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# On mappings into group-like spaces

by GEORGE W. WHITEHEAD

*Dedicated to H. Hopf on his 60<sup>th</sup> birthday*

**1. Introduction.** Let  $G$  be a space with continuous multiplication and inversion, which satisfies the group axioms up to homotopy. Then the homotopy classes of maps of any space  $X$  into  $G$  form a group  $\pi(X; G)$ . We shall first show that, under reasonable hypotheses on  $X$ ,  $G$ , the group  $\pi(X; G)$  is nilpotent; an upper bound for the class of nilpotency is  $c - 1$ , where  $c$  is the Lusternik-Schnirelmann category of  $X$ .

In particular, if  $X$  is the product of  $k$  spheres,  $\pi(X; G)$  has class  $\leq k$ ; an explicit central chain for  $\pi(X; G)$  can be constructed in this case; the successive factor groups are direct products of homotopy groups of  $G$ . In particular,  $\pi(S^p \times S^q; G)$  is a central extension of  $\pi_{p+q}(G)$  by  $\pi_p(G) \times \pi_q(G)$ , which is semi-split in the sense that each of the groups  $\pi_p(G)$ ,  $\pi_q(G)$  can be lifted to a subgroup of  $\pi(S^p \times S^q; G)$ . The group extension is then completely described by the commutators of elements of  $\pi_p(G)$  with elements of  $\pi_q(G)$ . These provide a bilinear map  $(\alpha, \beta) \rightarrow \langle \alpha, \beta \rangle$  of  $\pi_p(G) \times \pi_q(G)$  into  $\pi_{p+q}(G)$ . This map has been used by Samelson [3].

In the group  $\pi(S^p \times S^q \times S^r; G)$  the iterated commutators provide a trilinear map  $(\alpha, \beta, \gamma) \rightarrow \langle \alpha, \langle \beta, \gamma \rangle \rangle$  of the product

$$\pi_p(G) \times \pi_q(G) \times \pi_r(G)$$

into  $\pi_{p+q+r}(G)$ . Now if  $\Gamma$  is a group, the iterated commutators satisfy a "Jacobi congruence" modulo the fourth member of the descending central series of  $\Gamma$ ; since  $\pi(S^p \times S^q \times S^r; G)$  is nilpotent of class  $\leq 3$ , this congruence reduces to an identity. From this fact we deduce a Jacobi identity for the operation  $\langle \alpha, \beta \rangle$ .

Suppose that  $G$  is the space of loops in a space  $X$ ; then

$$\pi_n(G) \approx \pi_{n+1}(X) .$$

By this isomorphism the commutator mapping is transformed, except

for sign, into the Whitehead product. Thus the above results provide another proof that the Whitehead products satisfy a Jacoby identity<sup>1)</sup>.

**2. Nilpotency of  $\pi(X; G)$ .** Let  $X$  be a separable metric locally contractible space. A subset  $A$  of  $X$  is said to be *categorical* if and only if there is an open set  $U \supset A$  such that  $U$  is contractible in  $X$ . A covering of  $X$  by sets  $A_\alpha$  is said to be categorical if and only if each of the sets  $A_\alpha$  is categorical. The *category*  $\text{cat } X$  of  $X$  is the least of the cardinal numbers of the categorical coverings of  $X$ .

Let  $G$  be a space,  $e \in G$ , and let  $\mu: (G \times G, (e, e)) \rightarrow (G, e)$ ,  $i: (G, e) \rightarrow (G, e)$  be maps. We say that  $G$  is an  $H^*$ -space with respect to  $e, \mu, i$  if and only if the following conditions are satisfied:

2.1) The maps  $(x, y, z) \rightarrow \mu(x, \mu(y, z))$  and  $(x, y, z) \rightarrow \mu(\mu(x, y), z)$  of  $G \times G \times G$  into  $G$  are homotopic rel.  $(e, e, e)$ ;

2.2) The maps  $x \rightarrow \mu(x, e)$  and  $x \rightarrow \mu(e, x)$  of  $G$  into  $G$  are homotopic rel.  $e$  to the identity map;

2.3) The maps  $x \rightarrow \mu(x, i(x))$  and  $x \rightarrow \mu(i(x), x)$  of  $G$  into  $G$  are homotopic rel.  $e$  to the constant map.

