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On the Pontryagin product in spaces of paths

By R. Bott and H. Samelson [1])

Introduction

For a topological (arcwise connected) space X, let E be the function space consisting of the paths in X which start at a certain point x, and let Ω be the subspace of E consisting of the closed paths or loops; these spaces have been studied in particular by M. Morse [7] and J.-P. Serre [10]; E is a fiber space over X. Now Ω admits a natural multiplication: two loops in succession make a new loop (actually there is a more general operation between E and Ω). This multiplication gives rise to a multi-

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plication of the elements of the homology group of Ω; we call this the Pontryagin-multiplication, since an entirely similar concept for group spaces was introduced by Pontryagin [8] (cf. also [9]). In part II we study the relation between Leray's spectral sequence of E (in Serre's formulation [10]) and the Pontryagin product. A closely related situation, involving compact Lie groups and Leray's cohomology theory, has been considered by Leray [5, 6] and A. Borel [1]; our results are analogous to theirs. In part III we determine, as application, the Pontryagin ring of a space which is union of several spheres with a point in common. — We have to consider the homology of Cartesian product spaces. This makes necessary a modification of Serre's cubic homology theory; we present this in part I. Generally speaking we follow Serre's definitions, notation and conventions, with some minor deviations; we assume familiarity with his paper.

I. Cubic homology

I.1. We recall briefly some definitions from [10]. Let X be a 0-connected (i. e. arcwise connected) space. A singular n-cube in X is a map $u : I^n \to X$, where I denotes the unit interval $[0,1]$, and I^n means the Cartesian product of n factors I; 0-cubes are simply points of X. With a real ε, $0 \leqq \varepsilon \leqq 1$, and an integer i, $1 \leqq i \leqq n$, the operator λ_i^ε associates with each singular n-cube a singular $(n-1)$-cube by

$$(\lambda_i^\varepsilon u)(x_1, \ldots, x_{n-1}) = u(x_1, \ldots, x_{i-1}, \varepsilon, x_i, \ldots, x_{n-1}) .$$

The singular n-cubes are free generators of the (abelian) group $Q_n(X)$. The direct sum $Q(X)$ of all Q_n, $n \geqq 0$, is mapped into itself by the operator d, which on n-cubes is defined as $\sum_1^n (-1)^i (\lambda_1^0 - \lambda_i^1)$. To define homology groups, one has to introduce the d-stable subgroup $D = \Sigma D_n(X)$ where $D_n(X)$ is generated by the n-cubes which are degenerate along the last coordinate; the homology of X is that of Q/D. We now introduce, for each integer $p \geqq 1$, the subgroup $D^{(p)}$, generated by those cubes which are degenerate along any one of their last p coordinates (in particular $D^{(1)} = D$; if $p > n$, this means that $u \in Q_n$ is degenerate along some coordinate); with $D_n^{(p)} = D^{(p)} \cap Q_n$ we have $D^{(p)} = \Sigma D_n^{(p)}$. Clearly $D^{(1)} \subset D^{(2)} \subset \cdots$. We put $D^{(\infty)} = \cup_p D^{(p)}$; clearly $D_n^{(\infty)} = D_n^{(n)}$. We prove

Theorem I.1.A.: (a) For each p, with $1 \leqq p \leqq \infty$, $D^{(p)}$ is a d-stable subgroup of Q; (b) the natural map of the homology group of $Q/D^{(p)}$

into that of $Q/D^{(p+1)}$ and of $Q/D^{(\infty)}$, induced by $D^{(p)} \subset D^{(p+1)} \subset D^{(\infty)}$, is an isomorphism onto.

