Algebras of cohomologically finite dimension.

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by Samuel Eilenberg [1])

Dedicated to Heinz Hopf on his 60^{th} anniversary

1. Introduction and results

Let Λ be an algebra over a field K with $(\Lambda : K) < \infty$. The dimension of Λ (in the cohomology sense) is defined as the highest integer n for which the cohomology group $H^n(\Lambda, A)$ is non zero for some two-sided Λ-module A. If no such integer exists, then $\dim \Lambda = \infty$. Algebras of dimension zero are known to be those which are separable. A very interesting characterization of algebras of dimension n has been given recently by Ikeda, Nagao and Nakayama [4]. The purpose of this paper is to give a new treatment of the results of [4] within the framework of the Cartan-Eilenberg theory [1]. There result considerable simplifications of the proofs, and partly also a sharpening of the results. Relative cohomology and other ad hoc constructions used in [4] are eliminated.

Let A be a left Λ-module. The *dimension* of A (notation : $\mathrm{l.\,dim}_\Lambda A$) is defined as the least integer n for which there exists an exact sequence

$$0 \to X_n \to \ldots \to X_0 \to A \to 0$$

where the left Λ-modules $X_0, \ldots, X_n$ are projective. If no such sequence exists for any n, then $\mathrm{l.\,dim}_\Lambda A = \infty$. The *left global dimension* of Λ is

$$\mathrm{l.\,gl.\,dim}\, \Lambda = \sup\, \mathrm{l.\,dim}_\Lambda A$$

for all left Λ-modules A. Using the functors Ext, the condition $\mathrm{l.\,dim}_\Lambda A \leqq n$ is equivalent with $\mathrm{Ext}_\Lambda^{n+1}(A, C) = 0$ for all left Λ-modules C, while the condition $\mathrm{l.\,gl.\,dim}\, \Lambda \leqq n$ is equivalent with $\mathrm{Ext}_\Lambda^{n+1} = 0$.

As for two-sided Λ-modules A, the standard procedure will be to convert them into left modules over the algebra $\Lambda^e = \Lambda \otimes \Lambda^*$ where Λ^* is the algebra opposite to Λ and where $\otimes$ stands for the tensor product over K. Then (by definition)

[1]) Work done under contract AF–18 (600)–562.

$$H^n(\Lambda, \Lambda) = \operatorname{Ext}^n_{\Lambda^e}(\Lambda, \Lambda)$$

so that

$$\dim \Lambda = \mathrm{l.} \dim_{\Lambda^e} \Lambda \ .$$

We shall denote by N the radical of Λ and write $\Gamma = \Lambda/N$.

With these preliminaries, the main results may be stated.

Theorem I. $\dim \Lambda = \mathrm{l.\,gl.} \dim \Omega = \mathrm{l.} \dim_\Omega \Gamma$, where $\Omega = \Lambda \otimes \Gamma^*$.

Theorem II. If Γ is separable, then $\dim \Lambda = \mathrm{l.} \dim_\Lambda \Gamma$.

Theorem III. If $\dim \Lambda < \infty$ then Γ is separable.

Note that if K has characteristic zero or is algebraically closed, then Γ, being semi-simple, always is separable. Further, the separability of Γ is equivalent with $\dim \Gamma = 0$ and also is equivalent with the semi-simplicity of $\Gamma \otimes \Gamma^*$ (see [1], Ch. IX, prop. 7.9 and 7.10).

An immediate consequence of Theorem III is the following result of Hochschild ([3], p. 946):

Corollary 1. If Λ is semi-simple but inseparable then $\dim \Lambda = \infty$.

Combining Theorems II and III yields:

Corollary 2. In order that $\dim \Lambda = n \, (n < \infty)$ it is necessary and sufficient that the following conditions hold:

(1) Γ is separable.

(2) $\mathrm{l.} \dim_\Lambda \Gamma = n$.

If $\mathrm{l.} \dim_\Lambda \Gamma > 0$ then the exact sequence $0 \to N \to \Lambda \to \Gamma \to 0$ implies ([1], Ch. VI, prop. 2.3) that

$$\mathrm{l.} \dim_\Lambda \Gamma = 1 + \mathrm{l.} \dim_\Lambda N \ .$$

Thus for $n > 0$, condition (2) may be replaced by

$$\text{(2')} \qquad \mathrm{l.} \dim N = n - 1 \ .$$

The characterization given by Ikeda-Nagao-Nakayama in [4] (for $n > 0$) utilizes conditions (1) and (2′), except that condition (2′) is stated in a more explicit but also more involved form, which however is equivalent. For a proof of this equivalence we refer the reader to Eilenberg-Ikeda-Nakayama [2] where several questions related to this paper are discussed.

