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Motions with Maximal Displacements

by HERBERT BUSEMANN, Los Angeles

To Paul Finsler on his sixtieth birthday

This note deals with the surprisingly strong implications of a nearly trivial remark. We consider an abstract Finsler space, that is, a space in which geodesics with the usual geometric properties, apart from differentiability, exist. The exact requirements are found below. The remark is this: *If for a motion Φ of a Finsler space a point z exists at which the displacement, or the distance $xx\Phi$ from a point x to its image $x\Phi$ under Φ , attains a maximum which is not too large, then the shortest geodesic arc from z to $z\Phi^2$ passes through $z\Phi$.*

Among the facts which we deduce from this observation we mention the following: A closed group of motions of a compact space (without any differentiability conditions) is a Lie group. In a compact space without conjugate points and with an abelian fundamental group no geodesic has multiple points, and the closed geodesics in a given free homotopy class have the same length and cover the space simply.

1. The axioms. Proof of the remark.

The space is assumed to be a G -space, see [1] or [2]. The axioms for a G -space R are:

I R is metric. The distance of x and y is denoted by xy .

II R is finitely compact, i. e. the Theorem of Bolzano-Weierstrass holds.

III R is convex in Menger's sense, see [3]. If we introduce the notation (xyz) to indicate that x, y, z are distinct and $xy + yz = xz$, the last condition means: if $x \neq z$ then y with (xyz) exists. $S(p, \varrho)$ will denote the set of points x satisfying $px < \varrho$.

IV Prolongation is locally possible: every point p has a neighborhood $S(p, \varrho_p)$, $\varrho_p > 0$, such that for any two distinct points x, y in $S(p, \varrho_p)$ a point z with (xyz) exists.

V Prolongation is unique: If (xyz_1) , (xyz_2) and $yz_1 = yz_2$ then $z_1 = z_2$.

It follows from I, II, III that any two points y, z can be connected by a segment $T(y, z)$, i. e. a curve $x(t)$, $\alpha \leq t \leq \beta = \alpha + yz$ such that $x(\alpha) = y$, $x(\beta) = z$ and $x(t_1)x(t_2) = |t_1 - t_2|$, see [3] or [7, p. 12]. A geodesic is a curve $x(t)$, $-\infty < t < \infty$, with the property that for every real t_0 a positive $\varepsilon(t_0)$ exists such that $x(t_1)x(t_2) = |t_1 - t_2|$ for $|t_i - t_0| \leq \varepsilon(t_0)$ $i = 1, 2$. Axioms I to IV imply the existence of geodesics: a representation $x(t)$, $\alpha \leq t \leq \beta$, $\alpha < \beta$, of a segment can be extended to all real t to represent a geodesic. This extension is unique if V holds.

The function ϱ_p in IV may be erratic but it can be replaced by a continuous function: if $\varrho(p) = \sup \varrho_p$, where ϱ_p satisfies IV at p , then $S(p, \varrho(p))$ also satisfies IV. If $\varrho(p) = \infty$, then for any two distinct points x, y a point z with (xyz) exists. Therefore $\varrho(q) = \infty$ for any other point q and if $x(t)$ represents a geodesic, then

$$x(t_1)x(t_2) = |t_1 - t_2|$$

for any t_1, t_2 . We call a geodesic with this property a *straight line*. Thus for $\varrho(p) = \infty$ all geodesics are straight lines, and the G -space is called *straight*. In the terminology of the calculus of variations the straight spaces are the simply connected spaces without conjugate points.

If the space is not straight, then $0 < \varrho(p) < \infty$ and

$$|\varrho(p) - \varrho(q)| \leq pq.$$

For if $\varrho(p) > \varrho(q)$ and $\varrho(p) > pq$ then the triangle inequality yields $S(q, \varrho(p) - pq) \subset S(p, \varrho(p))$ hence $\varrho(p) - \varrho(q) \leq pq$. The number $\varepsilon(t_0)$ occurring in the definition of a geodesic $x(t)$ may be chosen as $\varrho(x(t_0))$.

