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On admissibility of sequences and a theorem of Pólya

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1. Introduction. Le K be the space of entire functions which obey the growth conditions $f(z) = O(e^{Ar})$, $f(iy) = O(e^{c|y|})$ for some $A < \infty$ and $c < \pi$. By a theorem of Carlson [7] any such function is completely determined by its values at the positive integers. A sequence $\{a_n\}$ of complex numbers is said to be admissible for the sequence of functionals T_n and the function space C if there exists $f \in C$ such that $T_n(f) = a_n$ for $n = 0,1,\ldots$. [3] For the functionals $T_n(f) = f(n)$ and $C = K$, admissibility is a delicate property; if one term of an admissible sequence is altered the result is inadmissible. More generally, if two sequences agree except for a non-void set of indices of density zero, only one can be admissible. A necessary and sufficient condition for admissibility in this case has been given. [Buck 2, Theorem 2.3.] The present paper deals with the closely related problem of admissibility for the functionals $T_n^*(f) = \Delta^n f(0)$ and the class K. Since $T_n^* = (-1)^n \sum_0^n \binom{n}{k} (-1)^k T_k$ and $T_n = \sum_0^n \binom{n}{k} T_k^*$, a sequence $\{a_n\}$ is admissible for $\{T_n\}$ if and only if the sequence $b_n = \Delta^n a_0$ is admissibility $\{T_n^*\}$. In replacing $\{T_n\}$ by $\{T_n^*\}$ much is gained. Admissibility no longer depends as much on the precise structure of a sequence but rather on matters of size and angular distribution. For this reason, it is much easier to discuss many questions relative to $\{T_n\}$ admissibility in terms of $\{T_n^*\}$. This approach has been used with success in the characterization problem for integral-valued entire functions. [Buck, 5] In the present paper, we discuss several other applications. In particular, Theorem 2 answers a number of questions raised by the "even difference" theorem of Agnew and Fuchs. The last section, which is somewhat independent, contains a new and extremely brief proof of the classical theorem of Pólya on functions of zero type.

2. Admissibility $\{T_n^*\}$. The first theorem gives a convenient necessary and sufficient condition for the $\{T_n^*\}$ admissibility of a sequence $\{b_n\}$.

Since the proof follows closely the pattern of that for the corresponding theorem for $\{T_n\}$, we omit most of the details. (See [2].)

Theorem 1. Given a complex sequence $\{b_n\}$, let $g(z) = \sum_0^\infty b_n z^n$, Then:

(i) $\{b_n\}$ is admissible $\{T_n^*\}$ if and only if g is analytic at zero and can be continued to the interval $-1 \leqq x \leqq 0$.

(ii) when $\{b_n\}$ is admissible, the interpolating function f for which $\Delta^n f(0) = b_n$ is given by

$$f(z) = (ML) - \sum_0^\infty \binom{z}{n} b_n \tag{2.1}$$

where $\binom{z}{n} = z(z-1)\cdots(z-n+1)/n!$ and ML denotes Mittag-Leffler summability.

(iii) $g(z)$ is entire if and only if $f(z)$ is of zero type

(iv) if $g(z)$ is a polynomial of degree N, so is $f(z)$, and conversely.

Proof. If $f \in K$ then $g(z) = \sum \Delta^n f(0) z^n$ is given by

$$g(z) = \frac{1}{2\pi i} \int_\Gamma \Phi(w)\,[1 - (e^w - 1)z]^{-1}\,dw \tag{2.2}$$

where $\Phi(w)$ is the Borel transform of f and Γ encloses the indicator set $D(f)$ of $f \cdot$ [4] [2]. Let E be the image of the boundary of $D(f)$ under the map $w \to (e^w - 1)^{-1}$. $g(z)$ is then analytic at zero and may be continued via (2.2) to the component of the complement of E which contains zero; this set in particular contains the interval $[-1, 0]$. If f is of zero type, $D(f)$ is the origin, and g is entire. Conversely, let $g(z) = \sum b_n z^n$ be analytic on $[-1, 0]$ and consider the function $f(z)$ defined by

$$f(z) = \frac{1}{2\pi i} \int_\Gamma t^{-1} g(t)\,[(1+t)/t]^z\,dt \quad .$$

where Γ is now a path enclosing the interval $[-1, 0]$. Calculation shows that $f \in K$ and that $\Delta^n f(0) = b_n$. Moreover, if g is entire, f is of zero type. Statement (ii) follows from a known theorem concerning the expansion of functions into Newton series [Buck 4, Theorem 4.3] and implies (iv) immediately. As an illustration, all "small" sequences are admissible $\{T_n^*\}$. [18, p. 52, Thm. 10].

Corollary 1. If lim sup $|b_n|^{1/n} < 1$, $\{b_n\}$ is admissible $\{T_n^*\}$.

