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Autor: Whitehead, George W.
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On mappings into group-like spaces

by George W. Whitehead

Dedicated to H. Hopf on his 60^th birthday

1. Introduction. Let G be a space with continuous multiplication and inversion, which satisfies the group axioms up to homotopy. Then the homotopy classes of maps of any space X into G form a group $\pi(X; G)$. We shall first show that, under reasonable hypotheses on X, G, the group $\pi(X; G)$ is nilpotent; an upper bound for the class of nilpotency is $c - 1$, where c is the Lusternik-Schnirelmann category of X.

In particular, if X is the product of k spheres, $\pi(X; G)$ has class $\leqq k$; an explicit central chain for $\pi(X; G)$ can be constructed in this case; the successive factor groups are direct products of homotopy groups of G. In particular, $\pi(S^p \times S^q; G)$ is a central extension of $\pi_{p+q}(G)$ by $\pi_p(G) \times \pi_q(G)$, which is semi-split in the sense that each of the groups $\pi_p(G)$, $\pi_q(G)$ can be lifted to a subgroup of $\pi(S^p \times S^q; G)$. The group extension is then completely described by the commutators of elements of $\pi_p(G)$ with elements of $\pi_q(G)$. These provide a bilinear map $(\alpha, \beta) \to \langle \alpha, \beta \rangle$ of $\pi_p(G) \times \pi_q(G)$ into $\pi_{p+q}(G)$. This map has been used by Samelson [3].

In the group $\pi(S^p \times S^q \times S^r; G)$ the iterated commutators provide a trilinear map $(\alpha, \beta, \gamma) \to \langle \alpha, \langle \beta, \gamma \rangle \rangle$ of the product

$$\pi_p(G) \times \pi_q(G) \times \pi_r(G)$$

into $\pi_{p+q+r}(G)$. Now if Γ is a group, the iterated commutators satisfy a "Jacobi congruence" modulo the fourth member of the descending central series of Γ; since $\pi(S^p \times S^q \times S^r; G)$ is nilpotent of class $\leqq 3$, this congruence reduces to an identity. From this fact we deduce a Jacobi identity for the operation $\langle \alpha, \beta \rangle$.

Suppose that G is the space of loops in a space X; then

$$\pi_n(G) \approx \pi_{n+1}(X) .$$

By this isomorphism the commutator mapping is transformed, except for sign, into the Whitehead product. Thus the above results provide another proof that the Whitehead products satisfy a Jacoby identity[1]).

2. **Nilpotency of** $\pi(X;G)$. Let X be a separable metric locally contractible space. A subset A of X is said to be *categorical* if and only if there is an open set $U \supset A$ such that U is contractible in X. A covering of X by sets A_α is said to be categorical if and only if each of the sets A_α is categorical. The *category* cat X of X is the least of the cardinal numbers of the categorical coverings of X.

Let G be a space, $e \in G$, and let $\mu: \bigl(G\times G, (e,e)\bigr) \to (G,e)$, $i: (G,e) \to (G,e)$ be maps. We say that G is an H^*-space with respect to e, μ, i if and only if the following conditions are satisfied:

2.1) The maps $(x,y,z) \to \mu\bigl(x,\mu(y,z)\bigr)$ and $(x,y,z) \to \mu\bigl(\mu(x,y),z\bigr)$ of $G\times G\times G$ into G are homotopic rel. (e,e,e);

2.2) The maps $x \to \mu(x,e)$ and $x \to \mu(e,x)$ of G into G are homotopic rel. e to the identity map;

2.3) The maps $x \to \mu\bigl(x,i(x)\bigr)$ and $x \to \mu\bigl(i(x),x\bigr)$ of G into G are homotopic rel. e to the constant map.

Hereafter we abbreviate $\mu(x,y)$ to $x\cdot y$ and $i(x)$ to x^{-1}.

Let $\Phi: G\times G \to G$ be the commutator map, defined by

$$\Phi(x,y) = (xy)(x^{-1}y^{-1}) ,$$

and let $G \vee G$ be the subset $G\times e \cup e\times G$ of $G\times G$. Then from 2.1 to 2.3 we conclude easily:

2.4) The map $\Phi \mid G\vee G$ is homotopic rel. (e,e) to the constant map.