Hereafter we abbreviate  $\mu(x, y)$  to  $x \cdot y$  and  $i(x)$  to  $x^{-1}$ .

Let  $\Phi: G \times G \rightarrow G$  be the commutator map, defined by

$$\Phi(x, y) = (xy)(x^{-1}y^{-1}),$$

and let  $G \vee G$  be the subset  $G \times e \cup e \times G$  of  $G \times G$ . Then from 2.1 to 2.3 we conclude easily:

2.4) The map  $\Phi|G \vee G$  is homotopic rel.  $(e, e)$  to the constant map.

Let  $X$  be a space,  $A \subset X$ , and let  $G$  be an  $H^*$ -space. If  $f, g: (X, A) \rightarrow (G, e)$  are maps, we define  $f \cdot g: (X, A) \rightarrow (G, e)$  by

$$(f \cdot g)(x) = f(x) \cdot g(x).$$

This operation preserves the relation of homotopy, and therefore induces an operation in the set  $\pi(X, A; G)$  of homotopy classes of maps of  $(X, A)$  into  $(G, e)$ . It is easily verified that  $\pi(X, A; G)$  is a group under this operation. Moreover, if  $h: (Y, B) \rightarrow (X, A)$  is a map, then composition with  $h$  induces a homomorphism

$$h: \pi(X, A; G) \rightarrow \pi(Y, B; G);$$

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<sup>1)</sup> Proofs of the Jacobi identity for the Whitehead product have been obtained recently by Hilton and by Massey-Uehara (to appear in Ann. of Math.). The author believes the present proof to be more elementary. The author is informed by Uehara that proofs have also been found by Nakaoka and by Toda.

the homomorphism induced by a composite map  $k \circ h$  is the composition  $h \circ k$  in the opposite order.

We further have :

2.5) If  $S$  is an  $n$ -sphere,  $x \in S$ , then  $\pi(S, x; G) = \pi_n(G)$ .

2.6) Let  $(X', a)$  be the pair obtained from  $(X, A)$  by collapsing  $A$  to a point,  $j: (X, A) \rightarrow (X', a)$  the identification map. Then  $j: \pi(X', a; G) \approx \pi(X, A; G)$ .

The proof of 2.5 is similar to the proof in the case where  $G$  is a topological group [1]; 2.6 follows from standard properties of identification spaces.

**Lemma 2.7.** Let  $(X; A_1, \dots, A_n)$  be an

$$(n+1) - \text{ad} \ , \quad A = A_1 \cup \dots \cup A_n \ .$$

Suppose that there are retractions  $\varrho_i: X \rightarrow A_i$  ( $i = 1, \dots, n$ ) such that  $\varrho_i(A_j) \subset A_j$  for all  $i, j$ . Suppose further that either  $(X; A_1, \dots, A_n)$  is triangulable or  $X$  is separable metric and  $G$  is an  $ANR^2$ . Then, if  $j: X \subset (X, A)$ , the homomorphism  $j: \pi(X, A; G) \rightarrow \pi(X; G)$  is an isomorphism into.

*Proof.* We must show that, if  $f_0, f_1: X \rightarrow G$  are homotopic maps with  $f_0(A) = f_1(A) = e$ , then  $f_0$  and  $f_1$  are homotopic rel.  $A$ . Let  $B_r = A_1 \cup \dots \cup A_r$ ; then it suffices to show that  $f_0 \simeq f_1$  rel.  $B_r$  implies  $f_0 \simeq f_1$  rel.  $B_{r+1}$ . Let  $F: X \times I \rightarrow G$  be a homotopy of  $f_0$  to  $f_1$  rel.  $B_r$ , and define  $F_1: X \times I \rightarrow G$  by

$$F_1(x, t) = F(x, t) \cdot F[\varrho_{r+1}(x), t]^{-1} \ .$$

Then  $x \in B_r$  implies  $\varrho_{r+1}(x) \in B_r$ , and therefore  $F_1(x, t) = e \cdot e^{-1} = e$ ; and  $x \in A_{r+1}$  implies  $\varrho_{r+1}(x) = x$ . Thus  $x \in B_{r+1}$  implies that  $F_1(x, t) = F(x, t) \cdot F(x, t)^{-1}$ . Furthermore, if  $t = 0$  or  $1$ ,  $F_1(x, t) = f_t(x) \cdot e$ .