Sections 2 and 3 contain a sequence of propositions leading to proofs of Theorem I and II. The technique of proofs fully conforms to the system developed in [1]. It should be noted that the results of sections 2 and 3 are obtained under weaker hypotheses than stated above. Indeed, the assumption that $(\Lambda : K) < \infty$ is dropped and K need not be a field.

This added generality is abandoned in sections 4, 5 and 6 devoted to the proof of Theorem III.

2. Preliminaries

We shall consider algebras Λ over a commutative ring K. We shall have the oppurtunity to discuss K-modules, left Λ-modules, right Λ-modules and two-sided Λ-modules. It will always be assumed that K and Λ have unit elements, and that these unit elements act as the identity on all modules.

For each left ideal l in Λ we set $l^0 = \Lambda$ and $l^n = l l^{n-1}$ for $n > 0$. We say that l is *nilpotent* if $l^k = 0$ for some integer k.

Proposition 3. Let A be a left Λ-module such that $\operatorname{Ext}^n_\Lambda(A, C) = 0$ for each left Λ-module C with $lC = 0$. If l is nilpotent, then $\operatorname{Ext}^n_\Lambda(A, C) = 0$ for all left Λ-modules C, i. e. $\mathrm{l.}\dim_\Lambda A < n$.

Proof. For each integer $i > 0$ consider the exact sequence

$$0 \to l^{i+1} C \to l^i C \to l^i C / l^{i+1} C \to 0 \ .$$

Since $l(l^i C / l^{i+1} C) = 0$ it follows that

$$\operatorname{Ext}^n_\Lambda(A, l^i C / l^{i+1} C) = 0$$

and therefore the homomorphism

$$\operatorname{Ext}^n_\Lambda(A, l^{i+1} C) \to \operatorname{Ext}^n(A, l^i C)$$

induced by inclusion $l^{i+1} C \subset l^i C$, is an epimorphism. Since $l^0 C = C$ and $l^k C = 0$ for k sufficiently large, it follows that $\operatorname{Ext}^n_\Lambda(A, C) = 0$.

In the sequel we shall use the following proposition established in [1] (Ch. IX, prop. 4.3).

Proposition 4. Let Λ and Γ be K-algebras where K is a commutative ring. If Λ is K-projective and Γ is semi-simple then we have the natural isomorphism

$$H^n(\Lambda, \operatorname{Hom}_\Gamma(B, C)) \approx \operatorname{Ext}_{\Lambda \otimes \Gamma^*}(B, C)$$

for any left Λ- and right Γ-modules B and C.

We shall need some corollaries (also derived in [1], Ch. IX, § 7). First taking $\Gamma = K$ we obtain

Corollary 5. If Λ is a K-algebra with K semi-simple then

$$\mathrm{l.\,gl.}\dim \Lambda \leqq \dim \Lambda \ .$$

Taking $\Gamma = \Lambda$ and noting the inequality

$$\dim \Lambda \leqq \mathrm{gl.}\dim \Lambda \otimes \Lambda^*$$

we obtain

Corollary 6. If the K-algebra Λ is K-projective and semi-simple then

$$\text{gl. dim } \Lambda \otimes \Lambda^* = \dim \Lambda \ .$$

Finally combining these two corollaries we obtain

Corollary 7. If Λ is a K-algebra with K semi-simple, then $\dim \Lambda = 0$ if and only if $\Lambda \otimes \Lambda^*$ is semi-simple.

3. Proofs of Theorems I and II

In this section we shall consider an algebra Λ over a commutative ring K. In Λ a two sided ideal l will be given with $\Gamma = \Lambda/l$. Every left Γ-module A will be regarded also as a left Λ-module with $lA = 0$. Similarly for right and two-sided modules.

Proposition 8. If Λ is K-projective and Γ is semi-simple then

$$H^n(\Lambda, C) \approx \text{Ext}^n_{\Lambda \otimes \Gamma^*}(\Gamma, C)$$

for every left Λ- and right Γ-module C.

