

<b>Zeitschrift:</b>	Commentarii Mathematici Helvetici
<b>Herausgeber:</b>	Schweizerische Mathematische Gesellschaft
<b>Band:</b>	27 (1953)
<b>Artikel:</b>	On admissibility of sequences and a theorem of Pólya.
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<b>DOI:</b>	<a href="https://doi.org/10.5169/seals-21887">https://doi.org/10.5169/seals-21887</a>

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# On admissibility of sequences and a theorem of Pólya

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**1. Introduction.** Let  $K$  be the space of entire functions which obey the growth conditions  $f(z) = O(e^{Ar})$ ,  $f(iy) = O(e^{c|y|})$  for some  $A < \infty$  and  $c < \pi$ . By a theorem of Carlson [7] any such function is completely determined by its values at the positive integers. A sequence  $\{a_n\}$  of complex numbers is said to be admissible for the sequence of functionals  $T_n$  and the function space  $C$  if there exists  $f \in C$  such that  $T_n(f) = a_n$  for  $n = 0, 1, \dots$  [3]. For the functionals  $T_n(f) = f(n)$  and  $C = K$ , admissibility is a delicate property; if one term of an admissible sequence is altered the result is inadmissible. More generally, if two sequences agree except for a non-void set of indices of density zero, only one can be admissible. A necessary and sufficient condition for admissibility in this case has been given. [Buck 2, Theorem 2.3.] The present paper deals with the closely related problem of admissibility for the functionals  $T_n^*(f) = \Delta^n f(0)$  and the class  $K$ . Since  $T_n^* = (-1)^n \sum_0^n \binom{n}{k} (-1)^k T_k$  and  $T_n = \sum_0^n \binom{n}{k} T_k^*$ , a sequence  $\{a_n\}$  is admissible for  $\{T_n\}$  if and only if the sequence  $b_n = \Delta^n a_0$  is admissible for  $\{T_n^*\}$ . In replacing  $\{T_n\}$  by  $\{T_n^*\}$  much is gained. Admissibility no longer depends as much on the precise structure of a sequence but rather on matters of size and angular distribution. For this reason, it is much easier to discuss many questions relative to  $\{T_n\}$  admissibility in terms of  $\{T_n^*\}$ . This approach has been used with success in the characterization problem for integral-valued entire functions. [Buck, 5] In the present paper, we discuss several other applications. In particular, Theorem 2 answers a number of questions raised by the “even difference” theorem of Agnew and Fuchs. The last section, which is somewhat independent, contains a new and extremely brief proof of the classical theorem of Pólya on functions of zero type.

**2. Admissibility  $\{T_n^*\}$ .** The first theorem gives a convenient necessary and sufficient condition for the  $\{T_n^*\}$  admissibility of a sequence  $\{b_n\}$ .

Since the proof follows closely the pattern of that for the corresponding theorem for  $\{T_n\}$ , we omit most of the details. (See [2].)

**Theorem 1.** Given a complex sequence  $\{b_n\}$ , let  $g(z) = \sum_0^{\infty} b_n z^n$ , Then:

(i)  $\{b_n\}$  is admissible  $\{T_n^*\}$  if and only if  $g$  is analytic at zero and can be continued to the interval  $-1 \leq x \leq 0$ .

(ii) when  $\{b_n\}$  is admissible, the interpolating function  $f$  for which  $\Delta^n f(0) = b_n$  is given by

$$f(z) = (ML) - \sum_0^{\infty} \binom{z}{n} b_n \quad (2.1)$$

where  $\binom{z}{n} = z(z-1) \cdots (z-n+1)/n!$  and  $ML$  denotes Mittag-Leffler summability.

(iii)  $g(z)$  is entire if and only if  $f(z)$  is of zero type

(iv) if  $g(z)$  is a polynomial of degree  $N$ , so is  $f(z)$ , and conversely.