It follows from 2.3 that  $F_1|B_{r+1} \times I$  is homotopic rel  $B_{r+1} \times \dot{I}$  to the constant map. It follows from 2.2 that  $F_1|X \times \dot{I}$  is homotopic rel  $A \times \dot{I}$  to the map  $(x, t) \rightarrow f_t(x)$ . Let  $C = X \times \dot{I} \cup B_{r+1} \times I$ ; then  $F_1|C$  is homotopic to the map  $F_2: C \rightarrow G$  such that

$$\begin{aligned} F_2(x, t) &= f_t(x) & (x \in X, t \in \dot{I}) ; \\ F_2(x, t) &= e & (x \in B_{r+1}) \ . \end{aligned}$$

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<sup>2</sup>) In the weak sense.



By the homotopy extension theorem,  $F_2$  has an extension  $F' : X \times I \rightarrow G$ ;  $F'$  is a homotopy of  $f_0$  to  $f_1$  rel.  $B_{r+1}$ .

Let  $\Gamma$  be a group. We recall that the *descending central series* of  $\Gamma$  is the sequence of subgroups defined inductively by

$$Z_0 = \Gamma; \quad Z_i = [\Gamma, Z_{i-1}] \quad \text{for } i > 0,$$

and that  $\Gamma$  is *nilpotent* if and only if there is a  $c$  such that  $Z_c = \{1\}$ ; the least such  $c$  is called the *class* of  $\Gamma$ . A *central chain* of length  $k$  of  $\Gamma$  is a sequence

$$\Gamma = \Gamma_0 \supset \dots \supset \Gamma_k = \{1\}$$

of subgroups such that  $[\Gamma, \Gamma_i] \subset \Gamma_{i+1}$  ( $i = 0, \dots, k-1$ ).  $\Gamma$  has a central chain of length  $k$  if and only if  $\Gamma$  is nilpotent of class  $\leq k$ .

We further recall [6, pp. 60—63]:

$$2.8) \quad [a, bc] \equiv [a, b] \cdot [a, c] \pmod{Z_2},$$

$$2.9) \quad [a, [b, c]] \cdot [b, [c, a]] \cdot [c, [a, b]] \equiv 1 \pmod{Z_3}.$$

**Theorem 2.10.** *Let  $X$  be a separable metric locally contractible space,  $G$  a 0-connected  $H^*$ -space. Suppose that  $X$  has finite category  $c$ , and that either a)  $X$  is triangulable or b)  $G$  is an ANR. Then the group  $\pi(X; G)$  is nilpotent of class  $\leq c-1$ .*

To prove Theorem 2.10, we recall from [2, p. 336] that  $X$  has a closed categorical covering  $A_1, \dots, A_c$ . If  $X$  is triangulable we may assume that the sets  $A_i$  are all subcomplexes of a certain simplicial decomposition of  $X$ . Theorem 2.10 will then follow from

**Lemma 2.11.** *Let  $\Gamma'_i$  be the set of all homotopy classes of maps  $f : X \rightarrow G$  such that  $f|_{A_1 \cup \dots \cup A_{i+1}}$  is inessential ( $i = 0, \dots, c-1$ ). Then the  $\Gamma'_i$  form a central chain for  $\pi(X; G)$ .*

*Proof.* Clearly the  $\Gamma'_i$  form a decreasing sequence of subgroups of  $\Gamma = \pi(X; G)$ ;  $\Gamma'_0 = \Gamma$  since  $A_1$  is contractible in  $X$ , and  $\Gamma'_{c-1} = \{1\}$  since  $A_1 \cup \dots \cup A_c = X$ . It remains to prove that  $[\Gamma, \Gamma'_{i-1}] \subset \Gamma'_i$ .