Let X be a space, $A \subset X$, and let G be an H^*-space. If $f, g: (X,A) \to (G,e)$ are maps, we define $f\cdot g: (X,A) \to (G,e)$ by

$$(f\cdot g)(x) = f(x)\cdot g(x) .$$

This operation preserves the relation of homotopy, and therefore induces an operation in the set $\pi(X,A;G)$ of homotopy classes of maps of (X,A) into (G,e). It is easily verified that $\pi(X,A;G)$ is a group under this operation. Moreover, if $h: (Y,B) \to (X,A)$ is a map, then composition with h induces a homomorphism

$$\boldsymbol{h}: \pi(X,A;G) \to \pi(Y,B;G) ;$$

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[1]) Proofs of the Jacobi identity for the Whitehead product have been obtained recently by Hilton and by Massey-Uehara (to appear in Ann. of Math.). The author believes the present proof to be more elementary. The author is informed by Uehara that proofs have also been found by Nakaoka and by Toda.

the homomorphism induced by a composite map $k \circ h$ is the composition $\boldsymbol{h} \circ \boldsymbol{k}$ in the opposite order.

We further have:

2.5) If S is an n-sphere, $x \in S$, then $\pi(S, x; G) = \pi_n(G)$.

2.6) Let (X', a) be the pair obtained from (X, A) by collapsing A to a point, $j: (X, A) \to (X', a)$ the identification map. Then $\boldsymbol{j}: \pi(X', a; G) \approx \pi(X, A; G)$.

The proof of 2.5 is similar to the proof in the case where G is a topological group [1]; 2.6 follows from standard properties of identification spaces.

Lemma 2.7. *Let $(X; A_1, \ldots, A_n)$ be an*

$$(n+1) - ad \ , \quad A = A_1 \cup \ldots \cup A_n \ .$$

Suppose that there are retractions ϱ_i: $X \to A_i$ $(i = 1, \ldots, n)$ such that $\varrho_i(A_j) \subset A_j$ for all i, j. Suppose further that either $(X; A_1, \ldots, A_n)$ is triangulable or X is separable metric and G is an ANR[2]). Then, if $j: X \subset (X, A)$, the homomorphism $\boldsymbol{j}: \pi(X, A; G) \to \pi(X; G)$ is an isomorphism into.

Proof. We must show that, if $f_0, f_1: X \to G$ are homotopic maps with $f_0(A) = f_1(A) = e$, then f_0 and f_1 are homotopic rel. A. Let $B_r = A_1 \cup \ldots \cup A_r$; then it suffices to show that $f_0 \simeq f_1$ rel. B_r implies $f_0 \simeq f_1$ rel. B_{r+1}. Let $F: X \times I \to G$ be a homotopy of f_0 to f_1 rel. B_r, and define $F_1: X \times I \to G$ by

$$F_1(x, t) = F(x, t) \cdot F[\varrho_{r+1}(x), t]^{-1} \ .$$

Then $x \in B_r$ implies $\varrho_{r+1}(x) \in B_r$, and therefore $F_1(x, t) = e \cdot e^{-1} = e$; and $x \in A_{r+1}$ implies $\varrho_{r+1}(x) = x$. Thus $x \in B_{r+1}$ implies that $F_1(x, t) = F(x, t) \cdot F(x, t)^{-1}$. Furthermore, if $t = 0$ or 1, $F_1(x, t) = f_t(x) \cdot e$.

It follows from 2.3 that $F_1 \mid B_{r+1} \times I$ is homotopic rel $B_{r+1} \times \dot{I}$ to the constant map. It follows from 2.2 that $F_1 \mid X \times \dot{I}$ is homotopic rel. $A \times \dot{I}$ to the map $(x, t) \to f_t(x)$. Let $C = X \times \dot{I} \cup B_{r+1} \times I$; then $F_1 \mid C$ is homotopic to the map $F_2: C \to G$ such that

$$F_2(x, t) = f_t(x) \qquad (x \in X, t \in \dot{I}) \ ;$$
$$F_2(x, t) = e \qquad (x \in B_{r+1}) \ .$$

[2]) In the weak sense.

By the homotopy extension theorem, F_2 has an extension $F' : X \times I \to G$; F' is a homotopy of f_0 to f_1 rel. B_{r+1}.