Froof: First take $p < \infty$. (a) is verified easily from the formula for d: If u belongs to $D^{(p)}$, and is degenerate along the r-th coordinate, then the terms $\lambda_r^0 u$ and $\lambda_r^1 u$ in du cancel; all other terms belong to $D^{(p)}$ or even to $D^{(p-1)}$. For (b), the usual application of the exact sequence of the triple $(Q, D^{(p+1)}, D^{(p)})$ [3, p. 28] shows that we have to prove that the relative homology group $H(D^{(p+1)}, D^{(p)})$ is 0. To do this, we consider the operator Δ, a linear map of Q into itself, raising dimension by 1, defined by $\Delta u(x_1, \ldots, x_{n+1}) = u(x_2, \ldots, x_n, x_1 x_{n+1})$ for $u \in Q_n$, $n > 0$, $\Delta u(x_1) = u$, for $u \in Q_0$. We state the following properties of Δ:

1. $\Delta(D^{(p)}) \subset D^{(p)}$ for all $p \geqq 1$.
2. Defining $\tau = 1 - d\Delta - \Delta d$ (1 = identity map), we have $\tau(D^{(p+1)}) \subset D^{(p)}$.

Property 1. is clear; if u does not depend on x_r, Δu does not depend on x_{r+1}. For 2. a direct computation shows that for $u \in Q_n$

$$\tau u = \lambda_1^0 \Delta u \pm (\lambda_{n+1}^1 \Delta u - \lambda_{n+1}^0 \Delta u) \pm (\Delta \lambda_n^1 u - \Delta \lambda_n^0 u).$$

If u belongs to $D^{(p+1)}$, then all the terms in τu belong to $D^{(p)}$ or cancel if $u \in D^{(1)}$. If now x is a cycle of $D^{(p+1)}$ mod $D^{(p)}$, i. e. $x \in D^{(p+1)}$, $dx \in D^{(p)}$, then $d\Delta x = x - \tau x - \Delta dx$ with $\Delta x \in D^{(p+1)}$ and τx and $\Delta dx \in D^{(p)}$, i. e. x is ~ 0 in $D^{(p+1)}$ mod $D^{(p)}$, and $H(D^{(p+1)}, D^{(p)})$ is 0. The assertion of I.1.A. for $p = \infty$ follows now easily from $D^{(\infty)} = \bigcup D^{(p)}$. The identification of the homology groups of the various $Q/D^{(p)}$ is natural, i. e. it commutes with the maps induced by a map of one space into another. Any one of the $Q/D^{(p)}$ will be called the group $C(X)$ of chains of X, and its homology group will be called the homology group $H(X)$ of X; we shall always use $Q/D^{(\infty)}$. Clearly all the chain groups are free; an application of a known theorem [3, p. 155] tells us that I.1.A. actually holds with arbitrary coefficients.

Since X is 0-connected, we can and shall, as in [10], restrict ourselves to cubes, all of whose vertices lie at a chosen point x_0. The resulting homology group is canonically isomorphic with the earlier one.

I.2 Let P be a fiber space over the space B, with projection p, in the sense of Serre [10]. We recall that p determines a filtration of $Q(P)$: a cube u in $Q(P)$ is of filtration $\leqq r$ if the cube $p \circ u$ in $Q(B)$ depends only on its first r coordinates; such cubes generate the subgroup

$T^r = T^r(P)$. The group of chains $C(P)$ is filtered by the canonical images $A^r(P) = T^r(P) + D^{(\infty)}/D^{(\infty)}$ of the $T^r(P)$. This sets up the spectral sequence of P, consisting of the groups $E_0, E_1, E_2, \ldots$ and the associated differentials $d_0, d_1, d_2, \ldots$ The terms E_0, E_1, E_2, and the differentials d_0, d_1 have been determined explicitly by Serre, with the group $Q(P)/D^{(1)}$ as chain group. The results remain exactly the same for the new definition $Q(P)/D^{(\infty)}$ of the chain group; the necessary changes in the reasoning are the following:

On p. 447 of [10], property 2) becomes

2′) If $u \in D^{(\infty)}$, then either Bu or $Fu \in D^{(\infty)}$; this permits construction of the map φ. The construction K of p. 448 of [10] can be made in such a fashion that condition 3) is replaced by