This follows directly from Proposition 4 by taking $B = \Gamma$ and observing that C is isomorphic with $\text{Hom}_\Gamma(\Gamma, C)$ as a left Λ- and right Γ-module.

Proposition 9. If Λ is K-projective and Γ is semi-simple then

$$\text{l. dim}_\Omega \Gamma \leqq \text{l. gl. dim } \Omega \leqq \dim \Lambda \ , \qquad \Omega = \Lambda \otimes \Gamma^* \ .$$

If further l is nilpotent, then equalities hold.

Proof. The first part is an immediate consequence of Proposition 4. To prove the second half assume $\text{l. dim}_\Omega \Gamma < n$. Then, by Proposition 8, $H^n(\Lambda, C) = 0$ for each Ω-module C. Since the sequence

$$\Lambda \otimes l^* \to \Lambda \otimes \Lambda^* \to \Lambda \otimes \Gamma^* \to 0$$

is exact and l is nilpotent, it follows that the kernel of the mapping $\Lambda \otimes \Lambda^* \to \Lambda \otimes \Gamma^* = \Omega$ is nilpotent. Thus Proposition 3 implies that $H^n(\Lambda, C) = 0$ for all two sided Λ-modules C, i. e. $\dim \Lambda < n$.

Theorem I is an immediate consequence of Proposition 9.

Proposition 10. If Γ is K-projective and $\Gamma \otimes \Gamma^*$ is semi-simple, then

$$\text{Ext}^n_{\Lambda \otimes \Gamma^*}(B, C) \approx \text{Hom}_{\Gamma \otimes \Gamma^*}(\text{Tor}^\Lambda_n(\Gamma, B), C)$$

for any left Λ- and right Γ-module B and any two-sided Γ-module C.

Proof. We first note the abvious natural isomorphism

$$\operatorname{Hom}_{\Lambda\otimes\Gamma^*}(B, C) \approx \operatorname{Hom}_{\Gamma\otimes\Gamma^*}(\Gamma \otimes_\Lambda B, C)$$

Now let X be a $\Lambda \otimes \Gamma^*$-projective resolution of B. Replacing B by X and passing to homology we obtain

$$H^n(\operatorname{Hom}_{\Lambda\otimes\Gamma^*}(X, C)) \approx H^n(\operatorname{Hom}_{\Gamma\otimes\Gamma^*}(\Gamma \otimes_\Lambda B, C)) .$$

The left hand side is $\operatorname{Ext}^n_{\Lambda\otimes\Gamma^*}(B, C)$. To calculate the right hand side we first observe that since $\Gamma \otimes \Gamma^*$ is semi-simple, the functor $\operatorname{Hom}_{\Gamma\otimes\Gamma^*}$ is exact. Therefore

$$H^n(\operatorname{Hom}_{\Gamma\otimes\Gamma^*}(\Gamma \otimes_\Lambda B, C)) \approx \operatorname{Hom}_{\Gamma\otimes\Gamma^*}(H_n(\Gamma \otimes_\Lambda B), C) .$$

Further, since Γ is K-projective, $\Lambda \otimes \Gamma^*$ is Λ-projective. Consequently X is also a Λ-projective resolution of B and thus $H_n(\Gamma \otimes_\Lambda B) = \operatorname{Tor}^\Lambda_n(\Gamma, B)$. This completes the proof.

Proposition 11. If Λ and Γ are K-projective and Γ and $\Gamma \otimes \Gamma^*$ are semi-simple then

$$H^n(\Lambda, C) \approx \operatorname{Hom}_{\Gamma\otimes\Gamma^*}(\operatorname{Tor}^\Lambda_n(\Gamma, \Gamma), C)$$

for every two sided Γ-module C.

This follows directly from Proposition 8 and Proposition 10 with $B = \Gamma$.

Let γ denote the smallest integer n such that $\operatorname{Tor}^\Lambda_{n+1}(\Gamma, \Gamma) = 0$. If no such integer exists then $\gamma = \infty$.