Let Γ be a group. We recall that the *descending central series* of Γ is the sequence of subgroups defined inductively by

$$Z_0 = \Gamma ; \quad Z_i = [\Gamma, Z_{i-1}] \quad \text{for} \quad i > 0 ,$$

and that Γ is *nilpotent* if and only if there is a c such that $Z_c = \{1\}$; the least such c is called the *class* of Γ. A *central chain* of length k of Γ is a sequence

$$\Gamma = \Gamma_0 \supset \cdots \supset \Gamma_k = \{1\}$$

of subgroups such that $[\Gamma, \Gamma_i] \subset \Gamma_{i+1}$ $(i = 0, \ldots, k-1)$. Γ has a central chain of length k if and only if Γ is nilpotent of class $\leqq k$.

We further recall [6, pp. 60—63]:

2.8) $[a, bc] \equiv [a, b] \cdot [a, c]$ $(\text{mod}\, Z_2)$,

2.9) $[a, [b, c]] \cdot [b, [c, a]] \cdot [c, [a, b]] \equiv 1$ $(\text{mod}\, Z_3)$.

Theorem 2.10. *Let X be a separable metric locally contractible space, G a 0-connected H^*-space. Suppose that X has finite category c, and that either* a) *X is triangulable or* b) *G is an ANR. Then the group $\pi(X; G)$ is nilpotent of class $\leqq c - 1$.*

To prove Theorem 2.10, we recall from [2, p. 336] that X has a closed categorical covering $A_1, \ldots, A_c$. If X is triangulable we may assume that the sets A_i are all subcomplexes of a certain simplicial decomposition of X. Theorem 2.10 will then follow from

Lemma 2.11. *Let Γ'_i be the set of all homotopy classes of maps $f : X \to G$ such that $f \mid A_1 \cup \cdots \cup A_{i+1}$ is inessential* $(i = 0, \ldots, c-1)$ *Then the Γ'_i form a central chain for $\pi(X; G)$.*

Proof. Clearly the Γ'_i form a decreasing sequence of subgroups of $\Gamma = \pi(X; G)$; $\Gamma'_0 = \Gamma$ since A_1 is contractible in X, and $\Gamma'_{c-1} = \{1\}$ since $A_1 \cup \cdots \cup A_c = X$. It remains to prove that $[\Gamma, \Gamma'_{i-1}] \subset \Gamma'_i$.

Let $f \in \alpha \in \Gamma$, $g \in \beta \in \Gamma'_{i-1}$; then $f \mid A_{i+1}$ is inessential since A_{i+1} is contractible in X, and $g \mid A_1 \cup \cdots \cup A_i$ is inessential by hypothesis. Hence we may assume $f(A_{i+1}) = g(A_1 \cup \cdots \cup A_i) = e$. Define $h : X \to G$ by $h = \Phi \circ (f \times g)$, i. e.,

$$h(x) = \Phi\big(f(x), g(x)\big) ;$$

then $h \in [\alpha, \beta]$. Now if $x \in A_1 \cup \cdots \cup A_{i+1}$, then $\big(f(x), g(x)\big) \in G \vee G$; it follows from 2.4 that $h \mid A_1 \cup \cdots \cup A_{i+1}$ is inessential, i. e., $[\alpha, \beta] \in \Gamma'_i$.

Corollary 2.12. *If X has finite dimension n, then $\pi(X;G)$ is nilpotent of class $\leqq n$.*

For cat $X \leqq 1 + \dim X$ [2, (5.4)].

Corollary 2.13. *If X is the product of k spheres, then $\pi(X;G)$ is nilpotent of class $\leqq k$.*

For cat $X = k + 1$ [2, p. 350].

The following variant of Lemma 2.11 will be more appropriate in what follows [3]).

Lemma 2.14. *Let X be a CW-complex [5] and let*

$$A_0 \subset \cdots \subset A_k = X$$

be subcomplexes such that, if E is any cell of A_i, then $\dot{E} \subset A_{i-1}$. Let Γ_i be the set of all homotopy classes of maps $f : X \to G$ such that $f \mid A_i$ is inessential. Then $\Gamma_0, \ldots, \Gamma_k$ is a central chain for $\pi(X;G)$.