Let  $f \in \alpha \in \Gamma$ ,  $g \in \beta \in \Gamma'_{i-1}$ ; then  $f|_{A_{i+1}}$  is inessential since  $A_{i+1}$  is contractible in  $X$ , and  $g|_{A_1 \cup \dots \cup A_i}$  is inessential by hypothesis. Hence we may assume  $f(A_{i+1}) = g(A_1 \cup \dots \cup A_i) = e$ . Define  $h : X \rightarrow G$  by  $h = \Phi \circ (f \times g)$ , i. e.,

$$h(x) = \Phi(f(x), g(x));$$

then  $h \in [\alpha, \beta]$ . Now if  $x \in A_1 \cup \dots \cup A_{i+1}$ , then  $(f(x), g(x)) \in G \vee G$ ; it follows from 2.4 that  $h|_{A_1 \cup \dots \cup A_{i+1}}$  is inessential, i. e.,  $[\alpha, \beta] \in \Gamma'_i$ .

**Corollary 2.12.** *If  $X$  has finite dimension  $n$ , then  $\pi(X; G)$  is nilpotent of class  $\leq n$ .*

For  $\text{cat } X \leq 1 + \dim X$  [2, (5.4)].

**Corollary 2.13.** *If  $X$  is the product of  $k$  spheres, then  $\pi(X; G)$  is nilpotent of class  $\leq k$ .*

For  $\text{cat } X = k + 1$  [2, p. 350].

The following variant of Lemma 2.11 will be more appropriate in what follows<sup>3</sup>).

**Lemma 2.14.** *Let  $X$  be a CW-complex [5] and let*

$$A_0 \subset \cdots \subset A_k = X$$

*be subcomplexes such that, if  $E$  is any cell of  $A_i$ , then  $\dot{E} \subset A_{i-1}$ . Let  $\Gamma_i$  be the set of all homotopy classes of maps  $f: X \rightarrow G$  such that  $f|A_i$  is inessential. Then  $\Gamma_0, \dots, \Gamma_k$  is a central chain for  $\pi(X; G)$ .*

*Proof.* Let  $f \in \alpha \in \Gamma = \pi(X; G)$ ,  $g \in \beta \in \Gamma_{i-1}$ . For each cell  $E_\lambda$  of  $A_i$ , let  $x_\lambda$  be an interior point of  $E_\lambda$ . Then we can find closed cells  $F_\lambda \subset \text{Int } E_\lambda$  such that  $x_\lambda \in \text{Int } F_\lambda$  and  $\dot{E}_\lambda$  is a deformation retract of  $\overline{E_\lambda - F_\lambda}$ ; it follows that  $A_{i-1}$  is a deformation retract of

$$Q = \bigcup_\lambda \overline{E_\lambda - F_\lambda} \cup A_{i-1}$$

and the set  $P_0 = \{x_\lambda\}$  is a deformation retract of  $P = \bigcup_\lambda F_\lambda$ . Since  $G$  is 0-connected,  $f|P_0$ , and therefore also  $f|P$ , is inessential. Since  $g|A_{i-1}$  is inessential, it follows that  $g|Q$  is inessential. Hence we may assume  $f(P) = g(Q) = e$ . The remainder of the proof follows the pattern of Lemma 2.11.

**3. The case  $X = S^{n_1} \times \cdots \times S^{n_k}$ .** We now examine in more detail the case  $X = S^{n_1} \times \cdots \times S^{n_k}$ . We first make some conventions about orientation. Let  $I^p$  be the unit cube in Euclidean  $p$ -space; we orient  $I^p$  by choosing the generator  $\omega_p$  of  $H_p(I^p, \dot{I}^p)$  represented by the identity map of  $I^p$  (we are using cubical singular homology theory as in [4]). Let  $x_p$  be a fixed point of  $S^p$  and choose a fixed map  $\psi_p: (I^p, \dot{I}^p) \rightarrow (S^p, x_p)$

<sup>3</sup>) The hypotheses of Lemma 2.14 imply that  $\text{cat } X \leq k + 1$ , so that  $\pi(X; G)$  is nilpotent of class  $\leq k$ . The conclusion follows formally from Lemma 2.11 and the following statement, which is easily proved by induction on  $k$ : *there are closed categorical subsets  $B_0, \dots, B_k$  of  $X$  such  $A_i$  is a deformation retract of  $B_0 \cup \dots \cup B_i$  ( $i = 0, \dots, k$ ).* However, it seemed simpler to prove Lemma 2.14 directly. Actually, only Lemma 2.14 is used in what follows; we state Theorem 2.10 because it seems to give the most natural upper bound for the class of nilpotency.

which maps the interior of  $I^p$  topologically onto  $S^p - x_p$ ; we then orient  $S^p$  by requiring that  $\psi_p$  have degree 1. Next we make the natural identification of  $I^{n_1} \times \dots \times I^{n_k}$  with  $I^{n_1 + \dots + n_k}$  and then orient  $S^{n_1} \times \dots \times S^{n_k}$  by requiring that  $\psi = \psi_{n_1} \times \dots \times \psi_{n_k}$  have degree 1.