Proposition 12. If K and $\Gamma \otimes \Gamma^*$ are semi-simple and l is nilpotent then

$$\dim \Lambda = \operatorname{l.dim}_\Lambda \Gamma = \gamma .$$

Proof. The inequality

$$\gamma \leqq \operatorname{l.dim}_\Lambda \Gamma$$

holds without any assumptions. Since K is semi-simple, Corollary 5 implies

$$\operatorname{l.dim}_\Lambda \Gamma \leqq \dim \Lambda .$$

To prove the inequality

$$\dim \Lambda \leqq \gamma$$

assume $\gamma < \infty$ and set $n = \gamma + 1$; then $\operatorname{Tor}^\Lambda_n(\Gamma, \Gamma) = 0$. Next observe that the semi-simplicity of $\Gamma \otimes \Gamma^*$ implies $\dim \Gamma = 0$ and therefore, by Corollary 5, implies the semi-simplicity of Γ. Thus the conditions of Proposition 11 are satisfied and we have $H^n(\Lambda, C) = 0$ for all two sided Γ-modules C. Now we utilize the exact sequence

$$l \otimes \Lambda^* + \Lambda \otimes l^* \to \Lambda \otimes \Lambda^* \to \Gamma \otimes \Gamma^* \to 0 .$$

Since l is nilpotent, it follows that the kernel of $\Lambda \otimes \Lambda^* \to \Gamma \otimes \Gamma^*$ is nilpotent. Thus Proposition 3 implies that $H^n(\Lambda, C) = 0$ for all two sided Λ-modules C. Hence $\dim \Lambda < n$ and the proof is complete.

Theorem II is an immediate consequence of Proposition 12.

4. Idempotents

From now on we assume that Λ is an algebra over a field K with $(\Lambda : K) < \infty$. We denote by N the radical of Λ, set $\Gamma = \Lambda/N$, and denote by $\varphi : \Lambda \to \Gamma$ the natural factorization homomorphism. All Λ-modules will be assumed to be finitely generated.

We shall be concerned with primitive idempotents in Λ. For any two such primitive idempotents e and f the following four conditions are equivalent:

(l.1) the left Λ-modules Λe and Λf are isomorphic,

(r.1) the right Λ-modules $e\Lambda$ and $f\Lambda$ are isomorphic,

(l.2) the left Γ-modules $\Gamma(\varphi e)$ and $\Gamma(\varphi f)$ are isomorphic,

(r.2) the right Γ-modules $(\varphi e)\Gamma$ and $(\varphi f)\Gamma$ are isomorphic.

For the equivalence (l.1)$\Longleftrightarrow$(l.2) see Artin-Nesbitt-Thrall, Rings with Minimum Conditions, p. 99. Analogously (r.1)$\Longleftrightarrow$(r.2). There remains the equivalence (r. 1)$\Longleftrightarrow$(r. 2), or what amounts to the same, the equivalence (l.1)$\Longleftrightarrow$(l.2) for Λ semi-simple. In this case either (l.1) or (l.2) signify that e and f are in the same simple component of Λ.

The primitive idempotents e and f in Λ are said to be *isomorphic* if either of the four conditions listed above is satisfied. A set consisting of one idempotent out of each isomorphism class will be called a *maximal set* (abbreviated for "maximal set of non-isomorphic primitive idempotents").

A *decomposition of unity* is a sequence $e_1, \ldots, e_n$ of mutally orthogonal primitive idempotents such that $e_1 + \cdots + e_n = 1$.

Proposition 13. Each decomposition of unity $e_1, \ldots, e_n$ contains a maximal set.

Proposition 14. If $e_1, \ldots, e_n$ is a maximal set then each projective left Λ-module is a direct sum of modules isomorphic with Λe_i.

We prove both propositions jointly. First let $e_1, \ldots, e_n$ be a decomposition of unity. Then $\Lambda = \Lambda e_1 + \cdots + \Lambda e_n$ is a representation of Λ as a direct sum of indecomposable left Λ-modules. Consequently every free Λ-module F is a direct sum of modules isomorphic with Λe_i. From the Krull-Remak-Schmidt theorem it then follows the same for each direct summand of F, i. e. for each projective left Λ-module.

In particular, for each primitive idempotent f, the left Λ-module Λf is indecomposable and therefore isomorphic with Λe_i for some $i = 1, \ldots, n$. It follows that $e_1, \ldots, e_n$ contains a maximal set. Further for this maximal set the conclusion of Proposition 14 holds. Therefore Proposition 14 holds in general.

Remark. Proposition 14 was established only for finitely generated projective left Λ-modules, and will be used in the sequel in this form only. However, the conclusion is valid for arbitrary projective left Λ-modules as was ingenieously proved by Nagao-Nakayama [5].