. *Proof.* If  $f \in K$  then  $g(z) = \sum \Delta^n f(0) z^n$  is given by

$$g(z) = \frac{1}{2\pi i} \int_{\Gamma} \Phi(w) [1 - (e^w - 1)z]^{-1} dw \quad (2.2)$$

where  $\Phi(w)$  is the Borel transform of  $f$  and  $\Gamma$  encloses the indicator set  $D(f)$  of  $f$  [4] [2]. Let  $E$  be the image of the boundary of  $D(f)$  under the map  $w \rightarrow (e^w - 1)^{-1}$ .  $g(z)$  is then analytic at zero and may be continued via (2.2) to the component of the complement of  $E$  which contains zero; this set in particular contains the interval  $[-1, 0]$ . If  $f$  is of zero type,  $D(f)$  is the origin, and  $g$  is entire. Conversely, let  $g(z) = \sum b_n z^n$  be analytic on  $[-1, 0]$  and consider the function  $f(z)$  defined by

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma} t^{-1} g(t) [(1+t)/t]^z dt$$

where  $\Gamma$  is now a path enclosing the interval  $[-1, 0]$ . Calculation shows that  $f \in K$  and that  $\Delta^n f(0) = b_n$ . Moreover, if  $g$  is entire,  $f$  is of zero type. Statement (ii) follows from a known theorem concerning the expansion of functions into Newton series [Buck 4, Theorem 4.3] and implies (iv) immediately. As an illustration, all “small” sequences are admissible  $\{T_n^*\}$ . [18, p. 52, Thm. 10].

**Corollary 1.** If  $\limsup |b_n|^{1/n} < 1$ ,  $\{b_n\}$  is admissible  $\{T_n^*\}$ .

Any theorem connecting the presence of singular points of a power series with its coefficients may be used to yield characterization theorems for sequences  $\{b_n\}$  and in turn for  $\{a_n\}$ . At this point we insert a

generalization of the familiar theorem concerning power series with positive coefficients. [15, p. 215].

**Lemma.** Let  $\limsup |c_n|^{1/n} = 1/R$ . Let  $S_n(z) = \sum_0^n c_k z^k$  and suppose that there is a sequence of points  $z_j$  approaching a point  $\beta = Re^{i\theta}$  from outside the circle  $|z| = R$  such that for each  $j$ ,  $S_n(z_j)$  approaches the point at infinity in an angle of opening less than  $\pi$ . Then,  $\beta$  is a singular point for  $f(z) = \sum c_n z^n$ .

If  $f(z)$  is regular at  $\beta$ , it is regular in a circular neighborhood  $N$  of  $\beta$  and  $(ML) - \lim S_n(z) = f(z)$  for all  $z$  in  $N$ . But,  $N$  contains a point  $z_j$ , and since Mittag-Leffler summability is totally regular,  $(ML) - \lim S_n(z_j) = \infty$ .

We note that Borel summability could have been used in place of Mittag-Leffler, if  $N$  is slightly modified; also, the same method yields an analogous result for Dirichlet series and for Laplace transforms. In the classical case,  $c_n \geq 0$  so that  $\lim S_n(x) = +\infty$  for all  $x > R$ .

**Corollary 2.** Let  $\{c_n\}$  be a complex sequence with  $\limsup |c_n|^{1/n} = 1$ , and obeying the condition described in the lemma, with  $R = 1$ . Then, the sequence  $b_n = (-1)^n c_n$  is not admissible  $\{T_n^*\}$ .

**Corollary 3.** If  $\limsup |b_n|^{1/n} \geq 1$  but  $(-1)^n b_n \geq 0$ ,  $\{b_n\}$  is not admissible  $\{T_n^*\}$ .

For  $b_n = \Delta^n a_0$ , the oscillation conditions  $(-1)^n b_n \geq 0$  and  $(-1)^n a_n \geq 0$  are closely related; in fact, the latter implies the former. In particular, we obtain again the following theorem for  $\{T_n\}$  admissibility [2, Theorem 4.1].

**Corollary 4.** If  $a_0 \neq 0$  and  $(-1)^n a_n \geq 0$ , then there is no function  $f \in K$  such that  $f(n) = a_n$  for  $n = 0, 1, \dots$ .