3′) If either u or $v \in D^{(\infty)}$, then $K(u, v) \in D^{(\infty)}$. The construction S of lemma 5, p. 448 ibid. can be made such that condition 5) becomes

5′) If $u \in D^{(\infty)}$, then $Su \in D^{(\infty)}$. One has to define K and S in the degenerate cases such that if u and v, resp. w, do not depend on certain coordinates, then $K(u, v)$, resp. Sw, do not depend on the corresponding coordinates. This is possible since the maps g of the sets A [10, p. 461, 462] do not depend on these coordinates and one can extend g to X by first collapsing A along these coordinates. The operation K defines then a map ψ of $C(B) \otimes C(F)$ into E_0, which commutes with the appropriate differentials; the operation S maps E_0 into itself and provides a chain homotopy of $\psi \circ \varphi$ with 1. The only change required for the determination of d_1 is the substitution of "$\in D^{(\infty)}$" for "dégénérés" on line 10, p. 450 in [10].

I.3. We consider the Cartesian product $X \times Y$ of two 0-connected spaces X and Y. With an m-cube u in X and an n-cube v in Y we associate the $(m+n)$-cube $u \times v$ in $X \times Y$, defined by

I.3.1 $u \times v(x_1, \ldots, x_{m+n}) = (u(x_1, \ldots, x_m), v(x_{m+1}, \ldots, x_{m+n}))$. This induces a map μ of the tensor product $Q(X) \otimes Q(Y)$ into $Q(X \times Y)$; we write also $x \times y$ for $\mu(x \otimes y)$. In the tensor product we consider the usual differential $d = d \otimes 1 + \omega \otimes d$, with $\omega(x) = (-1)^m x$ for $x \in Q_m(X)$; we have then $d^2 = 0$, and μ commutes with d. If either $x \in D^{(\infty)}(X)$ or $y \in D^{(\infty)}(Y)$, then clearly $x \times y \in D^{(\infty)}(X \times Y)$; by passage to the quotient groups we get therefore [3, p. 159] a map, also called μ, of $C(X) \otimes C(Y)$ into $C(X \times Y)$, which commutes with d, and induces a map μ_* of the homology groups.

Theorem I. 3. A: The map $\mu: C(X) \otimes C(Y) \to C(X \times Y)$ is a chain equivalence. We give a proof, following a suggestion of J.-P. Serre: We filter $C(X) \otimes C(Y)$ by the subgroups $C^p = \sum_{i \leq p} C_i(X) \otimes C(Y)$; this defines the spectral sequence (E'_r). We consider $X \times Y$ as fiber space over X, with respect to the natural projection; this defines the subgroups A^p and the spectral sequence (E_r), as in I.2. (The operations K and S are of course quite elementary now. We can put $K(u, v) = u \times v$. An n-cube u of $X \times Y$ is a pair (u_1, u_2) of n-cubes of X, resp. Y; if u_1 depends on its first p coordinates only, we put $S^p u = (u', u'')$ with

$u'(x_1, \ldots, x_p, t, y_1, \ldots, y_q) = u_1(x_1, \ldots, x_p, y_1, \ldots, y_q)$ and

$u''(x_1, \ldots, x_p, 1 - t, y_1, \ldots, y_q) = u_2(tx_1, \ldots, tx_p, y_1, \ldots, y_q)$). Clearly μ maps C^p into A^p, and we get an induced map $\mu_r: E'_r \to E_r$ of the spectral sequences. Now μ_0 is nothing else but the map ψ of [10, p. 448] and is therefore a chain equivalence. It follows that all μ_r, $r \geq 1$, and μ^* are isomorphisms onto [1, p. 122]. But then μ is a chain equivalence by [3, theorem 13.3, p. 154].

