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# Algebras of cohomologically finite dimension

by SAMUEL EILENBERG <sup>1)</sup>

*Dedicated to Heinz Hopf on his 60<sup>th</sup> anniversary*

## 1. Introduction and results

Let  $A$  be an algebra over a field  $K$  with  $(A : K) < \infty$ . The dimension of  $A$  (in the cohomology sense) is defined as the highest integer  $n$  for which the cohomology group  $H^n(A, A)$  is non zero for some two-sided  $A$ -module  $A$ . If no such integer exists, then  $\dim A = \infty$ . Algebras of dimension zero are known to be those which are separable. A very interesting characterization of algebras of dimension  $n$  has been given recently by Ikeda, Nagao and Nakayama [4]. The purpose of this paper is to give a new treatment of the results of [4] within the framework of the Cartan-Eilenberg theory [1]. There result considerable simplifications of the proofs, and partly also a sharpening of the results. Relative cohomology and other ad hoc constructions used in [4] are eliminated.

Let  $A$  be a left  $A$ -module. The *dimension* of  $A$  (notation:  $\text{l. dim}_A A$ ) is defined as the least integer  $n$  for which there exists an exact sequence

$$0 \rightarrow X_n \rightarrow \dots \rightarrow X_0 \rightarrow A \rightarrow 0$$

where the left  $A$ -modules  $X_0, \dots, X_n$  are projective. If no such sequence exists for any  $n$ , then  $\text{l. dim}_A A = \infty$ . The *left global dimension* of  $A$  is

$$\text{l. gl. dim } A = \sup \text{l. dim}_A A$$

for all left  $A$ -modules  $A$ . Using the functors  $\text{Ext}$ , the condition  $\text{l. dim}_A A \leq n$  is equivalent with  $\text{Ext}_A^{n+1}(A, C) = 0$  for all left  $A$ -modules  $C$ , while the condition  $\text{l. gl. dim } A \leq n$  is equivalent with  $\text{Ext}_A^{n+1} = 0$ .

As for two-sided  $A$ -modules  $A$ , the standard procedure will be to convert them into left modules over the algebra  $A^e = A \otimes A^*$  where  $A^*$  is the algebra opposite to  $A$  and where  $\otimes$  stands for the tensor product over  $K$ . Then (by definition)

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<sup>1)</sup> Work done under contract AF-18 (600)-562.

$$H^n(A, A) = \text{Ext}_{A^e}^n(A, A)$$

so that

$$\dim A = 1. \dim_{A^e} A .$$

We shall denote by  $N$  the radical of  $A$  and write  $\Gamma = A/N$ .

With these preliminaries, the main results may be stated.

**Theorem I.**  $\dim A = 1. \text{gl. dim } \Omega = 1. \dim_{\Omega} \Gamma$ , where  $\Omega = A \otimes \Gamma^*$ .

**Theorem II.** If  $\Gamma$  is separable, then  $\dim A = 1. \dim_A \Gamma$ .

**Theorem III.** If  $\dim A < \infty$  then  $\Gamma$  is separable.

Note that if  $K$  has characteristic zero or is algebraically closed, then  $\Gamma$ , being semi-simple, always is separable. Further, the separability of  $\Gamma$  is equivalent with  $\dim \Gamma = 0$  and also is equivalent with the semi-simplicity of  $\Gamma \otimes \Gamma^*$  (see [1], Ch. IX, prop. 7.9 and 7.10).

An immediate consequence of Theorem III is the following result of Hochschild ([3], p. 946):

**Corollary 1.** If  $A$  is semi-simple but inseparable then  $\dim A = \infty$ .

Combining Theorems II and III yields:

**Corollary 2.** In order that  $\dim A = n (n < \infty)$  it is necessary and sufficient that the following conditions hold:

- (1)  $\Gamma$  is separable.
- (2)  $1. \dim_A \Gamma = n$ .