Let  $X = S^{n_1} \times \dots \times S^{n_k}$ . We take  $S_i = S^{n_i}$  with the  $CW$ -decomposition consisting of the 0-cell  $e_i = x_{n_i}$  and the  $n_i$ -cell  $S_i - e_i$ . The product of these  $CW$ -decompositions gives a  $CW$ -decomposition of  $X$ . Let  $A_i$  be the set of all  $x \in X$  such that  $x_j \neq e_j$  for at most  $i$  values of  $j$ ; thus

$$\begin{aligned} A_0 &= e_1 \times \dots \times e_k, \\ A_1 &= S_1 \times e_2 \times \dots \times e_k \cup \dots \cup e_1 \times \dots \times e_{k-1} \times S_k, \\ &\dots \\ A_k &= S_1 \times \dots \times S_k = X. \end{aligned}$$

The  $A_i$  clearly satisfy the hypotheses of Lemma 2.14.

For each subset  $\alpha$  of  $\{1, \dots, k\}$ , let  $S_\alpha = \{x \in X \mid x_i = e_i \text{ for } i \notin \alpha\}$ ,  $|\alpha|$  = the cardinal of  $\alpha$ ,  $n(\alpha) = \sum_{i \in \alpha} n_i$ .  $S_\alpha$  is homeomorphic with  $\prod_{i \in \alpha} S_i$ , and we orient  $S_\alpha$  by requiring that the natural homeo-

morphism (the one preserving the order of the coordinates) be orientation-preserving. Finally, let  $p_\alpha: (S_\alpha, S_\alpha \cap A_{|\alpha|-1}) \rightarrow (S^{n(\alpha)}, e_{n(\alpha)})$  be an orientation-preserving map which is topological on  $S_\alpha - A_{|\alpha|-1}$ .

Let  $q_\alpha: X \rightarrow S_\alpha$  be the natural retraction:  $q_\alpha(x)$  is the point  $y$  such that  $y_i = x_i$  if  $i \in \alpha$  and  $y_i = e_i$  if  $i \notin \alpha$ . Then  $\beta \neq \alpha$  implies  $q_\beta(S_\alpha) \subset S_\alpha$ . We may therefore apply Lemma 2.7 to obtain:

**Lemma 3.1.** *Let  $j: X \subset (X, A_i)$ . Then  $j: \pi(X, A_i; G) \approx \Gamma_i$ .*

**Theorem 3.2.** *The group  $\Gamma_{i-1}/\Gamma_i$  is isomorphic with the direct product*

$$\prod_{\alpha} \pi_{n(\alpha)}(G),$$

where  $\alpha$  ranges over all  $i$ -element subsets of  $\{1, \dots, k\}$ .

*Proof.* Let  $j_\alpha: (S_\alpha, S_\alpha \cap A_{i-1}) \subset (X, A_{i-1})$ . Then

$$j: \pi(X, A_{i-1}; G) \approx \Gamma_{i-1}, j_\alpha: \pi(X, A_{i-1}; G) \rightarrow \pi(S_\alpha, S_\alpha \cap A_{i-1}; G),$$

and  $p_\alpha: \pi_{n(\alpha)}(G) \approx \pi(S_\alpha, S_\alpha \cap A_{i-1}; G)$ ; hence we may define  $\eta_\alpha: \Gamma_{i-1} \rightarrow \pi_{n(\alpha)}(G)$  by  $\eta_\alpha = p_\alpha^{-1} j_\alpha j^{-1}$ . The homomorphisms  $\eta_\alpha$  then define a homomorphism  $\eta: \Gamma_{i-1} \rightarrow \Delta = \prod_{\alpha} \pi_{n(\alpha)}(G)$ . Clearly  $\Gamma_i \subset$

Kernel  $\eta$ . Conversely, if  $f \in \gamma \in \text{Kernel } \eta$ , then, for each  $\alpha$ ,  $f|S_\alpha$  is

inessential rel.  $S_\alpha \cap A_{i-1}$ ; hence  $f|A_i$  is inessential. Therefore  $\eta$  induces an isomorphism  $\eta$  of  $\Gamma_{i-1}/\Gamma_i$  into  $\Delta$ .

