any other point of  $R$  and  $x_i = x\Phi^i$  then

$$n \cdot zz\Phi = zz_n \leq zx + \sum_{i=1}^n x_{i-1}x_i + x_n z_n = 2zx + n \cdot xx\Phi$$

or  $xx\Phi \geq zz\Phi - 2zx/n$ . Since  $n$  is arbitrary  $xx\Phi \geq zz\Phi$ , hence  $xx\Phi = zz\Phi$ .

Thus every point  $x$  of  $R$  is a point of „maximal“ displacement for  $\Phi$ , therefore (1) shows that the points  $x_i$  lie on a line  $\mathfrak{g}_x$ .

Clearly for any two points  $x, y$  either  $\mathfrak{g}_x = \mathfrak{g}_y$  or  $\mathfrak{g}_x \cap \mathfrak{g}_y = 0$ , since  $u \in \mathfrak{g}_x \cap \mathfrak{g}_y$  implies  $u\Phi^i \in \mathfrak{g}_x \cap \mathfrak{g}_y$  hence  $\mathfrak{g}_x = \mathfrak{g}_y$ .

Let the universal covering space  $R$  of the  $G$ -space  $R'$  be straight. There is a wellknown correspondence between the classes of conjugate elements in the fundamental group  $\mathfrak{F}$  of  $R'$  and the classes of freely homotopic curves in  $R'$ , see for instance [8, § 49]. If, as in [1],  $\mathfrak{F}$  is realized as the group of motions of  $R$  which lie over the identity of  $R'$  then the closed geodesics in a free homotopy class  $K_\Phi$  determined by a motion  $\Phi \neq E$  in  $\mathfrak{F}$  correspond to the straight lines in  $R$  which are taken into themselves by  $\Phi$ , the so-called axes of  $\Phi$ , see [2]. If  $x$  lies on an axis of  $\Phi$  then  $xx\Phi$  is the length of the corresponding geodesic.

If  $\Phi \neq E$  possesses a point of maximal displacement then we conclude from (6) that every point  $x'$  of  $R'$  lies on a closed geodesic of length  $xx\Phi$  in  $K_\Phi$  and that two such geodesics do not intersect. It is now easy to prove:

(7) **Theorem.** *Let  $R'$  be a compact  $G$ -space with an abelian fundamental group and a straight universal covering space  $R$ . Then the closed geodesics in any (non-trivial) free homotopy class of  $R'$  have the same length and cover  $R'$  simply. No geodesic in  $R'$  has multiple points.*

For let  $\Phi$  be any motion in the fundamental group  $\mathfrak{F}$  of  $R'$  different from the identity (such motions exist because  $R$  is non-compact, hence different from  $R'$ ). There is a compact subset  $C$  of  $R$  such that

$$\cup C\Phi_\nu = R ,$$

where  $\Phi_\nu$  traverses  $\mathfrak{F}$ , see [2, p. 267]. The Function  $yy\Phi$  attains on  $C$  a maximum at some point  $z \in C$ . If  $x$  is an arbitrary point of  $R$  then a  $\Phi_\nu \in \mathfrak{F}$  exists such that  $y = x\Phi_\nu \in C$ . Because  $\mathfrak{F}$  is abelian

$$xx\Phi = x\Phi_\nu x\Phi\Phi_\nu = x\Phi_\nu x\Phi_\nu \Phi = yy\Phi \leq zz\Phi ,$$

so that  $zz\Phi = \sup_{x \in R} xx\Phi$ .

The preceding discussion shows that the closed geodesics in  $K_\Phi$  all have length  $xx\Phi$  and cover  $R'$  simply.  $K_\Phi$  is, owing to the arbitrariness of  $\Phi$ , an arbitrary non-trivial free homotopy class in  $R'$ .

There can be no geodesic monogon with a proper vertex  $x'$ . For such a monogon would lie in some free homotopy class  $K_\Phi$ , not trivial because  $R$  is straight. If  $x$  lies over  $x'$  then the points  $x\Phi^i$  would not lie on a straight line. The absence of proper monogons means that the geodesics in  $R'$  have no multiple points.

Theorem (7) brings a result of E. Hopf [6] to mind, namely that a two-dimensional torus  $T'$  with a Riemannian metric is euclidean, if its universal covering plane  $T$  is straight. In that case (7) is therefore trivial. However, when the condition that the metric be Riemannian is omitted, then  $T'$  possesses a great number of essentially different metrizations for which  $T$  is straight. The geodesics in  $T$  need not satisfy Desargues' Theorem, but they always satisfy the parallel axiom.

#### 4. A characterization of Minkowskian geometry.

The translations of the euclidean space are obvious examples für (6). When there are enough motions satisfying (6) these motions are necessarily ordinary translations:

**(8) Theorem.** *If a straight space possesses a transitive group of motions such that for each motion  $\Phi$  in  $\Gamma$  a point exists whose displacement unter  $\Phi$  is maximal, then  $R$  is Minkowskian and  $\Gamma$  the group of translations of  $R$ .*

We deduce from (6) that  $xx\Phi$  is constant for each  $\Phi$  in  $\Gamma$ . Hence no motion  $\Phi \neq E$  in  $\Gamma$  has fixed points and  $\Gamma$  is simply transitive on  $R$ , see [7, p. 220]. The motion in  $\Gamma$  that takes  $a$  into  $b$  may therefore be denoted by  $(a \rightarrow b)$ . Because of (6) the line  $g(a, b)$  through  $a$  and  $b$ ,  $a \neq b$ , is an axis of  $(a \rightarrow b)$ . The proof of (8) consists of several steps the first of which is:

(a)  $R$  satisfies the parallel axiom, for the terminology see [7].