In the usual way μ induces also a map of $H(X) \otimes H(Y)$ into $H(X \times Y)$, for coefficients in a commutative ring R with unit. If R is a principal ideal ring, then I.3.A and the algebraic Künneth formula imply that the "Künneth formula for singular homology" holds: μ_* imbeds $H(X) \otimes H(Y)$ isomorphically into $H(X \times Y)$ as direct summand, and the factor group is Tor $[H(X), H(Y)]$ (cf. [3, p. 161]); as regards dimension, we have

$$H_n(X \times Y) = \sum_{p+q=n} H_p(X) \otimes H_q(Y) \oplus \sum_{r+s=n-1} \operatorname{Tor}[H_r(X), H_s(Y)].$$

For completeness sake we sketch the known algebraic reasoning: Let K and L be two free chain groups; denote by Z and B the cycles and boundaries of L, and by W a subgroup of L supplementary to Z. From the homology sequence of the pair $(K \otimes (B + W), K \otimes B)$ one finds that $H(K \otimes (B + W)) = 0$. From the sequence of the pair $(K \otimes (Z + W)$, $K \otimes (B + W))$ one finds that $H(K \otimes L)$ is isomorphic with $H(K \otimes H(L))$ (note $Z + W = L$ and $Z + W/B + W = H(L)$). To the latter group one applies the universal coefficient theorem [3, p. 161].

There are obvious associativity relations in the case of products of more than two spaces, and other elementary relations. With two maps $f: P \to X$, $g: Q \to Y$ are associated the maps $f \times g: P \times Q \to X \times Y$, $f \otimes g: C(P) \otimes C(Q) \to C(X) \otimes C(Y)$, $f_* \otimes g_*: H(P) \otimes H(Q) \to H(X) \otimes H(Y)$, with the relation $\mu_* \circ f_* \otimes g_* = (f \times g)_* \circ \mu_*$. If h and k are cochains on X and Y, and h' and k' are their images under the natural projections of $X \times Y$ onto X and Y, then we have the relation

$$(h' \smile g')(u \times v) = h(u) \cdot g(v) .$$

II. The Pontryagin product

II.1. Let X be 1-connected, i. e. arcwise connected and simply connected (the latter assumption is made in order to avoid local coefficients in the spectral sequence); choose a point x_0; all vertices of all singular cubes are to be at x_0. We consider now the spaces E and Ω of paths and loops in X, with the compact-open topology, as in [10], with the modification that we require all paths to end at x_0, i. e. $f(1) = x_0$ for the path $f: I \to X$, and that the projection $p: E \to X$ is defined as the starting point of the path, $p(f) = f(0)$; the reason is that we want Ω to operate on the right on E. This operation of Ω on E is the map γ of $E \times \Omega$ into E, defined by associating with the pair of paths (x, y), $x \in E$, $y \in \Omega$ the path $\gamma(x, y)$, also written $x \cdot y$, defined by

II.1.1 $$x \cdot y(t) = x(2t), \qquad 0 \leqq t \leqq \tfrac{1}{2}$$
$$x \cdot y(t) = y(2t-1), \qquad \tfrac{1}{2} \leqq t \leqq 1.$$

γ is obviously a continuous map; it is related to the projection p by

II.1.2 $$p(x \cdot y) = p(x),$$

($x \cdot y$ and x start at the same point).

Vertices of cubes in X are to be at x_0, in E (and in Ω) at e, defined by $e(t) = x_0$, $0 \leqq t \leqq 1$, the constant path. The composition of μ: $Q(E) \otimes Q(\Omega) \to Q(E \times \Omega)$ defined in I.2, with γ: $Q(E \times \Omega) \to Q(E)$ determines a map ϱ: $Q(E) \otimes Q(\Omega) \to Q(E)$, which commutes with d, related chain maps ϱ of the chain groups and the map $\varrho_* = \gamma_* \circ \mu_*$ of the homology groups. For $\varrho(u \otimes v)$ resp. $\varrho_*(z \otimes w)$ we write also $u * v$, resp. $z * w$, and call this the Pontryagin multiplication. The map γ has the property that it maps the subset $\Omega \times \Omega$ into the subset Ω; γ, restricted in this fashion, will be denoted by $\bar{\gamma}$; we have a corresponding $\bar{\varrho}$ and $\bar{\varrho}_*$; but we continue to use the symbols $\cdot$ and $*$. The point $e \in E$ satisfies $e \cdot e = e$. The map r_e: $E \to E$, defined by $r_e(x) = x \cdot e$, is homotopic to the identity, with e stationary (cf. [5, p. 475]). This is also true for r_e: $\Omega \to \Omega$ and for l_e: $\Omega \to \Omega$ by $l_e(x) = e \cdot x$. The two maps $(x, y, z) \to (x \cdot y) \cdot z$, resp. $x \cdot (y \cdot z)$ of $\Omega^3 \to \Omega$ are not identical, but homotopic (cf. II.5); this means that Pontryagin multiplication in $H(\Omega)$ is associative; we speak then of the Pontryagin ring (better-algebra); the 0-homology class, defined and denoted by e, is unit for this ring.