Proof. Let $f \in \alpha \in \Gamma = \pi(X;G)$, $g \in \beta \in \Gamma_{i-1}$. For each cell E_λ of A_i, let x_λ be an interior point of E_λ. Then we can find closed cells $F_\lambda \subset \operatorname{Int} E_\lambda$ such that $x_\lambda \in \operatorname{Int} F_\lambda$ and $\dot{E}_\lambda$ is a deformation retract of $\overline{E_\lambda - F_\lambda}$; it follows that A_{i-1} is a deformation retract of

$$Q = \bigcup_\lambda \overline{E_\lambda - F_\lambda} \cup A_{i-1}$$

and the set $P_0 = \{x_\lambda\}$ is a deformation retract of $P = \bigcup_\lambda F_\lambda$. Since G is 0-connected, $f \mid P_0$, and therefore also $f \mid P$, is inessential. Since $g \mid A_{i-1}$ is inessential, it follows that $g \mid Q$ is inessential. Hence we may assume $f(P) = g(Q) = e$. The remainder of the proof follows the pattern of Lemma 2.11.

3. The case $X = S^{n_1} \times \cdots \times S^{n_k}$. We now examine in more detail the case $X = S^{n_1} \times \cdots \times S^{n_k}$. We first make some conventions about orientation. Let I^p be the unit cube in Euclidean p-space; we orient I^p by choosing the generator ω_p of $H_p(I^p, \dot{I}^p)$ represented by the identity map of I^p (we are using cubical singular homology theory as in [4]). Let x_p be a fixed point of S^p and choose a fixed map $\psi_p : (I^p, \dot{I}^p) \to (S^p, x_p)$

[3]) The hypotheses of Lemma 2.14 imply that cat $X \leqq k + 1$, so that $\pi(X;G)$ is nilpotent of class $\leqq k$. The conclusion follows formally from Lemma 2.11 and the following statement, which is easily proved by induction on k: *there are closed categorical subsets $B_0, \ldots, B_k$ of X such A_i is a deformation retract of $B_0 \cup \ldots B \cup_i$ $(i = 0, \ldots, k)$.* However, it seemed simpler to proved Lemma 2.14 directly. Actually, only Lemma 2.14 is used in what follows; we state Theorem 2.10 because it seems to give the most natural upper bound for the class of nilpotency.

which maps the interior of I^p topologically onto $S^p - x_p$; we then orient S^p by requiring that ψ_p have degree 1. Next we make the natural identification of $I^{n_1} \times \cdots \times I^{n_k}$ with $I^{n_1 + \cdots + n_k}$ and then orient $S^{n_1} \times \cdots \times S^{n_k}$ by requiring that $\psi = \psi_{n_1} \times \cdots \times \psi_{n_k}$ have degree 1.

Let $X = S^{n_1} \times \cdots \times S^{n_k}$. We take $S_i = S^{n_i}$ with the CW-decomposition consisting of the 0-cell $e_i = x_{n_i}$ and the n_i-cell $S_i - e_i$. The product of these CW-decompositions gives a CW-decomposition of X. Let A_i be the set of all $x \in X$ such that $x_j \neq e_j$ for at most i values of j; thus

$$A_0 = e_1 \times \cdots \times e_k \ ,$$

$$A_1 = S_1 \times e_2 \times \cdots \times e_k \cup \cdots \cup e_1 \times \cdots \times e_{k-1} \times S_k \ ,$$

$$\cdots$$

$$A_k = S_1 \times \cdots \times S_k = X \ .$$

The A_i clearly satisfy the hypotheses of Lemma 2.14.

For each subset α of $\{1, \ldots, k\}$, let $S_\alpha = \{x \in X \mid x_i = e_i$ for $i \notin \alpha\}$, $|\alpha| =$ the cardinal of α, $n(\alpha) = \sum_{i \in \alpha} n_i \cdot S_\alpha$ is homeomorphic with $\prod_{i \in \alpha} S_i$, and we orient S_α by requiring that the natural homeomorphism (the one preserving the order of the coordinates) be orientation-preserving. Finally, let p_α: $(S_\alpha, S_\alpha \cap A_{|\alpha|-1}) \to (S^{n(\alpha)}, e_{n(\alpha)})$ be an orientation-preserving map which is topological on $S_\alpha - A_{|\alpha|-1}$.