If  $1. \dim_A \Gamma > 0$  then the exact sequence  $0 \rightarrow N \rightarrow A \rightarrow \Gamma \rightarrow 0$  implies ([1], Ch. VI, prop. 2.3) that

$$1. \dim_A \Gamma = 1 + 1. \dim_A N .$$

Thus for  $n > 0$ , condition (2) may be replaced by

$$(2') \quad 1. \dim N = n - 1 .$$

The characterization given by Ikeda-Nagao-Nakayama in [4] (for  $n > 0$ ) utilizes conditions (1) and (2'), except that condition (2') is stated in a more explicit but also more involved form, which however is equivalent. For a proof of this equivalence we refer the reader to Eilenberg-Ikeda-Nakayama [2] where several questions related to this paper are discussed.

Sections 2 and 3 contain a sequence of propositions leading to proofs of Theorem I and II. The technique of proofs fully conforms to the system developed in [1]. It should be noted that the results of sections 2 and 3 are obtained under weaker hypotheses than stated above. Indeed, the assumption that  $(A : K) < \infty$  is dropped and  $K$  need not be a field.

This added generality is abandoned in sections 4, 5 and 6 devoted to the proof of Theorem III.

## 2. Preliminaries

We shall consider algebras  $A$  over a commutative ring  $K$ . We shall have the opportunity to discuss  $K$ -modules, left  $A$ -modules, right  $A$ -modules and two-sided  $A$ -modules. It will always be assumed that  $K$  and  $A$  have unit elements, and that these unit elements act as the identity on all modules.

For each left ideal  $l$  in  $A$  we set  $l^0 = A$  and  $l^n = ll^{n-1}$  for  $n > 0$ . We say that  $l$  is *nilpotent* if  $l^k = 0$  for some integer  $k$ .

**Proposition 3.** Let  $A$  be a left  $A$ -module such that  $\text{Ext}_A^n(A, C) = 0$  for each left  $A$ -module  $C$  with  $lC = 0$ . If  $l$  is nilpotent, then  $\text{Ext}_A^n(A, C) = 0$  for all left  $A$ -modules  $C$ , i. e.  $\text{l. dim}_A A < n$ .

*Proof.* For each integer  $i > 0$  consider the exact sequence

$$0 \rightarrow l^{i+1}C \rightarrow l^iC \rightarrow l^iC/l^{i+1}C \rightarrow 0.$$

Since  $l(l^iC/l^{i+1}C) = 0$  it follows that

$$\text{Ext}_A^n(A, l^iC/l^{i+1}C) = 0$$

and therefore the homomorphism

$$\text{Ext}_A^n(A, l^{i+1}C) \rightarrow \text{Ext}_A^n(A, l^iC)$$

induced by inclusion  $l^{i+1}C \subset l^iC$ , is an epimorphism. Since  $l^0C = C$  and  $l^kC = 0$  for  $k$  sufficiently large, it follows that  $\text{Ext}_A^n(A, C) = 0$ .

In the sequel we shall use the following proposition established in [1] (Ch. IX, prop. 4.3).

**Proposition 4.** Let  $A$  and  $\Gamma$  be  $K$ -algebras where  $K$  is a commutative ring. If  $A$  is  $K$ -projective and  $\Gamma$  is semi-simple then we have the natural isomorphism

$$H^n(A, \text{Hom}_\Gamma(B, C)) \approx \text{Ext}_{A \otimes \Gamma^*}(B, C)$$

for any left  $A$ - and right  $\Gamma$ -modules  $B$  and  $C$ .

We shall need some corollaries (also derived in [1], Ch. IX, § 7). First taking  $\Gamma = K$  we obtain

**Corollary 5.** If  $A$  is a  $K$ -algebra with  $K$  semi-simple then

$$\text{l. gl. dim } A \leq \dim A.$$

Taking  $\Gamma = A$  and noting the inequality

$$\dim A \leq \text{gl. dim } A \otimes A^*$$

we obtain



**Corollary 6.** If the  $K$ -algebra  $A$  is  $K$ -projective and semi-simple then

$$\text{gl. dim } A \otimes A^* = \dim A .$$

Finally combining these two corollaries we obtain

**Corollary 7.** If  $A$  is a  $K$ -algebra with  $K$  semi-simple, then  $\dim A = 0$  if and only if  $A \otimes A^*$  is semi-simple.

