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The class number of an imaginary quadratic field

By L. CARLITZ

1. Introduction. Let $h(D)$ denote the number of classes of properly primitive forms $ax^2 + 2bxy + cy^2$ with $b^2 - ac = -D$, $D > 0$. In his paper [3] Hurwitz expressed the residue (mod p) of $h(D)$ in terms of the coefficients of certain power series; here p is an odd prime divisor of D such that $D \not\equiv 0 \pmod{p^2}$. In some cases the residue was expressed explicitly in terms of Bernoulli numbers.

Let now $h(d)$ denote the class number of the imaginary quadratic field $R(\sqrt{d})$ of discriminant d . (We prefer to use the terminology of quadratic fields rather than quadratic forms in order to stress the analogy with certain other results on the class number of cyclic fields [1]). Let p be an odd prime divisor of d and $n \geq 0$; then we show that for $d < -4$.

$$h(d) \equiv -2c \left(\frac{q}{p}\right) \sum_{1 \leq s < q/2} \left(\frac{q_0}{s}\right) B_k \left(\frac{s}{q}\right) \pmod{p^{n+1}}, \quad (1.1)$$

where $d = (-1)^{(p-1)/2} p q_0$, $q = |q_0|$, (q_0/s) is the Kronecker symbol, $B_k(x)$ is the Bernoulli polynomial of degree $k = \frac{1}{2}(p-1)p^n + 1$ and $c = 1 + \frac{1}{2}p^n$ for $n \geq 1$, while $c = 2$ for $n = 0$. A particularly simple special case of (1.1) is

$$h(-4p) \equiv \frac{1}{2} E_{k-1} \pmod{p^{n+1}}, \quad (1.2)$$

where $p \equiv 1 \pmod{4}$ and E_{k-1} is an Euler number.

2. Kronecker's symbol. We recall a few properties of Kronecker's symbol (see for example [4, p. 51]). If $d \equiv 0$ or $1 \pmod{4}$ and is not a square and if $m > 0$, we define

$$\begin{aligned} \left(\frac{d}{1}\right) &= 1, \quad \left(\frac{d}{p}\right) = 0 \quad (p \mid d), \\ \left(\frac{d}{2}\right) &= \begin{cases} 1 & (d \equiv 1 \pmod{8}) \\ -1 & (d \equiv 5 \pmod{8}) \end{cases}, \\ \left(\frac{d}{p}\right) &= \text{the Legendre symbol when } p > 2, d \not\equiv 0 \pmod{p}, \end{aligned}$$

$$\left(\frac{d}{p_1 \dots p_r}\right) = \left(\frac{d}{p_1}\right) \dots \left(\frac{d}{p_r}\right).$$

It follows from the definition that

$$\left(\frac{d}{m}\right) = 0 \quad \text{for } (d, m) > 1; \quad \left(\frac{d}{m_1 m_2}\right) = \left(\frac{d}{m_1}\right) \left(\frac{d}{m_2}\right); \quad (2.1)$$

also for odd d we have

$$\left(\frac{d}{m}\right) = \left(\frac{m}{|d|}\right), \quad (2.2)$$

where the quantity on the right is a Jacobi symbol. Another useful property is

$$\left(\frac{d}{m_1}\right) = \left(\frac{d}{m_2}\right) \quad (m_1 \equiv m_2 \pmod{d}). \quad (2.3)$$

We shall also require

$$\left(\frac{d}{|d| - m}\right) = \left(\frac{d}{m}\right) \operatorname{sgn} d \quad (m < |d|). \quad (2.4)$$

It is sometimes convenient to define $\left(\frac{d}{0}\right) = 0$.

Thus it is clear that if d is the discriminant of a quadratic field then $\left(\frac{d}{m}\right)$ is a character (mod d).

The letter p will denote a positive odd prime. We define

$$p_0 = (-1)^{(p-1)/2} p, \quad (2.5)$$

so that $p_0 \equiv 1 \pmod{4}$. If $p \mid d$ we put

$$d = p_0 q_0, \quad q = |q_0|. \quad (2.6)$$

It follows that the Kronecker symbols (p_0/m) and (q_0/m) are defined; moreover

$$\left(\frac{d}{m}\right) = \left(\frac{p_0}{m}\right) \left(\frac{q_0}{m}\right). \quad (2.7)$$

Hereafter d will denote the discriminant of an imaginary quadratic field. Hence $d < 0$ and is not divisible by the square of any odd prime.