Let q_α: $X \to S_\alpha$ be the natural retraction: $q_\alpha(x)$ is the point y such that $y_i = x_i$ if $i \in \alpha$ and $y_i = e_i$ if $i \notin \alpha$. Then $\beta \neq \alpha$ implies $q_\beta(S_\alpha) \subset S_\alpha$. We may therefore apply Lemma 2.7 to obtain:

Lemma 3.1. *Let j: $X \subset (X, A_i)$. Then $\boldsymbol{j} : \pi(X, A_i ; G) \approx \Gamma_i$.*

Theorem 3.2. *The group Γ_{i-1}/Γ_i is isomorphic with the direct product*

$$\prod_\alpha \pi_{n(\alpha)}(G) \ ,$$

where α ranges over all i-element subsets of $\{1, \ldots, k\}$.

Proof. Let $j_\alpha : (S_\alpha, S_\alpha \cap A_{i-1}) \subset (X, A_{i-1})$. Then

$$\boldsymbol{j} : \pi(X, A_{i-1} ; G) \approx \Gamma_{i-1}, \boldsymbol{j}_\alpha : \pi(X, A_{i-1} ; G) \to \pi(S_\alpha, S_\alpha \cap A_{i-1} ; G) \ ,$$

and $\boldsymbol{p}_\alpha : \pi_{n(\alpha)}(G) \approx \pi(S_\alpha, S_\alpha \cap A_{i-1} ;\ G)$; hence we may define η_α: $\Gamma_{i-1} \to \pi_{n(\alpha)}(G)$ by $\eta_\alpha = \boldsymbol{p}_\alpha^{-1} \, \boldsymbol{j}_\alpha \, \boldsymbol{j}^{-1}$. The homomorphisms η_α then define a homomorphism $\eta : \Gamma_{i-1} \to \Delta = \prod_\alpha \pi_{n(\alpha)}(G)$. Clearly $\Gamma_i \subset$ Kernel η. Conversely, if $f \in \gamma \in$ Kernel η, then, for each α, $f \mid S_\alpha$ is inessential rel. $S_\alpha \cap A_{i-1}$; hence $f \mid A_i$ is inessential. Therefore η induces an isomorphism η of Γ_{i-1}/Γ_i into Δ.

To show that η is onto, define i_α: $X \to S^{n(\alpha)}$ by $i_\alpha = p_\alpha \circ q_\alpha$. Since q_α can be factored through (X, A_{i-1}), it follows that $\boldsymbol{i}_\alpha : \pi_{n(\alpha)}(G) \to \Gamma_{i-1}$. Now

$$\eta_\alpha \circ \boldsymbol{i}_\alpha = \boldsymbol{p}_\alpha^{-1} \circ \boldsymbol{j}_\alpha \circ \boldsymbol{j}^{-1} \circ \boldsymbol{q}_\alpha \circ \boldsymbol{p}_\alpha \;;$$

since q_α is a retraction, $\boldsymbol{j}_\alpha \circ \boldsymbol{j}^{-1} \circ \boldsymbol{q}_\alpha$ is the identity. Hence $\eta_\alpha \circ \boldsymbol{i}_\alpha$ is the identity. If $\alpha \neq \beta$, then $q_\alpha(S_\beta) \subset S_\alpha \cap S_\beta \subset A_{i-1}$ and therefore $\boldsymbol{j}_\beta \circ \boldsymbol{j}^{-1} \circ \boldsymbol{q}_\alpha = 0$; thus

$$\eta_\beta \circ \boldsymbol{i}_\alpha = \boldsymbol{p}_\beta^{-1} \circ \boldsymbol{j}_\beta \circ \boldsymbol{j}^{-1} \circ \boldsymbol{q}_\alpha \circ \boldsymbol{p}_\alpha = 0 \;.$$

It follows that η is onto, and that the $\boldsymbol{i}_\alpha$ are isomorphisms into.