### 3. Proofs of Theorems I and II

In this section we shall consider an algebra  $A$  over a commutative ring  $K$ . In  $A$  a two sided ideal  $l$  will be given with  $\Gamma = A/l$ . Every left  $\Gamma$ -module  $A$  will be regarded also as a left  $A$ -module with  $lA = 0$ . Similarly for right and two-sided modules.

**Proposition 8.** If  $A$  is  $K$ -projective and  $\Gamma$  is semi-simple then

$$H^n(A, C) \approx \text{Ext}_{A \otimes \Gamma^*}^n(\Gamma, C)$$

for every left  $A$ - and right  $\Gamma$ -module  $C$ .

This follows directly from Proposition 4 by taking  $B = \Gamma$  and observing that  $C$  is isomorphic with  $\text{Hom}_{\Gamma}(\Gamma, C)$  as a left  $A$ - and right  $\Gamma$ -module.

**Proposition 9.** If  $A$  is  $K$ -projective and  $\Gamma$  is semi-simple then

$$1. \dim_{\Omega} \Gamma \leq 1. \text{ gl. dim } \Omega \leq \dim A , \quad \Omega = A \otimes \Gamma^* .$$

If further  $l$  is nilpotent, then equalities hold.

*Proof.* The first part is an immediate consequence of Proposition 4. To prove the second half assume  $1. \dim_{\Omega} \Gamma < n$ . Then, by Proposition 8,  $H^n(A, C) = 0$  for each  $\Omega$ -module  $C$ . Since the sequence

$$A \otimes l^* \rightarrow A \otimes A^* \rightarrow A \otimes \Gamma^* \rightarrow 0$$

is exact and  $l$  is nilpotent, it follows that the kernel of the mapping  $A \otimes A^* \rightarrow A \otimes \Gamma^* = \Omega$  is nilpotent. Thus Proposition 3 implies that  $H^n(A, C) = 0$  for all two sided  $A$ -modules  $C$ , i. e.  $\dim A < n$ .

Theorem I is an immediate consequence of Proposition 9.

**Proposition 10.** If  $\Gamma$  is  $K$ -projective and  $\Gamma \otimes \Gamma^*$  is semi-simple, then

$$\text{Ext}_{A \otimes \Gamma^*}^n(B, C) \approx \text{Hom}_{\Gamma \otimes \Gamma^*}(\text{Tor}_n^A(\Gamma, B), C)$$

for any left  $A$ - and right  $\Gamma$ -module  $B$  and any two-sided  $\Gamma$ -module  $C$ .

*Proof.* We first note the obvious natural isomorphism

$$\operatorname{Hom}_{\Lambda \otimes \Gamma^*}(B, C) \approx \operatorname{Hom}_{\Gamma \otimes \Gamma^*}(\Gamma \otimes_{\Lambda} B, C)$$

Now let  $X$  be a  $\Lambda \otimes \Gamma^*$ -projective resolution of  $B$ . Replacing  $B$  by  $X$  and passing to homology we obtain

$$H^n(\operatorname{Hom}_{\Lambda \otimes \Gamma^*}(X, C)) \approx H^n(\operatorname{Hom}_{\Gamma \otimes \Gamma^*}(\Gamma \otimes_{\Lambda} B, C)) .$$

The left hand side is  $\operatorname{Ext}_{\Lambda \otimes \Gamma^*}^n(B, C)$ . To calculate the right hand side we first observe that since  $\Gamma \otimes \Gamma^*$  is semi-simple, the functor  $\operatorname{Hom}_{\Gamma \otimes \Gamma^*}$  is exact. Therefore

$$H^n(\operatorname{Hom}_{\Gamma \otimes \Gamma^*}(\Gamma \otimes_{\Lambda} B, C)) \approx \operatorname{Hom}_{\Gamma \otimes \Gamma^*}(H_n(\Gamma \otimes_{\Lambda} B), C) .$$

Further, since  $\Gamma$  is  $K$ -projective,  $\Lambda \otimes \Gamma^*$  is  $\Lambda$ -projective. Consequently  $X$  is also a  $\Lambda$ -projective resolution of  $B$  and thus  $H_n(\Gamma \otimes_{\Lambda} B) = \operatorname{Tor}_n^{\Lambda}(\Gamma, B)$ . This completes the proof.