3. Bernoulli polynomials. We use the notation of Nörlund [5, Chapter 2] for the Bernoulli polynomials. The following formulas will be needed.

$$\sum_{a=0}^{m-1} (x+a)^{n-1} = \frac{B_n(x+m) - B_n(x)}{n}, \quad (3.1)$$

$$B_n(x+y) = \sum_{s=0}^n \binom{n}{s} x^{n-s} B_s(y); \quad (3.2)$$

$$B_n(1-x) = (-1)^n B_n(x). \quad (3.3)$$

In addition we recall a special case of Kummer's congruence [2, Theorem 5]

$$\frac{B_{n+t}(a)}{n+t} \equiv \frac{B_n(a)}{n} \pmod{p^e}, \quad (3.4)$$

where $p^{e-1}(p-1) \mid t$, $n \not\equiv 0 \pmod{p-1}$, $n > e$ and the rational number a is integral \pmod{p} . The following divisibility property will also be used.

$$B_m(a) \equiv 0 \pmod{p^r} \quad (p^r \mid m, m \not\equiv 0 \pmod{p-1}) \quad (3.5)$$

where again a is integral \pmod{p} .

For some purposes it is convenient to define the Bernoulli function $\bar{B}_m(x)$:

$$\begin{aligned} \bar{B}_m(x) &= \bar{B}_m(x) & (0 \leq x < 1) \\ \bar{B}_m(x+1) &= B_m(x). \end{aligned}$$

Then $\bar{B}_m(x)$ satisfies (3.3) as well as the multiplication formula

$$\sum_{s=0}^{r-1} \bar{B}_m\left(x + \frac{s}{r}\right) = r^{1-m} \bar{B}_m(rx); \quad (3.6)$$

the polynomial $B_m(x)$ also satisfies (3.6).

4. The main result. Let $d < -3$. It is familiar that

$$h(d) = \frac{1}{d} \sum_m \left(\frac{d}{m}\right), \quad (4.1)$$

where m runs through a complete residue system \pmod{d} . We assume $p \mid d$ and make use of (2.5), (2.6), (2.7). Let $q > 1$. Then (4.1) becomes

$$\begin{aligned} h(d) &= \sum_{r=0}^{p-1} \sum_{s=1}^{q-1} (rq+s) \left(\frac{p_0}{rq+s}\right) \left(\frac{q_0}{rq+s}\right) \\ &= \frac{1}{d} \sum_{s=1}^{q-1} \left(\frac{q_0}{s}\right) \sum_{r=0}^{p-1} (rq+s) \left(\frac{rq+s}{p}\right). \end{aligned} \quad (4.2)$$

Now it follows from

$$\left(\frac{a}{p}\right) \equiv a^{(p-1)/2} \pmod{p}$$

that

$$\left(\frac{a}{p}\right) \equiv a^{(p-1)p^{n/2}} \pmod{p^{n+1}}. \quad (4.3)$$

Then using (4.3) we have

$$\sum_{r=0}^{p-1} (rq+s) \left(\frac{rq+s}{p}\right) \equiv \sum_{r=0}^{p-1} (rq+s)^k \pmod{p^{n+1}}, \quad (4.4)$$

where for brevity we put

$$k = \frac{1}{2}(p-1)p^n + 1. \quad (4.5)$$

Then using (3.1) we get

$$\sum_{r=0}^{p-1} (rq + s)^k = q^k \sum_{r=0}^{p-1} \left(r + \frac{s}{q}\right)^k = q^k \frac{B_{k+1}\left(p + \frac{s}{q}\right) - B_{k+1}\left(\frac{s}{q}\right)}{k+1}. \quad (4.6)$$

In the next place by (3.2)

$$\begin{aligned} B_{k+1}\left(p + \frac{s}{q}\right) - B_{k+1}\left(\frac{s}{q}\right) &= (k+1)p B_k\left(\frac{s}{q}\right) + \binom{k+1}{2} p^2 B_{k-1}\left(\frac{s}{q}\right) \\ &\quad + \sum_{r=3}^{k+1} \binom{k+1}{r} p^r B_{k-r+1}\left(\frac{s}{q}\right). \end{aligned}$$

But it is easily verified that

$$\binom{k+1}{r} p^r B_{k-r+1}\left(\frac{s}{q}\right) \equiv 0 \pmod{p^{n+1}} \quad (r \geq 3).$$

Thus (4.6) becomes

$$\begin{aligned} \sum_{r=0}^{p-1} (rq + s)^k &\equiv q^k \left\{ p B_k\left(\frac{s}{q}\right) + \frac{1}{2} k p^2 B_{k-1}\left(\frac{s}{q}\right) \right\} \\ &\equiv p q \left(\frac{q}{p}\right) B_k\left(\frac{s}{q}\right) \pmod{p^{n+1}}, \end{aligned}$$

where we have used (4.3) and (3.5).