Similarly, Corollary 2 could be turned into a somewhat complicated theorem concerned with admissibility  $\{T_n\}$ . Some growth condition in Corollary 2 is needed since  $b_n = (-1)^n 2^{-n}$  is an oscillating sequence, achieved by the function  $2^{-z}$ . The condition given says essentially that infinitely many of the terms  $b_n$  are “large”, for example, bounded away from zero.

**3. Vanishing differences.** Agnew [1] proved that if  $\{a_n\}$  is a bounded sequence such that  $\Delta^n a_0 = 0$  for  $n = 0, 2, 4, \dots$ , then  $a_n = 0$  for all  $n = 0, 1, \dots$ . Pollard [11] gave a different proof of this and assuming that  $a_n = O(n^r)$  proved that  $a_n = f(n)$  where  $f(z)$  is a polynomial.

Fuchs [6] approached the problem from a different direction and proved the theorem with a weakened assumption on the set of  $n$  for which  $\Delta^n a_0 = 0$ , which in fact was shown to be best possible. We consider the effect of relaxing both this condition and the growth restriction on  $\{a_n\}$ . We use a simple relation connecting the functions  $g(z) = \sum_0^\infty b_k z^k$  and  $F(z) = \sum_0^\infty a_n z^n$ , namely

$$g(z) = (1+z)^{-1} F(z/(1+z)). \quad (3.1)$$

This is easily established, assuming that  $F$  is analytic at the origin. (See also [11].)

**Theorem 2.** Let  $\limsup |a_n|^{1/n} \leq 1$  and let  $\Delta^n a_0 = 0$  for  $n \in A$ , a set of integers of density  $d$ . If  $d > \frac{1}{3}$ , then the series  $\sum_0^\infty \binom{z}{n} \Delta^n a_0$  converges for all  $z$  to a function  $f(z)$  of exponential type not exceeding  $\log(1 + 2 \cos \pi d)$  such that  $f(n) = a_n$  for  $n = 0, 1, 2, \dots$ . In particular, if  $d \geq \frac{1}{2}$ ,  $f$  is of type zero. The value  $\frac{1}{3}$  as a lower bound for  $d$  is best possible.

*Proof.*  $F(t) = \sum_0^\infty a_n t^n$  is regular for  $|t| < 1$  so that by (3.1)  $g(z) = \sum_0^\infty \Delta^n a_0 z^n$  is regular in the half plane  $x > -\frac{1}{2}$ . Let the radius of convergence of this series be  $R$ . By Pólya's density theorem for power series [12], every arc of  $|z| = R$  of opening  $2\pi(1-d)$  contains a singularity of  $g(z)$ . Combining these, we see that if  $d > \frac{1}{3}$  then  $R > 1$ , and by Corollary 1,  $b_n = \Delta^n a_0$  is admissible  $\{T_n^*\}$ ; it then follows that  $\{a_n\}$  is admissible  $\{T_n\}$  and is therefore the sequence  $\{f(n)\}$  for a unique function  $f \in K$ . When  $d \geq \frac{1}{2}$ ,  $R$  is infinite,  $g(z)$  is entire and by Theorem 1,  $f$  is of zero type. If  $\frac{1}{3} < d < \frac{1}{2}$ ,  $R \geq (2 \cos \pi d)^{-1}$  and further calculation shows that the function  $\Phi(w)$  of (2.2) is regular at least for  $|w| > \log(1 + R^{-1})$ . The type of  $f$  does not exceed this value, and in particular is less than  $\log 2$ , so that the Newton series (2.1) is in fact convergent to  $f(z)$  [18, p. 52, Thm. 10]. That the number  $1/3$  cannot be improved follows from the fact that the sequence  $\{b_n\}$  defined by  $\sum_0^\infty b_n z^n = (z-1)/(z+1)(z^3-1)$  is not admissible  $\{T_n^*\}$  so that the corresponding sequence  $\{a_n\}$  has vanishing differences of density  $\frac{1}{3}$ , obeys the growth condition  $a_n = O(1)$ , but is not admissible  $\{T_n\}$ .