Axiom V implies that the segment $T(x, y)$ is unique if a point z with (xyz) exists, see [1, p. 216]. In particular $T(x, y)$ is always unique for $x, y \in S(p, \varrho(p))$.

The following is now an exact formulation of the remark mentioned in the introduction:

(1) If Φ is a motion of the G -space R which is not the identity E and if $zz\Phi = \sup_{x \in R} xx\Phi < \varrho(z)/2$, then $(zz\Phi z\Phi^2)$.¹⁾

Proof. Because $zz\Phi < \varrho(z)/2$ there is a point u such that $(zz\Phi u)$ and $zz\Phi = z\Phi u$, briefly $z\Phi$ is a midpoint of z and u . Then $z\Phi^2$ is a midpoint of $z\Phi$ and $u\Phi$ and the only one, because $\varrho(z\Phi) = \varrho(z)$. The relation

$$zz\Phi \geq uu\Phi \geq z\Phi u\Phi - z\Phi u = zu - z\Phi u = zz\Phi$$

¹⁾ It is also true, and has many applications (see [2]), that $0 < zz\Phi = \inf_{x \in R} xx\Phi < \varrho(z)/2$ implies $(zz\Phi z\Phi^2)$.

shows that u is a midpoint of $z\Phi$ and $u\Phi$, hence $u = z\Phi^2$, which proves the assertion.

2. Applications to compact spaces.

In the compact case the following additional statements can be made :

(2) For any motion $\Phi \neq E$ of a compact G -space R a point z of maximal displacement α (i. e. $\alpha = zz\Phi = \sup_{x \in R} xx\Phi$) exists. If k is the first integer for which $k\alpha \geq \varrho(z)/2$, then $zz\Phi^k = k\alpha$. If $k > 1$ then a geodesic $x(t)$ exists such that $x(i\alpha) = z\Phi^i$, $i = 0, \pm 1, \pm 2, \dots$ and $x(t)$ represents a segment for $i\alpha \leq t \leq (i+k)\alpha$.

Proof. The existence of z is obvious and for $k = 1$ there is nothing to prove. If $k > 1$ then $k\alpha < \varrho(z)$ and with $z_i = z\Phi^i$ it follows from (1) that (zz_1z_2) , hence $(z_{i-1}z_iz_{i+1})$ for all i . Since $\varrho(z_i) = \varrho(z)$ the segment $T(z_{i-1}, z_{i+1})$ is unique and passes through z_i . The existence of $x(t)$ follows, and $x(t)$ represents a segment for $i\alpha \leq t \leq (i+k)\alpha$ because $k\alpha < \varrho(z_i)$. In particular $x(0)x(k\alpha) = zz\Phi^k = k\alpha$.

We use the standard metric $\delta(\Phi, \Psi) = \sup_{x \in R} x\Phi x\Psi$ for motions Φ, Ψ of a compact space. Since $\varrho(x)$ is continuous and positive it has on a compact space R a positive minimum $\varrho(R)$. An immediate consequence of (2) is

(3) *A non-trivial group of motions of a compact G -space R has at least diameter $\varrho(R)/2$.*

„Non-trivial“ means that the group contains at least one motion $\Phi \neq E$, and (2) implies that $\delta(E, \Phi^k) \geq \varrho(z)/2 \geq \varrho(R)/2$ for a suitable positive k . Well known theorems on topological groups yield the further result:

(4) **Theorem.** *A closed group of motions of a compact G -space R is a Lie group. If the group Γ of all motions which R possesses is transitive on R , then R is a topological manifold and $\dim \Gamma \leq \dim R (\dim R + 1)/2$.*

The first statement follows from [4, Theorem 53] and the second from [5, Corollary 3', Theorem 9 and Theorem 12]. In spite of the recent result of Gleason it is an open question whether (4) extends to non-compact spaces, since no analogue to (3) is known, even when $\inf_{x \in R} \varrho(x) > 0$.