Any theorem connecting the presence of singular points of a power series with its coefficients may be used to yield characterization theorems for sequences $\{b_n\}$ and in turn for $\{a_n\}$. At this point we insert a generalization of the familiar theorem concerning power series with positive coefficients. [15, p. 215].

Lemma. Let $\limsup |c_n|^{1/n} = 1/R$. Let $S_n(z) = \sum_{0}^{n} c_k z^k$ and suppose that there is a sequence of points z_j approaching a point $\beta = Re^{i\Theta}$ from outside the circle $|z| = R$ such that for each j, $S_n(z_j)$ approaches the point at infinity in an angle of openig less than π. Then, β is a singular point for $f(z) = \sum c_n z^n$.

If $f(z)$ is regular at β, it is regular in a circular neighborhood N of β and $(ML) - \lim S_n(z) = f(z)$ for all z in N. But, N contains a point z_j, and since Mittag-Leffler summability is totally regular, $(ML) - \lim S_n(z_j) = \infty$.

We note that Borel summability could have been used in place of Mittag-Leffler, if N is slightly modified; also, the same method yields an analogous result for Dirichlet series and for Laplace transforms. In the classical case, $c_n \geqq 0$ so that $\lim S_n(x) = +\infty$ for all $x > R$.

Corollary 2. Let $\{c_n\}$ be a complex sequence with $\limsup |c_n|^{1/n} = 1$, and obeying the condition described in the lemma, with $R = 1$. Then, the sequence $b_n = (-1)^n c_n$ is not admissible $\{T_n^*\}$.

Corollary 3. If $\limsup |b_n|^{1/n} \geqq 1$ but $(-1)^n b_n \geqq 0$, $\{b_n\}$ is not admissible $\{T_n^*\}$.

For $b_n = \Delta^n a_0$, the oscillation conditions $(-1)^n b_n \geqq 0$ and $(-1)^n a_n \geqq 0$ are closely related; in fact, the latter implies the former. In particular, we obtain again the following theorem for $\{T_n\}$ admissibility [2, Theorem 4.1].

Corollary 4. If $a_0 \neq 0$ and $(-1)^n a_n \geqq 0$, then there is no function $f \in K$ such that $f(n) = a_n$ for $n = 0, 1, \ldots$.

Similarly, Corollary 2 could be turned into a somewhat complicated theorem concerned with admissibility $\{T_n\}$. Some growth condition in Corollary 2 is needed since $b_n = (-1)^n 2^{-n}$ is an oscillating sequence, achieved by the function 2^{-z}. The condition given says essentially that infinitely many of the terms b_n are "large", for example, bounded away from zero.

3. Vanishing differences. Agnew [1] proved that if $\{a_n\}$ is a bounded sequence such that $\Delta^n a_0 = 0$ for $n = 0, 2, 4, \ldots$, then $a_n = 0$ for all $n = 0, 1, \ldots$. Pollard [11] gave a different proof of this and assuming that $a_n = O(n^r)$ proved that $a_n = f(n)$ where $f(z)$ is a polynomial.

Fuchs [6] approached the problem from a different direction and proved the theorem with a weakened assumption on the set of n for which $\Delta^n a_0 = 0$, which in fact was shown to be best possible. We consider the effect of relaxing both this condition and the growth restriction on $\{a_n\}$. We use a simple relation connecting the functions $g(z) = \sum_0^\infty b_k z^k$ and $F(z) = \sum_0^\infty a_n z^n$, namely

$$g(z) = (1+z)^{-1} F(z/(1+z)) . \tag{3.1}$$

This is easily established, assuming that F is analytic at the origin. (See also [11].)

Theorem 2. Let $\limsup |a_n|^{1/n} \leqq 1$ and let $\Delta^n a_0 = 0$ for $n \in A$, a set of integers of density d. If $d > \frac{1}{3}$, then the serie $\sum_0^\infty \binom{z}{n} \Delta^n a_0$ converges for all z to a function $f(z)$ of exponential type not exceeding $\log(1 + 2\cos\pi d)$ such that $f(n) = a_n$ for $n = 0, 1, 2, \ldots$. In particular, if $d \geqq \frac{1}{2}$, f is of type zero. The value $\frac{1}{3}$ as a lower bound for d is best possible.