Proposition 15. The map $\varphi : \Lambda \to \Gamma$ maps the set of primitive idempotents in Λ onto the set of primitive idempotents of Γ and establishes a $1-1$ correspondence between the isomorphisms classes of primitive idempotents in Λ and in Γ.

Proof. It is known that the image in Γ of a primitive idempotent in Λ is primitive and that each primitive idempotent in Γ is the image of an idempotent in Λ, which must be primitive. The statement concerning isomorphism classes follows from the equivalence (1.1)$\Leftrightarrow$(1.2).

Proposition 16. Let Λ' and Λ'' be algebras such that Γ' and Γ'' are direct products of full matrix algebras over K. If $e'_1, \ldots, e'_m$ and $e''_1, \ldots, e''_n$ are maximal sets in Λ' and Λ'' respectively, then the set $\{e'_i \otimes e''_j\}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$, is a maximal set in the algebra $\Lambda = \Lambda' \otimes \Lambda''$.

Proof. Consider the algebra $\Gamma = \Gamma' \otimes \Gamma''$ and the map $\varphi : \Lambda \to \Gamma$ given by $\varphi = \varphi' \otimes \varphi''$. Then Γ is again a direct product of full matrix algebras and therefore is semi-simple. Since the kernel of φ is nilpotent, it follows that Γ may be regarded as the quotient of Λ by its radical. It follows from Proposition 15 that we may restrict ourselves to the case $\Lambda' = \Gamma'$, $\Lambda'' = \Gamma''$, $\Lambda = \Gamma$. By applying the direct product decompositions of Λ' and Λ'' we further reduce the proof to the case when Λ' and Λ'' are full matrix algebras. Maximal sets in Λ' and Λ'' are then given by single primitive idempotents e' and e'' which may be chosen to be matrices with one unit on the diagonal and zeros elsewhere. Then Λ is again a full matrix algebra and $e = e' \otimes e''$ has a similar form. Thus e is primitive and therefore is a maximal set for Λ.

5. Cartan matrices

Let A be a left Λ'- and right Λ''-module where Λ' and Λ'' are K-algebras. Given maximal sets $e'_1, \ldots, e'_m$ and $e''_1, \ldots, e''_n$ in Λ' and Λ'' respectively we consider the m by n matrix $M(A)$ of integers

$$a_{ij} = (e'_i \, A \, e''_j : K)$$

The isomorphism

$$e'_i \, A \, e''_j \approx e'_i \, \Lambda' \otimes_{\Lambda'} A \otimes_{\Lambda''} \Lambda'' e''_j$$

shows that the matrix does not change if the idempotents are replaced by isomorphic idempotents. Of course a change in the order of the idempotents e'_i (or e''_i) interchanges the rows (or columns) of the matrix. To eliminate this ambiguity it is appropriate the regard the matrix $M(A)$ as indexed by the pairs of isomorphism classes of primitive idempotents.

In particular an algebra Λ may be regarded as a two-sided Λ-module and this leads to the square matrix $M(\Lambda)$ called the *Cartan matrix* of Λ.

We have the following obvious proposition:

Proposition 17. If $\Lambda = \Lambda_1 + \Lambda_2$ is the direct product of algebras Λ_1 and Λ_2 then

$$M(\Lambda) = \begin{vmatrix} M(\Lambda_1) & 0 \\ 0 & M(\Lambda_2) \end{vmatrix}$$

If Λ is a simple algebra, then all primitive idempotents are isomorphic and thus a maximal set consists of one element e. The matrix $M(\Lambda)$ has then order 1 (and indeed consists of the integer $(\Lambda : K)/n^2$ where n is the length of a decomposition of unity in Λ). It is further well known that $(e \, \Lambda \, e : K) = 1$ if and only if Λ is a full matrix algebra over K. This yields

Proposition 18. If Λ is semi-simple then $M(\Lambda)$ is diagonal. Further (assuming Λ semi-simple) $M(\Lambda)$ is the unit matrix if and only if Λ is isomorphic with the direct product of full matrix algebras over K.

Proposition 19. If $0 \to A' \to A \to A'' = 0$ is an exact sequence of left Λ'- and right Λ''-modules then $M(A) \to M(A') + M(A'')$.

This follows readily from the exactness of the sequences

$$0 \to e' A' e'' \to e' A e'' \to e' A'' e'' \to 0$$

for any idempotents $e' \in \Lambda'$, $e'' \in \Lambda''$.