The rotations about the z -axis of the surface $z = (x^2 + y^2)^{-1/2}$ in E^3 , with the length of the shortest connection on the surface as distance, show that a one-parameter group of motions of a non-compact G -space

may not have any orbits which are geodesics. (1) and (2) imply the existence of such orbits on compact G -spaces:

(5) **Theorem.** *A one-parameter group of motions of a compact G -space possesses an orbit which is a geodesic.*

We assume that the one-parameter group is given in the form $\Phi(s)$ with $\Phi(s_1)\Phi(s_2) = \Phi(s_1 + s_2)$, and prove that a geodesic $x(t)$ and a positive α exist such that $x(t) = x(0)\Phi(\alpha t)$.

Choose $\varepsilon > 0$ such that $\delta(E, \Phi(s)) < \varrho(R)/2$ for $|s| < \varepsilon$. Let $0 < u < \varepsilon$. By (2) there are points z and z' of maximal displacement under $\Phi(u)$ and $\Phi(u/2)$ respectively. Then the choice of ε and (1) imply

$$\begin{aligned} z'z'\Phi(u) &= 2z'z'\Phi(u/2) \geq 2zz\Phi(u/2) \\ &= zz\Phi(u/2) + z\Phi(u/2)z\Phi(u) \geq zz\Phi(u) \geq z'z'\Phi(u). \end{aligned}$$

Hence z is also a point of maximal displacement for $\Phi(u/2)$ and generally for $\Phi(2^{-n}u)$. Moreover $(zz\Phi(u/2)z\Phi(u))$ and generally

$$(zz\Phi(2^{-n-1}u)z\Phi(2^{-n}u)) \text{ .}$$

If $x(t)$ is the geodesic with $x(0) = z$ which represents for

$$0 \leq t \leq zz\Phi(u) = \beta$$

the (unique) segment $T(z, z\Phi(u))$ then (2) yields

$$x(i \cdot 2^{-n} \beta) = z\Phi(i \cdot 2^{-n} u)$$

for all i and non-negative n . A trivial continuity argument shows that $x(\beta t) = z\Phi(ut)$ or $x(t) = x(0)\Phi(\alpha t)$ for all t , where $\alpha = u/\beta$.

3. Compact spaces without conjugate points and abelian fundamental groups.

For a G -space R' which satisfies the usual differentiability hypotheses of the calculus of variations the absence of conjugate points means that the universal covering space R of R' is straight.

The relation of the theorem mentioned in the introduction to motions with maximal displacements comes from:

(6) **Theorem.** *If R is straight and Φ is a motion of R for which a point z with $0 < zz\Phi = \sup_{x \in R} xx\Phi$ exists, then $xx\Phi$ is independent of x . The points $x\Phi^i$, $i = 0, \pm 1, \pm 2, \dots$ lie for each x on a straight line \mathfrak{g}_x .*

For it follows from (1) that the points $z_i = z\Phi^i$ satisfy

$$(z_{i-1}z_iz_{i+1})$$

hence lie on a straight line g_z . If x is any other point of R and $x_i = x\Phi^i$ then

$$n \cdot zz\Phi = zz_n \leq zx + \sum_{i=1}^n x_{i-1}x_i + x_n z_n = 2zx + n \cdot xx\Phi$$

or $xx\Phi \geq zz\Phi - 2zx/n$. Since n is arbitrary $xx\Phi \geq zz\Phi$, hence $xx\Phi = zz\Phi$.

Thus every point x of R is a point of „maximal“ displacement for Φ , therefore (1) shows that the points x_i lie on a line g_x .

Clearly for any two points x, y either $g_x = g_y$ or $g_x \cap g_y = 0$, since $u \in g_x \cap g_y$ implies $u\Phi^i \in g_x \cap g_y$ hence $g_x = g_y$.