The effect of the weakened growth condition  $\limsup |a_n|^{1/n} \leq 1$  is striking; in contrast with the Agnew-Fuchs result,  $d$  may exceed  $\frac{1}{2}$  and

may in fact be 1, without  $f(z)$  being a polynomial. Witness for example  $f(z) = \sum_0^{\infty} \binom{z}{n!} (1/n!)$ . In this connection, the following more detailed information may be of interest. As we have seen, the region of regularity of  $g(z)$  restricts the rate of growth of  $f(z)$ , and when  $g(z)$  is entire,  $f(z)$  is of growth at most order 1, type 0. In this case, the rate of growth of  $g$  might be expected to impose further restrictions.

**Theorem 3.** Let  $g(z) = \sum_0^{\infty} A^n f(0) z^n$ , where  $f \in K$ . If  $g$  is analytic in  $|z| < R$  and  $R > 1$ ,  $f$  is (at most) of order 1 type  $\log(1 + R^{-1})$ ; if  $g$  is entire, and of infinite order,  $f$  is of order 1 type 0; if  $g$  is of finite order  $\varrho$ ,  $f$  is of order  $\varrho/(1 + \varrho)$ .

Since the first two statements have already been discussed, we prove only the last. From (3.1)  $F(z) = \sum f(n) z^n = (1 - z)^{-1} g(z/(1 - z))$ . If  $\zeta = 1/(1 - z)$  this may be written as  $\zeta g(\zeta - 1)$  which is of order  $\varrho$  as a function of  $\zeta$ . By a theorem of Whittaker and Wilson [17]  $f$  is of order  $\varrho/(1 + \varrho)$ .

For the specific example,  $f(z) = \sum \binom{z}{n!} (1/n!)$   $g$  is of infinite order and  $f$  of zero type. In contrast,  $\sum \binom{z}{n!} (1/n^2!)$  is of order  $\frac{1}{2}$  with  $d$  again 1.

#### 4. The Theorem of Pólya. The theorem in question is the following [13]:

**Theorem 4.** Let  $f(z)$  be of order 1 type 0 and suppose that  $f(n) = O(1)$  for  $n = 0, 1, -1, 2, -2, \dots$ . Then,  $f$  is constant.

Many proofs of this have been given since it was first proposed. (See Szego [14], Tschakaloff [16], Paley and Wiener [10, p. 81], Levinson [9, p. 127], Korevaar [8]). The following proof is new and has the virtue of extreme simplicity, involving no interpolation series or delicate growth estimates. We make the initial observation, as in [8], that nothing is lost by the assumption that  $\sum_{-\infty}^{\infty} |f(n)| < \infty$ . Since  $f$  is of zero type,  $g(z)$  is entire and is given by (2.2). Expanding the kernel  $[1 - (e^w - 1)z]^{-1}$ , we have

$$\begin{aligned} g(z) &= \frac{1}{2\pi i} \int_I \Phi(w) (1+z)^{-1} \sum_0^{\infty} e^{nw} [z/(1+z)]^n dw \\ &= (1+z^{-1}) \sum_0^{\infty} f(n) [z/(1+z)]^n \end{aligned}$$

valid for  $|z| < |1+z|$ . (This is also another verification of (3.1).) From our assumption on  $\sum |f(n)|$ ,  $g(z)$  is bounded in the half plane  $x > -\frac{1}{2}$ . Expanding the kernel in the opposite fashion,

$$\begin{aligned}
g(z) &= \frac{1}{2\pi i} \int \Phi(w) (-1/z) \sum_0^{\infty} e^{-(n+1)w} [(z+1)/z]^n dw \\
&= (-1/z) \sum_0^{\infty} f(-n-1) [(1+z)/z]^n
\end{aligned}$$

valid for  $|1+z| < |z|$ . Again,  $g(z)$  is bounded in the half plane  $x < -\frac{1}{2}$ . Combining these,  $g$  and hence  $f$  is constant.

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Received 14th July 1952.