**Proposition 11.** If  $\Lambda$  and  $\Gamma$  are  $K$ -projective and  $\Gamma$  and  $\Gamma \otimes \Gamma^*$  are semi-simple then

$$H^n(\Lambda, C) \approx \operatorname{Hom}_{\Gamma \otimes \Gamma^*}(\operatorname{Tor}_n^{\Lambda}(\Gamma, \Gamma), C)$$

for every two sided  $\Gamma$ -module  $C$ .

This follows directly from Proposition 8 and Proposition 10 with  $B = \Gamma$ .

Let  $\gamma$  denote the smallest integer  $n$  such that  $\operatorname{Tor}_{n+1}^{\Lambda}(\Gamma, \Gamma) = 0$ . If no such integer exists then  $\gamma = \infty$ .

**Proposition 12.** If  $K$  and  $\Gamma \otimes \Gamma^*$  are semi-simple and  $l$  is nilpotent then

$$\dim \Lambda = l \cdot \dim_{\Lambda} \Gamma = \gamma .$$

*Proof.* The inequality

$$\gamma \leq l \cdot \dim_{\Lambda} \Gamma$$

holds without any assumptions. Since  $K$  is semi-simple, Corollary 5 implies

$$l \cdot \dim_{\Lambda} \Gamma \leq \dim \Lambda .$$

To prove the inequality

$$\dim \Lambda \leq \gamma$$

assume  $\gamma < \infty$  and set  $n = \gamma + 1$ ; then  $\operatorname{Tor}_n^{\Lambda}(\Gamma, \Gamma) = 0$ . Next observe that the semi-simplicity of  $\Gamma \otimes \Gamma^*$  implies  $\dim \Gamma = 0$  and therefore, by Corollary 5, implies the semi-simplicity of  $\Gamma$ . Thus the

conditions of Proposition 11 are satisfied and we have  $H^n(\Lambda, C) = 0$  for all two sided  $\Gamma$ -modules  $C$ . Now we utilize the exact sequence

$$l \otimes \Lambda^* + \Lambda \otimes l^* \rightarrow \Lambda \otimes \Lambda^* \rightarrow \Gamma \otimes \Gamma^* \rightarrow 0 .$$

Since  $l$  is nilpotent, it follows that the kernel of  $\Lambda \otimes \Lambda^* \rightarrow \Gamma \otimes \Gamma^*$  is nilpotent. Thus Proposition 3 implies that  $H^n(\Lambda, C) = 0$  for all two sided  $\Lambda$ -modules  $C$ . Hence  $\dim \Lambda < n$  and the proof is complete.

Theorem II is an immediate consequence of Proposition 12.

#### 4. Idempotents

From now on we assume that  $\Lambda$  is an algebra over a field  $K$  with  $(\Lambda : K) < \infty$ . We denote by  $N$  the radical of  $\Lambda$ , set  $\Gamma = \Lambda/N$ , and denote by  $\varphi : \Lambda \rightarrow \Gamma$  the natural factorization homomorphism. All  $\Lambda$ -modules will be assumed to be finitely generated.

We shall be concerned with primitive idempotents in  $\Lambda$ . For any two such primitive idempotents  $e$  and  $f$  the following four conditions are equivalent :

- (l.1) the left  $\Lambda$ -modules  $\Lambda e$  and  $\Lambda f$  are isomorphic,
- (r.1) the right  $\Lambda$ -modules  $e\Lambda$  and  $f\Lambda$  are isomorphic,
- (l.2) the left  $\Gamma$ -modules  $\Gamma(\varphi e)$  and  $\Gamma(\varphi f)$  are isomorphic,
- (r.2) the right  $\Gamma$ -modules  $(\varphi e)\Gamma$  and  $(\varphi f)\Gamma$  are isomorphic.