To see this let  $x(t)$  be any geodesic and  $y$  a point not on  $x(t)$ . Since  $x(t)$  is an axis of  $\Phi = (x(0) \rightarrow x(1))$  it suffices to show that  $g(y, x(t))$  tends for  $t \rightarrow \infty$  or  $t \rightarrow -\infty$  to the axis  $g_y$  of  $\Phi$  through  $y$ . For the statement that the line  $g_y$  is an axis of the same motion  $\Phi$  as  $g_x$ , is symmetric and transitive, hence the statement that  $g_y$  is parallel to  $g_x$  also has these properties.

Let  $y(t)$  represent the axis of  $\Phi$  through  $y$  with  $y(0) = y$  and  $y\Phi = y(1)$ . The limit sphere  $\mathcal{A}(y, r)$  through  $y$  to  $r$  (see [1, p. 240] or

[7, p. 98]), where  $r$  is the ray  $t \geq 0$  of  $x(t)$ , intersects  $x(t)$  in a point  $x(t_0)$  and  $x(t_0)\Phi = x(t_0 + 1)$ . Moreover  $\Lambda(y\Phi, r) = \Lambda(y, r)\Phi = \Lambda(x(t_0 + 1), r)$ . The asymptote  $a$  to  $r$  through  $y$  intersects  $\Lambda(y\Phi, r)$  in the unique foot  $f$  of  $y$  on  $\Lambda(y\Phi, r)$ . But, see [1, p. 242],

$$1 = x(t_0)x(t_0 + 1) = yf \leq yy\Phi = 1 ,$$

hence  $y\Phi$  is also a foot of  $y$  on  $\Lambda(y\Phi, r)$ , so that  $y\Phi = f$  and  $a = g_v$ , which proves (a).

We show next

(b) If  $y(t)$ ,  $t \geq 0$  represents a ray  $s$  and  $g$  is a straight line through  $y = y(0)$  not containing  $s$  then  $y(t)g \rightarrow \infty$  for  $t \rightarrow \infty$ .

For an indirect proof assume the existence of a sequence  $t_n$  with  $x(t_n)g < M$ . If  $f_n$  is a foot of  $x(t_n)$  on  $g$ , then  $f_n \neq y$  for large  $n$ , and  $q_n = x(t_n)(f_n \rightarrow y)$  has  $y$  as foot on  $g$ . Because  $q_n y = x(t_n)f_n < M$  there is a subsequence  $\{v\}$  of  $\{n\}$  for which  $q_v$  tends to a point  $q$ . ( $qy > 0$  because of [1, Theorem (11.14)]).

The line  $g(x(t_v), q_v)$  is an axis of  $(f_v \rightarrow y)$ , hence parallel to  $g$ . It tends therefore to the parallel  $g'$  to  $g$  through  $q$ . On the other hand, the line  $g(q_v, x(t_v))$  tends also to the parallel through  $q$  to the line  $h$  carrying  $s$  (see the definition of co-ray in [1]). Since parallelism is symmetric, it would follow that  $g$  and  $h$  are parallel to  $g'$ , which is impossible because  $g$  and  $h$  intersect.

(c)  $x_1g_2$  and  $x_2g_1$  are bounded for  $x_i \in g_i$  if and only if  $g_1$  and  $g_2$  are parallel.

If  $g_1$  and  $g_2$  are parallel then the fact that they are axes of the same motion in  $\Phi$  shows that  $x_1g_2$  and  $x_2g_1$  are bounded. The converse follows from (b), for a proof see [2, p. 278].

(d)  $\Gamma$  is abelian.

If  $\Phi$  and  $\Psi$  are two non-trivial motions in  $\Gamma$ , select an arbitrary point  $z$ . If the axes of  $\Phi$  and  $\Psi$  through  $z$  coincide, it is easily seen that  $\Phi$  and  $\Psi$  commute (this case can also be deduced by a limit process from the general case). We assume therefore that  $z$ ,  $p = z\Phi$  and  $q = z\Psi$  are not collinear. Put  $g(z, p) = g$ ,  $g(z, q) = h$  and  $h' = h\Phi$ . Then  $y' = y\Phi \in h'$  for  $y \in h$ . The relation  $yy' = zp$  shows that  $y'h$  and  $yh'$  are bounded, by (c) the lines  $h$  and  $h'$  are parallel. Therefore  $h'$  is an axis of  $\Psi$ , so that  $p\Psi$  is a point  $u$  of  $h'$  with  $zq = pu$ . On the other hand  $zq = z\Phi q\Phi = pq\Phi$ , hence  $q\Phi = u$ . Therefore  $\Phi = (q \rightarrow u)$ ,  $\Psi = (p \rightarrow u)$  and

$$\Phi\Psi = (z \rightarrow p)(p \rightarrow u) = (z \rightarrow u) = (z \rightarrow q)(q \rightarrow u) = \Psi\Phi.$$

It now follows readily from a wellknown result of Pontrjagin, see [4, p. 170], that the space is a finite dimensional Minkowski space. A simple proof which does not use the theory of topological groups is found in [7, pp. 229—231].

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