To show that  $\eta$  is onto, define  $i_\alpha: X \rightarrow S^{n(\alpha)}$  by  $i_\alpha = p_\alpha \circ q_\alpha$ . Since  $q_\alpha$  can be factored through  $(X, A_{i-1})$ , it follows that  $i_\alpha: \pi_{n(\alpha)}(G) \rightarrow \Gamma_{i-1}$ . Now

$$\eta_\alpha \circ i_\alpha = p_\alpha^{-1} \circ j_\alpha \circ j^{-1} \circ q_\alpha \circ p_\alpha ;$$

since  $q_\alpha$  is a retraction,  $j_\alpha \circ j^{-1} \circ q_\alpha$  is the identity. Hence  $\eta_\alpha \circ i_\alpha$  is the identity. If  $\alpha \neq \beta$ , then  $q_\alpha(S_\beta) \subset S_\alpha \cap S_\beta \subset A_{i-1}$  and therefore  $j_\beta \circ j^{-1} \circ q_\alpha = 0$ ; thus

$$\eta_\beta \circ i_\alpha = p_\beta^{-1} \circ j_\beta \circ j^{-1} \circ q_\alpha \circ p_\alpha = 0 .$$

It follows that  $\eta$  is onto, and that the  $i_\alpha$  are isomorphisms into.

**4. Commutator relations.** In this section we write  $\pi_{n_1, \dots, n_k}$  for  $\pi(S^{n_1} \times \dots \times S^{n_k}; G)$ . Let  $i_1: \pi_p \rightarrow \pi_{p,q}$ ,  $i_2: \pi_q \rightarrow \pi_{p,q}$ ,  $i_{1,2}: \pi_{p+q} \rightarrow \pi_{p,q}$  be the isomorphisms into of § 3. Then  $i_{1,2}$  maps  $\pi_{p+q}$  isomorphically onto  $\Gamma_1$ , and if  $\alpha \in \pi_p$ ,  $\beta \in \pi_q$ , we have  $[i_1(\alpha), i_2(\beta)] \in \Gamma_1$ . Hence we may define

$$\langle \alpha, \beta \rangle = i_{1,2}^{-1}[i_1(\alpha), i_2(\beta)] \in \pi_{p+q} .$$

**Lemma 4.1.** (a)  $\langle \alpha, \beta \rangle = (-1)^{p^q-1} \langle \beta, \alpha \rangle$ ; (b)  $\langle \alpha, \beta \rangle$  depends bilinearly on  $\alpha, \beta$ .

*Proof.* Let  $t: S^p \times S^q \rightarrow S^q \times S^p$  be the map which interchanges the coordinates. In virtue of our conventions about orientation,  $t$  has degree  $(-1)^{p^q}$ . Furthermore,  $t$  induces a map  $t': S^{p+q} \rightarrow S^{p+q}$  of the same degree. We verify easily that  $i_1 \circ t = t_2$ ,  $i_2 \circ t = i_1$ , and  $i_{1,2} \circ t = t' \circ i_{1,2}$ . Furthermore  $t'(u) = (-1)^{p^q} u$  for  $u \in \pi_{p+q}$ . Hence

$$\begin{aligned} \langle \beta, \alpha \rangle &= i_{1,2}^{-1}[i_1(\beta), i_2(\alpha)] = t'^{-1} i_{1,2}^{-1} t[i_1(\beta), i_2(\alpha)] \\ &= (-1)^{p^q} i_{1,2}^{-1}[ti_1(\beta), ti_2(\alpha)] \\ &= (-1)^{p^q} i_{1,2}^{-1}[i_2(\beta), i_1(\alpha)] \\ &= (-1)^{p^q-1} i_{1,2}^{-1}[i_1(\alpha), i_2(\beta)] \\ &= (-1)^{p^q-1} \langle \alpha, \beta \rangle . \end{aligned}$$