Proof. $F(t) = \sum_0^\infty a_n t^n$ is regular for $|t| < 1$ so that by (3.1) $g(z) = \sum_0^\infty \Delta^n a_0 z^n$ is regular in the half plane $x > -\frac{1}{2}$. Let the radius of convergence of this series be R. By Pólya's density theorem for power series [12], every arc of $|z| = R$ of opening $2\pi(1-\mathrm{d})$ contains a singularity of $g(z)$. Combining these, we see that if $d > \frac{1}{3}$ then $R > 1$, and by Corollary 1, $b_n = \Delta^n a_0$ is admissible $\{T_n^*\}$; it then follows that $\{a_n\}$ is admissible $\{T_n\}$ and is therefore the sequence $\{f(n)\}$ for a unique function $f \in K$. When $d \geqq \frac{1}{2}$. R is infinite, $g(z)$ is entire and by Theorem 1, f is of zero type. If $\frac{1}{3} < d < \frac{1}{2}$, $R \geqq (2\cos\pi d)^{-1}$ and further calculation shows that the function $\Phi(w)$ of (2.2) is regular at least for $|w| > \log(1 + R^{-1})$. The type of f does not exceed this value, and in particular is less than log 2, so that the Newton series (2.1) is in fact convergent to $f(z)$ [18, p. 52, Thm. 10]. That the number $1/3$ cannot be improved follows from the fact that the sequence $\{b_n\}$ defined by $\sum_0^\infty b_n z^n = (z-1)/(z+1)(z^3-1)$ is not admissible $\{T_n^*\}$ so that the corresponding sequence $\{a_n\}$ has vanishing differences of density $\frac{1}{3}$, obeys the growth condition $a_n = 0(1)$, but is not admissible $\{T_n\}$.

The effect of the weakened growth condition $\limsup |a_n|^{1/n} \leqq 1$ is striking; in contrast with the Agnew-Fuchs result, d may exceed $\frac{1}{2}$ and may in fact be 1, without $f(z)$ being a polynomial. Witness for example $f(z) = \sum_0^\infty \binom{z}{n} (1/n!)$. In this connection, the following more detailed information may be of interest. As we have seen, the region of regularity of $g(z)$ restricts the rate of growth of $f(z)$, and when $g(z)$ is entire, $f(z)$ is of growth at most order 1, type 0. In this case, the rate of growth of g might be expected to impose further restrictions.

Theorem 3. Let $g(z) = \sum_0^\infty \Delta^n f(0) z^n$, where $f \in K$. If g is analytic in $|z| < R$ and $R > 1$, f is (at most) of order 1 type $\log(1 + R^{-1})$; if g is entire, and of infinite order, f is of order 1 type 0; if g is of finite oder ϱ, f is of order $\varrho/(1+\varrho)$.

Since the first two statements have already been discussed, we prove only the last. From (3.1) $F(z) = \sum f(n) z^n = (1-z)^{-1} g(z/(1-z))$. If $\zeta = 1/(1-z)$ this may be written as $\zeta g(\zeta - 1)$ which is of order ϱ as a function of ζ. By a theorem of Whittaker and Wilson [17] f is of order $\varrho/(1+\varrho)$.

For the specific example, $f(z) = \sum \binom{z}{n} (1/n!)$ g is of infinite order and f of zero type. In contrast, $\sum \binom{z}{n} (1/n^2!)$ is of order $\frac{1}{2}$ with d again 1.

4. The Theorem of Pólya. The theorem in question is the following [13]:

Theorem 4. Let $f(z)$ be of order 1 type 0 and suppose that $f(n) = O(1)$ for $n = 0, 1, -1, 2, -2, \ldots$. Then, f is constant.

Many proofs of this have been given since it was first proposed. (See Szego [14], Tschakaloff [16], Paley and Wiener [10, p. 81], Levinson [9, p. 127], Korevaar [8]). The following proof is new and has the virtue of extreme simplicity, involving no interpolation series or delicate growth estimates. We make the initial observation, as in [8], than nothing is lost by the assumption that $\sum_{-\infty}^{\infty} |f(n)| < \infty$. Since f is of zero type, $g(z)$ is entire and is given by (2.2). Expanding the kernel $[1 - (e^w - 1)z]^{-1}$, we have

$$g(z) = \frac{1}{2\pi i} \int_\Gamma \Phi(w) (1+z)^{-1} \sum_0^\infty e^{nw} [z/(1+z)]^n \, dw$$

$$= (1 + z^{-1}) \sum_0^\infty f(n) [z/(1+z)]^n$$

valid for $|z| < |1+z|$. (This is also another verification of (3.1).) From our assumption on $\sum |f(n)|$, $g(z)$ is bounded in the half plane $x > -\frac{1}{2}$. Expanding the kernel in the opposite fashion,

$$g(z) = \frac{1}{2\pi i} \int_{\Gamma} \Phi(w) \, (-1/z) \sum_{0}^{\infty} e^{-(n+1)w} [(z+1)/z]^n \, dw$$

$$= (-1/z) \sum_{0}^{\infty} f(-n-1) \, [(1+z)/z]^n$$

valid for $|1+z| < |z|$. Again, $g(z)$ is bounded in the half plane $x < -\frac{1}{2}$. Combining these, g and hence f is constant.

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