4. Commutator relations. In this section we write $\pi_{n_1,\ldots,n_k}$ for $\pi(S^{n_1} \times \cdots \times S^{n_k}; G)$. Let $\boldsymbol{i}_1 : \pi_p \to \pi_{p,q}$, $\boldsymbol{i}_2 : \pi_q \to \pi_{p,q}$, $\boldsymbol{i}_{1,2} : \pi_{p+q} \to \pi_{p,q}$ be the isomorphisms into of § 3. Then $\boldsymbol{i}_{1,2}$ maps π_{p+q} isomorphically onto Γ_1, and if $\alpha \in \pi_p$, $\beta \in \pi_q$, we have $[\boldsymbol{i}_1(\alpha), \boldsymbol{i}_2(\beta)] \in \Gamma_1$. Hence we may define

$$\langle \alpha, \beta \rangle = \boldsymbol{i}_{1,2}^{-1}[\boldsymbol{i}_1(\alpha), \boldsymbol{i}_2(\beta)] \in \pi_{p+q} \;.$$

Lemma 4.1. (a) $\langle \alpha, \beta \rangle = (-1)^{pq-1}\langle \beta, \alpha \rangle$; (b) $\langle \alpha, \beta \rangle$ *depends bilinearly on* α, β.

Proof. Let t: $S^p \times S^q \to S^q \times S^p$ be the map which interchanges the coordinates. In virtue of our conventions about orientation, t has degree $(-1)^{pq}$. Furthermore, t induces a map t': $S^{p-q} \to S^{p+q}$ of the same degree. We verify easily that $i_1 \circ t = t_2$, $i_2 \circ t = i_1$, and $i_{1,2} \circ t = t' \circ i_{1,2}$. Furthermore $\boldsymbol{t}'(u) = (-1)^{pq} u$ for $u \in \pi_{p+q}$. Hence

$$\begin{aligned} \langle \beta, \alpha \rangle &= \boldsymbol{i}_{1,2}^{-1}[\boldsymbol{i}_1(\beta), \boldsymbol{i}_2(\alpha)] = \boldsymbol{t}'^{-1}\, \boldsymbol{i}_{1,2}^{-1} \boldsymbol{t}\, [\boldsymbol{i}_1(\beta), \boldsymbol{i}_2(\alpha)] \\ &= (-1)^{pq}\, \boldsymbol{i}_{1,2}^{-1}[\boldsymbol{t}\boldsymbol{i}_1(\beta), \boldsymbol{t}\boldsymbol{i}_2(\alpha)] \\ &= (-1)^{pq}\, \boldsymbol{i}_{1,2}^{-1}[\boldsymbol{i}_2(\beta), \boldsymbol{i}_1(\alpha)] \\ &= (-1)^{pq-1}\, \boldsymbol{i}_{1,2}^{-1}[\boldsymbol{i}_1(\alpha), \boldsymbol{i}_2(\beta)] \\ &= (-1)^{pq-1}\, \langle \alpha, \beta \rangle \;. \end{aligned}$$

Now if $\alpha \in \pi_p$, $\beta, \gamma \in \pi_q$, we have, by 2.8,

$$\begin{aligned} [\boldsymbol{i}_1(\alpha), \boldsymbol{i}_2(\beta \cdot \gamma)] &= [\boldsymbol{i}_1(\alpha), \boldsymbol{i}_2(\beta)\, \boldsymbol{i}_2(\gamma)] \\ &\equiv [\boldsymbol{i}_1(\alpha), \boldsymbol{i}_2(\beta)] \cdot [\boldsymbol{i}_1(\alpha), \boldsymbol{i}_2(\gamma)] \qquad (\text{mod } \Gamma_2) \;. \end{aligned}$$

But $\Gamma_2 = \{1\}$ and $\boldsymbol{i}_{1,2}$ is a homomorphism. Hence right linearity holds. Left linearity follows from right linearity and part (a).