For the equivalence (l.1)  $\Leftrightarrow$  (l.2) see Artin-Nesbitt-Thrall, Rings with Minimum Conditions, p. 99. Analogously (r.1)  $\Leftrightarrow$  (r.2). There remains the equivalence (r.1)  $\Leftrightarrow$  (r.2), or what amounts to the same, the equivalence (l.1)  $\Leftrightarrow$  (l.2) for  $\Lambda$  semi-simple. In this case either (l.1) or (l.2) signify that  $e$  and  $f$  are in the same simple component of  $\Lambda$ .

The primitive idempotents  $e$  and  $f$  in  $\Lambda$  are said to be *isomorphic* if either of the four conditions listed above is satisfied. A set consisting of one idempotent out of each isomorphism class will be called a *maximal set* (abbreviated for "maximal set of non-isomorphic primitive idempotents").

A *decomposition of unity* is a sequence  $e_1, \dots, e_n$  of mutually orthogonal primitive idempotents such that  $e_1 + \dots + e_n = 1$ .

**Proposition 13.** Each decomposition of unity  $e_1, \dots, e_n$  contains a maximal set.

**Proposition 14.** If  $e_1, \dots, e_n$  is a maximal set then each projective left  $\Lambda$ -module is a direct sum of modules isomorphic with  $\Lambda e_i$ .

We prove both propositions jointly. First let  $e_1, \dots, e_n$  be a decom-

position of unity. Then  $A = Ae_1 + \dots + Ae_n$  is a representation of  $A$  as a direct sum of indecomposable left  $A$ -modules. Consequently every free  $A$ -module  $F$  is a direct sum of modules isomorphic with  $Ae_i$ . From the Krull-Remak-Schmidt theorem it then follows the same for each direct summand of  $F$ , i. e. for each projective left  $A$ -module.

In particular, for each primitive idempotent  $f$ , the left  $A$ -module  $Af$  is indecomposable and therefore isomorphic with  $Ae_i$  for some  $i = 1, \dots, n$ . It follows that  $e_1, \dots, e_n$  contains a maximal set. Further for this maximal set the conclusion of Proposition 14 holds. Therefore Proposition 14 holds in general.

*Remark.* Proposition 14 was established only for finitely generated projective left  $A$ -modules, and will be used in the sequel in this form only. However, the conclusion is valid for arbitrary projective left  $A$ -modules as was ingeniously proved by Nagao-Nakayama [5].

**Proposition 15.** The map  $\varphi: A \rightarrow \Gamma$  maps the set of primitive idempotents in  $A$  onto the set of primitive idempotents of  $\Gamma$  and establishes a 1 — 1 correspondence between the isomorphism classes of primitive idempotents in  $A$  and in  $\Gamma$ .

*Proof.* It is known that the image in  $\Gamma$  of a primitive idempotent in  $A$  is primitive and that each primitive idempotent in  $\Gamma$  is the image of an idempotent in  $A$ , which must be primitive. The statement concerning isomorphism classes follows from the equivalence (1.1) $\Leftrightarrow$ (1.2).

**Proposition 16.** Let  $A'$  and  $A''$  be algebras such that  $\Gamma'$  and  $\Gamma''$  are direct products of full matrix algebras over  $K$ . If  $e'_1, \dots, e'_m$  and  $e''_1, \dots, e''_n$  are maximal sets in  $A'$  and  $A''$  respectively, then the set  $\{e'_i \otimes e''_j\}$ ,  $i = 1, \dots, m$ ,  $j = 1, \dots, n$ , is a maximal set in the algebra  $A = A' \otimes A''$ .