Now if  $\alpha \in \pi_p$ ,  $\beta, \gamma \in \pi_q$ , we have, by 2.8,

$$\begin{aligned} [i_1(\alpha), i_2(\beta \cdot \gamma)] &= [i_1(\alpha), i_2(\beta) i_2(\gamma)] \\ &\equiv [i_1(\alpha), i_2(\beta)] \cdot [i_1(\alpha), i_2(\gamma)] \pmod{\Gamma_2} . \end{aligned}$$

But  $\Gamma_2 = \{1\}$  and  $i_{1,2}$  is a homomorphism. Hence right linearity holds. Left linearity follows from right linearity and part (a).

Consider now the group  $\pi_{p,q,r}$  and the isomorphisms into

$i_1: \pi_p \rightarrow \pi_{p,q,r}$ ,  $i_2: \pi_q \rightarrow \pi_{p,q,r}$ ,  $i_3: \pi_r \rightarrow \pi_{p,q,r}$ , and  $i_{1,2,3}: \pi_{p+q+r} \rightarrow \pi_{p,q,r}$ . The latter is an isomorphism onto  $\Gamma_2$ . If  $\alpha \in \pi_p$ ,  $\beta \in \pi_q$ ,  $\gamma \in \pi_r$ , then  $[i_1(\alpha), [i_2(\beta), i_3(\gamma)]] \in \Gamma_2$ , and we have:

**Lemma 4.2.**  $\langle \alpha, \langle \beta, \gamma \rangle \rangle = i_{1,2,3}^{-1} [i_1(\alpha), [i_2(\beta), i_3(\gamma)]]$ .

*Proof.* We have, by definition,

$$\langle \alpha, \langle \beta, \gamma \rangle \rangle = i_{1,2}^{-1} [i_1(\alpha), i_2 i_{1,2}^{-1} [i_1(\beta), i_2(\gamma)]] .$$

Let  $f: S^p \times S^q \times S^r \rightarrow S^p \times S^{q+r}$  be the map given by

$$f(x, y, z) = (x, p_{1,2}(y, z)) .$$

The maps  $p_{1,2} \circ f$  and  $p_{1,2,3}$  of  $(S^p \times S^q \times S^r, A_2)$  into  $(S^{p+q+r}, x_{p+q+r})$  both preserve orientation. Hence

$$f \circ i_{1,2} = f \circ p_{1,2} = p_{1,2,3} = i_{1,2,3} .$$

On the other hand,  $i_2 \circ f = i_{2,3}$  and  $i_1 \circ f = i_1$ . Let  $h$  be the projection of  $S^p \times S^q \times S^r$  on  $S^q \times S^r$ . Then  $i_{1,2} \circ h = i_{2,3}$ ,  $i_1 \circ h = i_2$ , and  $i_2 \circ h = i_3$ . Thus

$$\begin{aligned} i_{1,2,3} \langle \alpha, \langle \beta, \gamma \rangle \rangle &= f i_{1,2} \langle \alpha, \langle \beta, \gamma \rangle \rangle \\ &= f [i_1(\alpha), i_2 i_{1,2}^{-1} [i_1(\beta), i_2(\gamma)]] \\ &= [f i_1(\alpha), f i_2 i_{1,2}^{-1} [i_1(\beta), i_2(\gamma)]] \\ &= [i_1(\alpha), i_{2,3} i_{1,2}^{-1} [i_1(\beta), i_2(\gamma)]] \\ &= [i_1(\alpha), h [i_1(\beta), i_2(\gamma)]] \\ &= [i_1(\alpha), [h i_1(\beta), h i_2(\gamma)]] \\ &= [i_1(\alpha), [i_2(\beta), i_3(\gamma)]] . \end{aligned}$$