Consider now the group $\pi_{p,q,r}$ and the isomorphisms into $\boldsymbol{i}_1 : \pi_p \to \pi_{p,q,r}$, $\boldsymbol{i}_2 : \pi_q \to \pi_{p,q,r}$, $\boldsymbol{i}_3 : \pi_r \to \pi_{p,q,r}$, and $\boldsymbol{i}_{1,2,3} : \pi_{p+q+r} \to \pi_{p,q,r}$. The latter is an isomorphism onto Γ_2. If $\alpha \in \pi_p$, $\beta \in \pi_q$, $\gamma \in \pi_r$, then $[\boldsymbol{i}_1(\alpha), [\boldsymbol{i}_2(\beta), \boldsymbol{i}_3(\gamma)]] \in \Gamma_2$, and we have:

Lemma 4.2. $\langle\alpha, \langle\beta, \gamma\rangle\rangle = \boldsymbol{i}_{1,2,3}^{-1}[\boldsymbol{i}_1(\alpha), [\boldsymbol{i}_2(\beta), \boldsymbol{i}_3(\gamma)]]$.

Proof. We have, by definition,

$$\langle\alpha, \langle\beta, \gamma\rangle\rangle = \boldsymbol{i}_{1,2}^{-1}[\boldsymbol{i}_1(\alpha), \boldsymbol{i}_2\boldsymbol{i}_{1,2}^{-1}[\boldsymbol{i}_1(\beta), \boldsymbol{i}_2(\gamma)]] .$$

Let f: $S^p \times S^q \times S^r \to S^p \times S^{q+r}$ be the map given by

$$f(x, y, z) = (x, p_{1,2}(y, z)) .$$

The maps $p_{1,2} \circ f$ and $p_{1,2,3}$ of $(S^p \times S^q \times S^r, A_2)$ into (S^{p+q+r}, x_{p+q+r}) both preserve orientation. Hence

$$\boldsymbol{f} \circ \boldsymbol{i}_{1,2} = \boldsymbol{f} \circ \boldsymbol{p}_{1,2} = \boldsymbol{p}_{1,2,3} = \boldsymbol{i}_{1,2,3} .$$

On the other hand, $i_2 \circ f = i_{2,3}$ and $i_1 \circ f = i_1$. Let h be the projection of $S^p \times S^q \times S^r$ on $S^q \times S^r$. Then $i_{1,2} \circ h = i_{2,3}$, $i_1 \circ h = i_2$, and $i_2 \circ h = i_3$. Thus

$$\begin{aligned}
\boldsymbol{i}_{1,2,3}\langle\alpha, \langle\beta, \gamma\rangle\rangle &= \boldsymbol{f}\boldsymbol{i}_{1,2}\langle\alpha, \langle\beta, \gamma\rangle\rangle \\
&= \boldsymbol{f}[\boldsymbol{i}_1(\alpha), \boldsymbol{i}_2\boldsymbol{i}_{1,2}^{-1}[\boldsymbol{i}_1(\beta), \boldsymbol{i}_2(\gamma)]] \\
&= [\boldsymbol{f}\boldsymbol{i}_1(\alpha), \boldsymbol{f}\boldsymbol{i}_2\boldsymbol{i}_{1,2}^{-1}[\boldsymbol{i}_1(\beta), \boldsymbol{i}_2(\gamma)]] \\
&= [\boldsymbol{i}_1(\alpha), \boldsymbol{i}_{2,3}\,\boldsymbol{i}_{1,2}^{-1}[\boldsymbol{i}_1(\beta), \boldsymbol{i}_2(\gamma)]] \\
&= [\boldsymbol{i}_1(\alpha), \boldsymbol{h}[\boldsymbol{i}_1(\beta), \boldsymbol{i}_2(\gamma)]] \\
&= [\boldsymbol{i}_1(\alpha), [\boldsymbol{h}\boldsymbol{i}_1(\beta), \boldsymbol{h}\boldsymbol{i}_2(\gamma)]] \\
&= [\boldsymbol{i}_1(\alpha), [\boldsymbol{i}_2(\beta), \boldsymbol{i}_3(\gamma)]] .
\end{aligned}$$

Lemma 4.3. *If* $\alpha \in \pi_p$, $\beta \in \pi_q$, $\gamma \in \pi_r$, *then*

$$\boldsymbol{i}_{1,2,3}^{-1}[\boldsymbol{i}_2(\beta), [\boldsymbol{i}_3(\gamma), \boldsymbol{i}_1(\alpha)]] = (-1)^{p(q+r)} \langle\beta, \langle\gamma, \alpha\rangle\rangle ;$$
$$\boldsymbol{i}_{1,2,3}^{-1}[\boldsymbol{i}_3(\gamma), [\boldsymbol{i}_1(\alpha), \boldsymbol{i}_2(\beta)]] = (-1)^{r(p+q)} \langle\gamma, \langle\alpha, \beta\rangle\rangle .$$