Substituting in (4.4) and (4.2) we therefore get

$$h(d) \equiv - \left(\frac{q}{p}\right) \sum_{s=1}^{q-1} \left(\frac{q_0}{s}\right) B_k\left(\frac{s}{q}\right) \pmod{p^n}.$$

Finally using (3.4) this becomes

$$h(d) \equiv - \frac{1}{1 - \frac{1}{2} p^{n-1}} \left(\frac{q}{p}\right) \sum_{s=1}^{q-1} \left(\frac{q_0}{s}\right) B_l\left(\frac{s}{q}\right) \pmod{p^n}.$$

where $l = \frac{1}{2}(p-1)p^{n-1} + 1$. Replacing n by $n+1$ we get

$$h(d) \equiv - c \left(\frac{q}{p}\right) \sum_{s=1}^{q-1} \left(\frac{q_0}{s}\right) B_k\left(\frac{s}{q}\right) \pmod{p^{n+1}}, \quad (4.7)$$

where k is defined by (4.5) and $c = 1 + \frac{1}{2} p^n$ for $n > 0$, $c = 2$ for $n = 0$.

In the next place if $p \equiv 1 \pmod{4}$, then k is odd so that by (3.3)

$$B_k\left(\frac{q-s}{q}\right) = -B_k\left(\frac{s}{q}\right).$$

Also since $p_0 > 0, q_0 < 0$, so that by (2.4) $\left(\frac{q_0}{q-s}\right) = -\left(\frac{q_0}{s}\right)$.

It follows that

$$\left(\frac{q_0}{q-s}\right) B_k\left(\frac{q-s}{q}\right) = \left(\frac{q_0}{s}\right) B_k\left(\frac{s}{q}\right). \quad (4.8)$$

If $p \equiv -1 \pmod{4}$, then k is even and $q > 0$, so that

$$B_k\left(\frac{q-s}{q}\right) = B_k\left(\frac{s}{q}\right), \quad \left(\frac{q_0}{q-s}\right) = \left(\frac{q_0}{s}\right).$$

and again (4.8) follows. Since for q_0 even the value $s = q_0/2$ in (4.7) may be ignored we get

$$h(d) \equiv -2c\left(\frac{q}{p}\right) \sum_{1 \leq s < q/2} \left(\frac{q_0}{s}\right) B_k\left(\frac{s}{q}\right) \pmod{p^{n+1}}, \quad (4.9)$$

where k and c are the same as in (4.7).

5. The case $q = 1$. While (4.9) does not hold for $q = 1$, it is easy to obtain a similar result in that case. We now have $d = -p$, where $p \equiv 3 \pmod{4}$. Thus (4.1) becomes

$$h(-p) = -\frac{1}{p} \sum_m m \left(\frac{-p}{m}\right) = -\frac{1}{p} \sum_m m \left(\frac{m}{p}\right).$$

Now

$$\sum_{m=1}^{p-1} m \left(\frac{m}{p}\right) \equiv \sum_{m=1}^{p-1} m^k \equiv \frac{B_{k+1}(p) - B_{k+1}}{k+1} \pmod{p^{n+1}},$$

where k is the same as in (4.5); note that k is even. A little manipulation leads to

$$h(-p) \equiv -B_k \pmod{p^n}. \quad (5.1)$$

In particular for $n = 1$, (5.1) becomes

$$h(-p) \equiv -B_{(p-1)p/2+1} \pmod{p},$$

which by (3.4) reduces to

$$h(-p) \equiv -2B_{(p+1)/2} \pmod{p}. \quad (5.2)$$

6. Some special cases. Returning to (4.9) we consider first the special case $d = -3p, p \equiv 1 \pmod{4}$. Thus $q_0 = -3, q = 3$ and (4.9) reduces to

$$h(-3p) \equiv -2c\left(\frac{3}{p}\right) B_k\left(\frac{1}{3}\right) \pmod{p^{n+1}}. \quad (6.1)$$