**Lemma 4.3.** If  $\alpha \in \pi_p$ ,  $\beta \in \pi_q$ ,  $\gamma \in \pi_r$ , then

$$\begin{aligned} i_{1,2,3}^{-1} [i_2(\beta), [i_3(\gamma), i_1(\alpha)]] &= (-1)^{p(q+r)} \langle \beta, \langle \gamma, \alpha \rangle \rangle ; \\ i_{1,2,3}^{-1} [i_3(\gamma), [i_1(\alpha), i_2(\beta)]] &= (-1)^{r(p+q)} \langle \gamma, \langle \alpha, \beta \rangle \rangle . \end{aligned}$$

*Proof.* Let  $g: S^p \times S^q \times S^r \rightarrow S^q \times S^r \times S^p$  be the map given by  $g(x, y, z) = (y, z, x)$ . Then  $g$  induces  $g': S^{p+q+r} \rightarrow S^{p+q+r}$ ;  $g$  and  $g'$  have the same degree  $(-1)^{p(q+r)}$ , and  $i_{1,2,3} \circ g = g' \circ i_{1,2,3}$ ,  $i_1 \circ g = i_2$ ,  $i_2 \circ g = i_3$ ,  $i_3 \circ g = i_1$ . Hence

$$\begin{aligned} i_{1,2,3} \{(-1)^{p(q+r)} \langle \beta, \langle \gamma, \alpha \rangle \rangle\} &= g i_{1,2,3} \langle \beta, \langle \gamma, \alpha \rangle \rangle \\ &= g [i_1(\beta), [i_2(\gamma), i_3(\alpha)]] \\ &= [g i_1(\beta), [g i_2(\gamma), g i_3(\alpha)]] \\ &= [i_2(\beta), [i_3(\gamma), i_1(\alpha)]] . \end{aligned}$$

The other statement is proved similarly.

**Theorem 4.4.** *If  $\alpha \in \pi_p$ ,  $\beta \in \pi_q$ ,  $\gamma \in \pi_r$ , then*

$$\langle \alpha, \langle \beta, \gamma \rangle \rangle + (-1)^{p(q+r)} \langle \beta, \langle \gamma, \alpha \rangle \rangle + (-1)^{r(p+q)} \langle \gamma, \langle \alpha, \beta \rangle \rangle = 0 .$$

*Proof.* Let  $\alpha' = i_1(\alpha)$ ,  $\beta' = i_2(\beta)$ ,  $\gamma' = i_3(\gamma)$ . Since  $\Gamma_3 = \{1\}$ , we have, by 2.9,

$$1 = [\alpha', [\beta', \gamma']] \cdot [\beta', [\gamma', \alpha']] \cdot [\gamma', [\alpha', \beta']]$$

and therefore

$$\begin{aligned} 0 &= i_{1,2,3}^{-1}[\alpha', [\beta', \gamma']] \cdot [\beta', [\gamma', \alpha']] \cdot [\gamma', [\alpha', \beta']] \\ &= \langle \alpha, \langle \beta, \gamma \rangle \rangle + (-1)^{p(q+r)} \langle \beta, \langle \gamma, \alpha \rangle \rangle + (-1)^{r(p+q)} \langle \gamma, \langle \alpha, \beta \rangle \rangle . \end{aligned}$$

Now suppose that  $X$  is a space,  $x \in X$ , and that  $G$  is the path-component of the constant map in the space of loops in  $X$  based at  $x$ . Then we have a natural isomorphism  $T: \pi_{n+1}(X) \approx \pi_n(G)$  for each  $n > 0$ . Furthermore, if  $\alpha \in \pi_{p+1}(X)$ ,  $\beta \in \pi_{q+1}(X)$ , and if  $[\alpha, \beta] \in \pi_{p+q+1}(X)$  is their Whitehead product, we have [3]:

$$T([\alpha, \beta]) = (-1)^p \langle T(\alpha), T(\beta) \rangle . \quad (4.5)$$

From 4.5 and Theorem 4.4 we then obtain:

**Theorem 4.6.** *If  $\alpha \in \pi_{p+1}(X)$ ,  $\beta \in \pi_{q+1}(X)$ ,  $\gamma \in \pi_{r+1}(X)$ , then*

$$(-1)^{r(p+1)} [\alpha, [\beta, \gamma]] + (-1)^{p(q+1)} [\beta, [\gamma, \alpha]] + (-1)^{q(r+1)} [\gamma, [\alpha, \beta]] = 0 .$$

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