Proposition 20. Let Λ' and Λ'' be algebras such that Γ' and Γ'' are direct products of full matrix algebras. If A is a left Λ'- and right Λ''-module such that

$$\text{l.}\dim_{\Omega} A < \infty \ , \qquad \Omega = \Lambda' \otimes \Lambda''^{*} \ ,$$

then

$$M(A) = M(\Lambda')D'' = D' M(\Lambda'')$$

where D' and D'' are integral matrices.

Proof. Let $\text{l.}\dim_{\Omega} A = n$ with $0 < n < \infty$. Then there exists an exact sequence $0 \to B \to X \to A \to 0$ with X Ω-projective and $\text{l.}\dim_{\Omega} B = n - 1$. By the preceding proposition

$$M(A) = M(X) - M(B) \ .$$

Thus if the conclusion applies to X and B it also applies to A. This reduces the proof to the case $n = 0$ i. e. to the case when A is Ω-projective.

Let $e_1', \ldots, e_m' \in \Lambda'$ and $e_1'', \ldots, e_n'' \in \Lambda''$ be maximal sets in Λ' and Λ''. Then, by Proposition 16, $\{e_i' \otimes e_j''^{*}\}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$, is a maximal set in Ω, and therefore, by Proposition 14, A is isomorphic with a direct sum of modules of the form $\Omega(e_u' \otimes e_v''^{*})$. Thus we may assume that A is one of these modules. Then

$$A = (\Lambda' \otimes \Lambda''^{*})(e_u' \otimes e_v''^{*}) = \Lambda' e_u' \otimes \cdot \Lambda''^{*} e_v''^{*}$$

or if we prefer to regard A as a left Λ'- and right Λ''-module

$$A = \Lambda' e_u' \otimes e_v'' \Lambda'' \ .$$

Consequently

$$\begin{aligned}
(e_i' A e_j'' : K) &= (e_i' \Lambda' e_u' \otimes e_v'' \Lambda'' e_j'' : K) \\
&= (e_i' \Lambda' e_u' : K)(e_v'' \Lambda'' e_j'' : K) \\
&= \sum_k (e_i' \Lambda' e_k' : K)\, \delta_{ku} (e_v'' \Lambda'' e_j'' : K) \\
&= \sum_k (e_i' \Lambda' e_u' : K)\, \delta_{vk} (e_k'' \Lambda'' e_j'' : K) \ .
\end{aligned}$$

Thus $M(A) = M(\Lambda')D'' = D' M(\Lambda'')$, where D' and D'' are the matrices of integers

$$d'_{ij} = (e'_i \Lambda' e'_u : K)\, \delta_{vj}, \qquad d''_{ij} = \delta_{iu}(e''_v \Lambda'' e''_j : K) .$$

The proof is now complete.

6. Proof of Theorem III

Let L be the algebraic closure of the field K. For a K-algebra Λ, we denote by Λ_L the ring $\Lambda \otimes_K L$ regarded as an L-algebra; thus Λ_L is the algebra obtained by extension of the ground field. It is known (see [1], Ch. IX, prop. 7.2) that $\dim \Lambda = \dim \Lambda_L$. Theorem III now follows from the following two propositions

Proposition 21. If $\dim \Lambda < \infty$ then $\det M(\Lambda_L) = \pm 1$.

Proposition 22. If $\det M(\Lambda_L) = \pm 1$ then $\Gamma = \Lambda/N$ is separable.

This last proposition was established by Ikeda, Nagao and Nakayama ([4], Lemma 6). No cohomology theory is involved in the proof which will not be reproduced here.

As for Proposition 21, since $\dim \Lambda = \dim \Lambda_L$, we may assume that K is algebraically closed. Then Γ is a direct product of full matrix algebras. By Theorem I

$$\dim \Lambda = \text{l.}\dim_{\Omega} \Gamma , \qquad \Omega = \Lambda \otimes \Gamma^* .$$

Therefore, if $\dim \Lambda < \infty$ then by Proposition 20

$$M(\Gamma) = M(\Lambda) D$$

where D is a matrix of integers. Here $M(\Gamma)$ is the matrix of Γ regarded as a left Λ- and right Γ-module. However, in view of Proposition 15, this matrix coincides with the Cartan matrix of Γ and therefore is the unit matrix I. Thus $M(\Lambda)D = I$ and the proof is complete.

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