Since k is odd it does not seem possible to further simplify the right member of (6.1). For $n = 0$, (6.1) becomes

$$h(-3p) \equiv -4\left(\frac{3}{p}\right) B_{(p+1)/2}\left(\frac{1}{3}\right) \pmod{p}. \quad (6.1)'$$

Next for $d = -4p$, $p \equiv 1 \pmod{4}$, $q_0 = -4$, $q = 4$, so that (4.9) becomes

$$h(-4p) \equiv -2c B_k\left(\frac{1}{4}\right) \pmod{p^{n+1}}. \quad (6.2)$$

Now we may use the formula [4, p. 29]

$$B_k\left(\frac{1}{4}\right) = -k \frac{E_{k-1}}{4^k} \quad (k \text{ odd}),$$

where E_{k-1} is an Euler number. Since

$$4^k = 2^{2k} = 2^{(p-1)p^{n+2}} \equiv 4 \pmod{p^{n+1}},$$

it is easily verified that (6.2) gives

$$h(-4p) \equiv \frac{1}{2} E_{k-1} \pmod{p^{n+1}}; \quad (6.3)$$

in particular for $n = 0$, we get

$$h(-4p) \equiv \frac{1}{2} E_{(p-1)/2} \pmod{p}. \quad (6.3)'$$

For example for $p = 5$, $h(-20) = 2$, $E_2 = -1$.

For $d = -5p$, $p \equiv 3 \pmod{4}$, $q = q_0 = 5$, we have

$$h(-5p) \equiv -2c \left(\frac{5}{p}\right) \left\{ B_k\left(\frac{1}{5}\right) - B_k\left(\frac{2}{5}\right) \right\} \pmod{p^{n+1}}; \quad (6.4)$$

in particular

$$h(-5p) \equiv -4 \left(\frac{5}{p}\right) \left\{ B_{(p+1)/2}\left(\frac{1}{5}\right) - B_{(p+1)/2}\left(\frac{2}{5}\right) \right\} \pmod{p}. \quad (6.4)'$$

For $d = -8p$, we have either (i) $p \equiv 1 \pmod{4}$, $q_0 = -8$, $q = 8$, or (ii) $p \equiv 3 \pmod{4}$, $q_0 = q = 8$. The two possibilities may be combined in the single formula

$$h(-8p) \equiv -2c \left(\frac{2}{p}\right) \left\{ B_k\left(\frac{1}{8}\right) + \left(\frac{-1}{p}\right) B_k\left(\frac{3}{8}\right) \right\} \pmod{p^{n+1}}, \quad (6.5)$$

which does not seem to reduce further. Using (3.3) we may however write

$$h(-8p) \equiv -2c \left(\frac{2}{p}\right) \left\{ B_k\left(\frac{1}{8}\right) - B_k\left(\frac{5}{8}\right) \right\} \pmod{p^{n+1}}, \quad (6.6)$$

as is easily verified.

We may also mention the case $d = -12p$, where $p \equiv 3 \pmod{4}$, $q = q_0 = 12$. Thus (4.9) becomes

$$h(-12p) \equiv -2c \left(\frac{3}{p}\right) \left\{ B_k\left(\frac{1}{12}\right) - B_k\left(\frac{5}{12}\right) \right\} \pmod{p^{n+1}}. \quad (6.7)$$

7. Some additional formulas. Formula (4.7) becomes somewhat more symmetrical if we introduce the Bernoulli function $\bar{B}_k(x)$ defined in § 3. For we may now write

$$h(d) \equiv -c \left(\frac{q}{p} \right) \sum_s \left(\frac{q_0}{s} \right) \bar{B}_k \left(\frac{s}{q} \right) \pmod{p^{n+1}}, \quad (7.1)$$

where s runs through a complete residue system $(\text{mod } q)$. In the next place, using (3.6), we have

$$\sum_{s=0}^{q-1} \bar{B}_k \left(\frac{s}{q} \right) = q^{1-k} \bar{B}_k(0) = q^{1-k} B_k. \quad (7.2)$$