Let the universal covering space R of the G -space R' be straight. There is a wellknown correspondence between the classes of conjugate elements in the fundamental group \mathfrak{F} of R' and the classes of freely homotopic curves in R' , see for instance [8, § 49]. If, as in [1], \mathfrak{F} is realized as the group of motions of R which lie over the identity of R' then the closed geodesics in a free homotopy class K_Φ determined by a motion $\Phi \neq E$ in \mathfrak{F} correspond to the straight lines in R which are taken into themselves by Φ , the so-called axes of Φ , see [2]. If x lies on an axis of Φ then $xx\Phi$ is the length of the corresponding geodesic.

If $\Phi \neq E$ possesses a point of maximal displacement then we conclude from (6) that every point x' of R' lies on a closed geodesic of length $xx\Phi$ in K_Φ and that two such geodesics do not intersect. It is now easy to prove:

(7) **Theorem.** *Let R' be a compact G -space with an abelian fundamental group and a straight universal covering space R . Then the closed geodesics in any (non-trivial) free homotopy class of R' have the same length and cover R' simply. No geodesic in R' has multiple points.*

For let Φ be any motion in the fundamental group \mathfrak{F} of R' different from the identity (such motions exist because R is non-compact, hence different from R'). There is a compact subset C of R such that

$$\cup C\Phi_\nu = R ,$$

where Φ_ν traverses \mathfrak{F} , see [2, p. 267]. The Function $yy\Phi$ attains on C a maximum at some point $z \in C$. If x is an arbitrary point of R then a $\Phi_\nu \in \mathfrak{F}$ exists such that $y = x\Phi_\nu \in C$. Because \mathfrak{F} is abelian

$$xx\Phi = x\Phi_\nu x\Phi\Phi_\nu = x\Phi_\nu x\Phi_\nu \Phi = yy\Phi \leq zz\Phi ,$$

so that $zz\Phi = \sup_{x \in R} xx\Phi$.

The preceding discussion shows that the closed geodesics in K_Φ all have length $xx\Phi$ and cover R' simply. K_Φ is, owing to the arbitrariness of Φ , an arbitrary non-trivial free homotopy class in R' .

There can be no geodesic monogon with a proper vertex x' . For such a monogon would lie in some free homotopy class K_Φ , not trivial because R is straight. If x lies over x' then the points $x\Phi^i$ would not lie on a straight line. The absence of proper monogons means that the geodesics in R' have no multiple points.

Theorem (7) brings a result of E. Hopf [6] to mind, namely that a two-dimensional torus T' with a Riemannian metric is euclidean, if its universal covering plane T is straight. In that case (7) is therefore trivial. However, when the condition that the metric be Riemannian is omitted, then T' possesses a great number of essentially different metrizations for which T is straight. The geodesics in T need not satisfy Desargues' Theorem, but they always satisfy the parallel axiom.

4. A characterization of Minkowskian geometry.

The translations of the euclidean space are obvious examples für (6). When there are enough motions satisfying (6) these motions are necessarily ordinary translations:

(8) Theorem. *If a straight space possesses a transitive group of motions such that for each motion Φ in Γ a point exists whose displacement under Φ is maximal, then R is Minkowskian and Γ the group of translations of R .*

We deduce from (6) that $xx\Phi$ is constant for each Φ in Γ . Hence no motion $\Phi \neq E$ in Γ has fixed points and Γ is simply transitive on R , see [7, p. 220]. The motion in Γ that takes a into b may therefore be denoted by $(a \rightarrow b)$. Because of (6) the line $g(a, b)$ through a and b , $a \neq b$, is an axis of $(a \rightarrow b)$. The proof of (8) consists of several steps the first of which is:

(a) R satisfies the parallel axiom, for the terminology see [7].

To see this let $x(t)$ be any geodesic and y a point not on $x(t)$. Since $x(t)$ is an axis of $\Phi = (x(0) \rightarrow x(1))$ it suffices to show that $g(y, x(t))$ tends for $t \rightarrow \infty$ or $t \rightarrow -\infty$ to the axis g_y of Φ through y . For the statement that the line g_y is an axis of the same motion Φ as g_x , is symmetric and transitive, hence the statement that g_y is parallel to g_x also has these properties.