Proof. Let g: $S^p \times S^q \times S^r \to S^q \times S^r \times S^p$ be the map given by $g(x, y, z) = (y, z, x)$. Then g induces g': $S^{p+q+r} \to S^{p+q+r}$; g and g' have the same degree $(-1)^{p(q+r)}$, and $i_{1,2,3} \circ g = g' \circ i_{1,2,3}$, $i_1 \circ g = i_2$, $i_2 \circ g = i_3$, $i_3 \circ g = i_1$. Hence

$$\begin{aligned}
\boldsymbol{i}_{1,2,3}\{(-1)^{p(q+r)} \langle\beta, \langle\gamma, \alpha\rangle\rangle\} &= \boldsymbol{g}\boldsymbol{i}_{1,2,3}\langle\beta, \langle\gamma, \alpha\rangle\rangle \\
&= \boldsymbol{g}[\boldsymbol{i}_1(\beta), [\boldsymbol{i}_2(\gamma), \boldsymbol{i}_3(\alpha)]] \\
&= [\boldsymbol{g}\boldsymbol{i}_1(\beta), [\boldsymbol{g}\boldsymbol{i}_2(\gamma), \boldsymbol{g}\boldsymbol{i}_3(\alpha)]] \\
&= [\boldsymbol{i}_2(\beta), [\boldsymbol{i}_3(\gamma), \boldsymbol{i}_1(\alpha)]] .
\end{aligned}$$

The other statement is proved similarly.

Theorem 4.4. *If* $\alpha \in \pi_p$, $\beta \in \pi_q$, $\gamma \in \pi_r$, *then*

$$\langle\alpha, \langle\beta, \gamma\rangle\rangle + (-1)^{p(q+r)} \langle\beta, \langle\gamma, \alpha\rangle\rangle + (-1)^{r(p+q)} \langle\gamma, \langle\alpha, \beta\rangle\rangle = 0 .$$

Proof. Let $\alpha' = \boldsymbol{i}_1(\alpha)$, $\beta' = \boldsymbol{i}_2(\beta)$, $\gamma' = \boldsymbol{i}_3(\gamma)$. Since $\Gamma_3 = \{1\}$, we have, by 2.9,

$$1 = [\alpha', [\beta', \gamma']]\cdot[\beta', [\gamma', \alpha']]\cdot[\gamma', [\alpha', \beta']]$$

and therefore

$$0 = i_{1,2,3}^{-1}[\alpha', [\beta', \gamma']]\cdot[\beta', [\gamma', \alpha']]\cdot[\gamma', [\alpha', \beta']]$$
$$= \langle\alpha, \langle\beta, \gamma\rangle\rangle + (-1)^{p(q+r)} \langle\beta, \langle\gamma, \alpha\rangle\rangle + (-1)^{r(p+q)} \langle\gamma, \langle\alpha, \beta\rangle\rangle .$$

Now suppose that X is a space, $x \in X$, and that G is the path-component of the constant map in the space of loops in X based at x. Then we have a natural isomorphism $T : \pi_{n+1}(X) \approx \pi_n(G)$ for each $n > 0$. Furthermore, if $\alpha \in \pi_{p+1}(X)$, $\beta \in \pi_{q+1}(X)$, and if $[\alpha, \beta] \in \pi_{p+q+1}(X)$ is their Whitehead product, we have [3]:

$$T([\alpha, \beta]) = (-1)^p \langle T(\alpha), T(\beta)\rangle . \qquad (4.5)$$

From 4.5 and Theorem 4.4 we then obtain:

Theorem 4.6. *If* $\alpha \in \pi_{p+1}(X)$, $\beta \in \pi_{q+1}(X)$, $\gamma \in \pi_{r+1}(X)$, *then*

$$(-1)^{r(p+1)} [\alpha, [\beta, \gamma]] + (-1)^{p(q+1)} [\beta, [\gamma, \alpha]] + (-1)^{q(r+1)} [\gamma, [\alpha, \beta]] = 0.$$

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