Also

$$q^{1-k} = q^{-(p-1)/2} p^n \equiv \left(\frac{q}{p} \right) \pmod{p^{n+1}},$$

so that (7.2) becomes

$$\sum_{s=1}^{q-1} \bar{B}_k \left(\frac{s}{q} \right) \equiv \left\{ \left(\frac{q}{p} \right) - 1 \right\} B_k.$$

Combining with (7.1) we get

$$h(d) \equiv -2c \left(\frac{q}{p} \right) \sum'_s \bar{B}_k \left(\frac{s}{q} \right) + c \left\{ 1 - \left(\frac{q}{p} \right) \right\} B_k \pmod{p^{n+1}}, \quad (7.3)$$

where the sum is now restricted to such s that $(q_0/s) = 1$. If $p \equiv 1 \pmod{4}$, so that k is odd, (7.3) reduces to

$$h(d) \equiv -2c \left(\frac{q}{p} \right) \sum'_s \bar{B}_k \left(\frac{s}{q} \right) \pmod{p^{n+1}}; \quad (7.4)$$

if $(q/p) = 1$ then (7.4) holds for all p . If q is a prime then (7.3) may also be written in the form

$$h(d) \equiv -c \left(\frac{q}{p} \right) \sum_{s=0}^{q-1} \bar{B}_k \left(\frac{s^2}{q} \right) + c B_k \pmod{p^{n+1}}. \quad (7.5)$$

The last formula suggests that it may be interesting to consider the sum

$$S_k(h, q) = \sum_{s=0}^{q-1} \bar{B}_k \left(\frac{s^2 h}{q} \right) - q^{1-k} B_k \quad (7.6)$$

for arbitrary positive k and q . In particular if q is an odd prime power then it follows from (7.2) and (7.6) that

$$S_k(h, q) = \sum_{s=1}^{q-1} \left(\frac{s}{q} \right) \bar{B}_k \left(\frac{sh}{q} \right) \quad ((h, q) = 1)$$

and therefore

$$S_k(h, q) = \left(\frac{h}{q} \right) S_k(1, q). \quad (7.7)$$

In the next place if $q = p^r, r \geq 3$, then

$$\begin{aligned}
 S_k(1, p^r) &= \sum_{b=0}^{p^{r-1}-1} \sum_{a=0}^{p-1} \bar{B}_k \left(\frac{b^2}{p^r} + \frac{2ab}{p} \right) \\
 &= p \sum_{b=0}^{p^{r-2}-1} \bar{B}_k \left(\frac{b^2}{p^{r-2}} \right) + \sum_{\substack{b=0 \\ p+b}}^{p^{r-1}-1} \sum_{a=0}^{p-1} \bar{B}_k \left(\frac{b^2}{p^r} + \frac{2ab}{p} \right) \\
 &= p \{ S_k(1, p^{r-2}) + p^{(r-2)(1-k)} B_k \} + p^{1-k} \sum_{\substack{b=0 \\ p+b}}^{p^{r-1}-1} \bar{B}_k \left(\frac{b^2}{p^{r-1}} \right) \\
 &= p \{ S_k(1, p^{r-2}) + p^{(r-2)(1-k)} B_k \} + p^{1-k} \sum_{b=0}^{p^{r-1}-1} \bar{B}_k \left(\frac{b^2}{p^{r-1}} \right) \\
 &\quad - p^{1-k} \sum_{b=0}^{p^{r-2}-1} \bar{B}_k \left(\frac{b^2}{p^{r-3}} \right) \\
 &= p \{ S_k(1, p^{r-2}) + p^{(r-2)(1-k)} B_k \} \\
 &\quad + p^{1-k} \{ S_k(1, p^{r-1}) + p^{(r-1)(1-k)} B_k \} \\
 &\quad - p^{2-k} \{ S_k(1, p^{r-3}) + p^{(r-3)(1-k)} B_k \},
 \end{aligned}$$

so that

$$\begin{aligned}
 S_k(1, p^r) &= p^{1-k} S_k(1, p^{r-1}) + p S_k(1, p^{r-2}) - p^{2-k} S_k(1, p^{r-3}) \\
 &\quad + p^{r(1-k)} B_k.
 \end{aligned} \tag{7.8}$$

For $r = 2$, we find that

$$S_k(1, p^2) = p^{1-k} S_k(1, p) + (p - p^{1-k} + p^{2(1-k)}) B_k. \tag{7.9}$$

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