Let $y(t)$ represent the axis of Φ through y with $y(0) = y$ and $y\Phi = y(1)$. The limit sphere $A(y, r)$ through y to r (see [1, p. 240] or

[7, p. 98]), where r is the ray $t \geq 0$ of $x(t)$, intersects $x(t)$ in a point $x(t_0)$ and $x(t_0)\Phi = x(t_0 + 1)$. Moreover $\Lambda(y\Phi, r) = \Lambda(y, r)\Phi = \Lambda(x(t_0 + 1), r)$. The asymptote a to r through y intersects $\Lambda(y\Phi, r)$ in the unique foot f of y on $\Lambda(y\Phi, r)$. But, see [1, p. 242],

$$1 = x(t_0)x(t_0 + 1) = yf \leq yy\Phi = 1,$$

hence $y\Phi$ is also a foot of y on $\Lambda(y\Phi, r)$, so that $y\Phi = f$ and $a = g_v$, which proves (a).

We show next

(b) If $y(t)$, $t \geq 0$ represents a ray s and g is a straight line through $y = y(0)$ not containing s then $y(t)g \rightarrow \infty$ for $t \rightarrow \infty$.

For an indirect proof assume the existence of a sequence t_n with $x(t_n)g < M$. If f_n is a foot of $x(t_n)$ on g , then $f_n \neq y$ for large n , and $q_n = x(t_n)(f_n \rightarrow y)$ has y as foot on g . Because $q_n y = x(t_n)f_n < M$ there is a subsequence $\{v\}$ of $\{n\}$ for which q_v tends to a point q . ($qy > 0$ because of [1, Theorem (11.14)]).

The line $g(x(t_v), q_v)$ is an axis of $(f_v \rightarrow y)$, hence parallel to g . It tends therefore to the parallel g' to g through q . On the other hand, the line $g(q_v, x(t_v))$ tends also to the parallel through q to the line h carrying s (see the definition of co-ray in [1]). Since parallelism is symmetric, it would follow that g and h are parallel to g' , which is impossible because g and h intersect.

(c) x_1g_2 and x_2g_1 are bounded for $x_i \in g_i$ if and only if g_1 and g_2 are parallel.

If g_1 and g_2 are parallel then the fact that they are axes of the same motion in Φ shows that x_1g_2 and x_2g_1 are bounded. The converse follows from (b), for a proof see [2, p. 278].

(d) Γ is abelian.

If Φ and Ψ are two non-trivial motions in Γ , select an arbitrary point z . If the axes of Φ and Ψ through z coincide, it is easily seen that Φ and Ψ commute (this case can also be deduced by a limit process from the general case). We assume therefore that z , $p = z\Phi$ and $q = z\Psi$ are not collinear. Put $g(z, p) = g$, $g(z, q) = h$ and $h' = h\Phi$. Then $y' = y\Phi \in h'$ for $y \in h$. The relation $yy' = zp$ shows that $y'h$ and yh' are bounded, by (c) the lines h and h' are parallel. Therefore h' is an axis of Ψ , so that $p\Psi$ is a point u of h' with $zq = pu$. On the other hand $zq = z\Phi q\Phi = pq\Phi$, hence $q\Phi = u$. Therefore $\Phi = (q \rightarrow u)$, $\Psi = (p \rightarrow u)$ and

$$\Phi\Psi = (z \rightarrow p)(p \rightarrow u) = (z \rightarrow u) = (z \rightarrow q)(q \rightarrow u) = \Psi\Phi.$$

It now follows readily from a wellknown result of Pontrjagin, see [4, p. 170], that the space is a finite dimensional Minkowski space. A simple proof which does not use the theory of topological groups is found in [7, pp. 229—231].

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