*Proof.* Consider the algebra  $\Gamma = \Gamma' \otimes \Gamma''$  and the map  $\varphi: A \rightarrow \Gamma$  given by  $\varphi = \varphi' \otimes \varphi''$ . Then  $\Gamma$  is again a direct product of full matrix algebras and therefore is semi-simple. Since the kernel of  $\varphi$  is nilpotent, it follows that  $\Gamma$  may be regarded as the quotient of  $A$  by its radical. It follows from Proposition 15 that we may restrict ourselves to the case  $A' = \Gamma'$ ,  $A'' = \Gamma''$ ,  $A = \Gamma$ . By applying the direct product decompositions of  $A'$  and  $A''$  we further reduce the proof to the case when  $A'$  and  $A''$  are full matrix algebras. Maximal sets in  $A'$  and  $A''$  are then given by single primitive idempotents  $e'$  and  $e''$  which may be chosen to be matrices with one unit on the diagonal and zeros elsewhere. Then  $A$  is again a full

matrix algebra and  $e = e' \otimes e''$  has a similar form. Thus  $e$  is primitive and therefore is a maximal set for  $A$ .

## 5. Cartan matrices

Let  $A$  be a left  $A'$ - and right  $A''$ -module where  $A'$  and  $A''$  are  $K$ -algebras. Given maximal sets  $e'_1, \dots, e'_m$  and  $e''_1, \dots, e''_n$  in  $A'$  and  $A''$  respectively we consider the  $m$  by  $n$  matrix  $M(A)$  of integers

$$a_{ij} = (e'_i A e''_j : K)$$

The isomorphism

$$e'_i A e''_j \approx e'_i A' \otimes_{A'} A \otimes_{A''} A'' e''_j$$

shows that the matrix does not change if the idempotents are replaced by isomorphic idempotents. Of course a change in the order of the idempotents  $e'_i$  (or  $e''_i$ ) interchanges the rows (or columns) of the matrix. To eliminate this ambiguity it is appropriate to regard the matrix  $M(A)$  as indexed by the pairs of isomorphism classes of primitive idempotents.

In particular an algebra  $A$  may be regarded as a two-sided  $A$ -module and this leads to the square matrix  $M(A)$  called the *Cartan matrix* of  $A$ .

We have the following obvious proposition :

**Proposition 17.** If  $A = A_1 + A_2$  is the direct product of algebras  $A_1$  and  $A_2$  then

$$M(A) = \begin{vmatrix} M(A_1) & 0 \\ 0 & M(A_2) \end{vmatrix}$$

If  $A$  is a simple algebra, then all primitive idempotents are isomorphic and thus a maximal set consists of one element  $e$ . The matrix  $M(A)$  has then order 1 (and indeed consists of the integer  $(A : K)/n^2$  where  $n$  is the length of a decomposition of unity in  $A$ ). It is further well known that  $(e A e : K) = 1$  if and only if  $A$  is a full matrix algebra over  $K$ . This yields

**Proposition 18.** If  $A$  is semi-simple then  $M(A)$  is diagonal. Further (assuming  $A$  semi-simple)  $M(A)$  is the unit matrix if and only if  $A$  is isomorphic with the direct product of full matrix algebras over  $K$ .

**Proposition 19.** If  $0 \rightarrow A' \rightarrow A \rightarrow A'' = 0$  is an exact sequence of left  $A'$ - and right  $A''$ -modules then  $M(A) \rightarrow M(A') + M(A'')$ .

This follows readily from the exactness of the sequences

$$0 \rightarrow e' A' e'' \rightarrow e' A e'' \rightarrow e' A'' e'' \rightarrow 0$$

for any idempotents  $e' \in A'$ ,  $e'' \in A''$ .

**Proposition 20.** Let  $A'$  and  $A''$  be algebras such that  $\Gamma'$  and  $\Gamma''$  are direct products of full matrix algebras. If  $A$  is a left  $A'$ - and right  $A''$ -module such that

$$1. \dim_{\Omega} A < \infty, \quad \Omega = A' \otimes A''^*,$$

then

$$M(A) = M(A')D'' = D'M(A'')$$

where  $D'$  and  $D''$  are integral matrices.

*Proof.* Let  $1. \dim_{\Omega} A = n$  with  $0 < n < \infty$ . Then there exists an exact sequence  $0 \rightarrow B \rightarrow X \rightarrow A \rightarrow 0$  with  $X$   $\Omega$ -projective and  $1. \dim_{\Omega} B = n - 1$ . By the preceding proposition

$$M(A) = M(X) - M(B).$$

Thus if the conclusion applies to  $X$  and  $B$  it also applies to  $A$ . This reduces the proof to the case  $n = 0$  i. e. to the case when  $A$  is  $\Omega$ -projective.

Let  $e'_1, \dots, e'_m \in A'$  and  $e''_1, \dots, e''_n \in A''$  be maximal sets in  $A'$  and  $A''$ . Then, by Proposition 16,  $\{e'_i \otimes e''_j^*\}$ ,  $i = 1, \dots, m$ ,  $j = 1, \dots, n$ , is a maximal set in  $\Omega$ , and therefore, by Proposition 14,  $A$  is isomorphic with a direct sum of modules of the form  $\Omega(e'_u \otimes e''_v^*)$ . Thus we may assume that  $A$  is one of these modules. Then

$$A = (A' \otimes A''^*)(e'_u \otimes e''_v^*) = A' e'_u \otimes A''^* e''_v^*$$

or if we prefer to regard  $A$  as a left  $A'$ - and right  $A''$ -module

$$A = A' e'_u \otimes e''_v A''.$$

Consequently

$$\begin{aligned} (e'_i A e''_j : K) &= (e'_i A' e'_u \otimes e''_v A'' e''_j : K) \\ &= (e'_i A' e'_u : K)(e''_v A'' e''_j : K) \\ &= \sum_k (e'_i A' e'_k : K) \delta_{ku} (e''_v A'' e''_j : K) \\ &= \sum_k (e'_i A' e'_u : K) \delta_{vk} (e''_k A'' e''_j : K). \end{aligned}$$

Thus  $M(A) = M(A')D'' = D'M(A'')$ , where  $D'$  and  $D''$  are the ma-

trices of integers

$$d'_{ij} = (e'_i \Lambda' e'_u : K) \delta_{vj}, \quad d''_{ij} = \delta_{iu} (e''_v \Lambda'' e''_j : K) .$$

The proof is now complete.

## 6. Proof of Theorem III

Let  $L$  be the algebraic closure of the field  $K$ . For a  $K$ -algebra  $\Lambda$ , we denote by  $\Lambda_L$  the ring  $\Lambda \otimes_K L$  regarded as an  $L$ -algebra; thus  $\Lambda_L$  is the algebra obtained by extension of the ground field. It is known (see [1], Ch. IX, prop. 7.2) that  $\dim \Lambda = \dim \Lambda_L$ . Theorem III now follows from the following two propositions

**Proposition 21.** If  $\dim \Lambda < \infty$  then  $\det M(\Lambda_L) = \pm 1$ .

**Proposition 22.** If  $\det M(\Lambda_L) = \pm 1$  then  $\Gamma = \Lambda/N$  is separable.

This last proposition was established by Ikeda, Nagao and Nakayama ([4], Lemma 6). No cohomology theory is involved in the proof which will not be reproduced here.

As for Proposition 21, since  $\dim \Lambda = \dim \Lambda_L$ , we may assume that  $K$  is algebraically closed. Then  $\Gamma$  is a direct product of full matrix algebras. By Theorem I

$$\dim \Lambda = l \cdot \dim_{\Omega} \Gamma, \quad \Omega = \Lambda \otimes \Gamma^* .$$

Therefore, if  $\dim \Lambda < \infty$  then by Proposition 20

$$M(\Gamma) = M(\Lambda)D$$

where  $D$  is a matrix of integers. Here  $M(\Gamma)$  is the matrix of  $\Gamma$  regarded as a left  $\Lambda$ - and right  $\Gamma$ -module. However, in view of Proposition 15, this matrix coincides with the Cartan matrix of  $\Gamma$  and therefore is the unit matrix  $I$ . Thus  $M(\Lambda)D = I$  and the proof is complete.

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