II.2. The basic fact concerning E is that relative to the map p it is a fiber space over X, with fiber Ω [10]. We recall that $Q(E)$ is filtered (cf. I.2), giving rise to a filtration of ${}'C(E)$ and to a spectral sequence; we will have to make use of the explicit form of E_0, E_1, E_2, as determined by Serre. Now II.1.2 has the obvious consequence that for $u \in Q_m(E)$, $v \in Q_n(\Omega)$, one has

II.2.1. $p(u(x_1, \ldots, x_m) \cdot v(x_{m+1}, \ldots, x_{m+n})) = p(u(x_1, \ldots, x_m))$.

This implies that the filtration of $u * v$ equals that of u. We filter $Q(E) \otimes Q(\Omega)$ by the subgroups $T^r(E) \otimes Q(\Omega)$ (these are actually subgroups since the T^r are direct summands of $Q(E)$); we have then $\varrho(T^r(E) \otimes Q(\Omega)) \subset T^r(E)$. We consider now $C(E) \otimes C(\Omega)$, which group we denote by Γ, with the standard differential $d = d_E \otimes 1 + \omega \otimes d_\Omega$, and filter it by the subgroups $A^p(\Gamma) = A^p(E) \otimes C(\Omega)$ (they are well defined since the $A^p(E)$ are direct summands of $C(E)$). The map ϱ gives then a map of $A^p(\Gamma)$ into $A^p(E)$, and induces therefore in standard algebraic fashion a map ϱ of the spectral sequence of Γ into that of E, i. e. maps ϱ_r of $E_r(\Gamma)$ into $E_r(E)$, which commute with the differentials d_r. Our main task will be to study these maps for $r = 0, 1, 2$ in terms of Serre's description of $E_0(E)$, $E_1(E)$, $E_2(E)$.

II.3. Since $A^{p-1}(E)$ is a direct summand of $A^p(E)$, we have a canonical isomorphism of $E_0(\Gamma)$ and $E_0(E) \otimes C(\Omega)$, with respect to the differentials d_0 and $d_0 \otimes 1 + \omega \otimes d$; this can be interpreted, via the map $\varrho_0 : E_0(\Gamma) \to E_0(E)$, as an operation of $C(\Omega)$ on $E_0(E)$. Going to the homology groups, we have a canonical imbedding of $E_1(E) \otimes H(\Omega)$ into $E_1(\Gamma) = H(E_0(E) \otimes C(\Omega))$ by the Künneth theorem ($E_0(E)$ and $C(\Omega)$ are free groups), with $d_1 \otimes 1 + \omega \otimes d$ going into d_1 (actually the term $\omega \otimes d$ can be dropped since we are dealing with $H(\Omega)$). Similarly $E_2(\Gamma)$ will contain the canonical image of $E_2(E) \otimes H(\Omega)$, but will contain additional terms from two sources: from the other summand of E_1, and from the universal coefficient theorem for $E_1(E) \otimes H(\Omega)$. If the coefficients are not integers, but taken from an arbitrary commutative ring, the situation is even more complicated. We therefore restrict ourselves to considering canonical maps of $E_r(E) \otimes H(\Omega)$ into $E_r(\Gamma)$ and via ϱ_r, into $E_r(E)$, and interpret this as an operation of $H(\Omega)$ on $E_r(E)$, as follows:

Let Z be the group of cycles, and B the group of boundaries of $C(\Omega)$. The identity map $C(E) \otimes C(\Omega) \to \Gamma$ induces obviously maps ν_r: $C_r^p(E) \otimes Z \to C_r^p(\Gamma)$, $r \geqq 1$; one verifies that $C_r^p(E) \otimes B$, $C_{r-1}^{p-1} \otimes Z$, $B_{r-1}^p \otimes Z$ are mapped respectively into $C_{r-1}^{p-1}(\Gamma) + B_{r-1}^p(\Gamma)$, $C_{r-1}^{p-1}(\Gamma)$, $B_{r-1}^p(\Gamma)$, and that therefore, passing to the quotients, one gets an induced map $\nu_r : E_r(E) \otimes H(\Omega) \to E_r(\Gamma)$; the relation $\nu_r \circ (d_r \otimes 1) = d_r \circ \nu_r$ is clear (for $r = 0$ this is modified to $\nu_0 : C_0^p(E) \otimes C(\Omega) \to C_0^p(\Gamma)$, inducing

$\nu_0: E_0(E) \otimes C(\Omega) \to E_0(\Gamma)$, with $d_0 \circ \nu_0 = \nu_0 \circ (d_0 \otimes 1 + \omega \otimes d))$. We form now $\varrho_r \circ \nu_r = \pi_r$, and have the induced maps

$$\pi_0: E_0(E) \otimes C(\Omega) \to E_0(E)$$
$$\pi_r: E_r(E) \otimes H(\Omega) \to E_r(E),\ r \geqq 1,$$

which commute with the d_r. If we write $z * v$ for $\pi_r(z \otimes v)$ $(r \geqq 0)$, we can express this by the formula

$$d_r(z * v) = d_r z * v \quad \text{for} \quad r \geqq 1,\ z \in E_r(E),\ v \in H(\Omega), \quad \text{while}$$
$$d_0(c * v) = d_0 c * v + \omega c * dv \quad \text{for} \quad c \in E_0(E),\ v \in C(\Omega)\ .$$

Moreover the operation $*$ commutes with the passage to homology groups; if [ ] denotes homology class, then we have $[z * v] = [z] * v$, if z is a cycle of $E_r(E)$, $r \geqq 1$, $v \in H(\Omega)$; for $r = 0$ this becomes $[z * v] = [z] * [v]$, if z is a cycle of $E_0(E)$ and $v \in Z$. This follows from the fact that the operation $*$ is derived from the original operation $\otimes$ in $C(E) \otimes C(\Omega)$ by passage to quotient groups; in more detail: if $\bar{z}$ is a chain in $C_r^p(E)$, representing $z \in E_r(E)$, and if $\bar{u}$ is a cycle of Ω, representing $u \in H(\Omega)$, then the chain $\bar{z} * \bar{u}$ represents $z * u$ in $E_r(E)$.

II.5. In order to study associativity relations, we consider the two maps f_0, f_1 of $E \times \Omega \times \Omega$ into E, defined by $f_0(q_1, q_2, q_3) = (q_1 \cdot q_2) \cdot q_3$, resp. $f_1(q_1, q_2, q_3) = q_1 \cdot (q_2 \cdot q_3)$. Of course the two maps are not equal, but they are homotopy-associative with stationary projection and even with $(e, e, e,)$ stationary, i. e. there exists a homotopy g $(g_t,\ 0 \leq t \leq 1)$, such that:

(1) $g_0 = f_0, g_1 = f_1,$

(2) $p \circ g_t(q_1, q_2, q_3) = p(q_1)$ for all $q_1, q_2, q_3, t,$

(3) $g_t(e, e, e) = e$ for all t,

defined as follows: q_1, q_2, q_3 are paths in X, i. e. maps of [0,1] into X; the homotopy to be constructed is just a shift in parametrization. We put

$$g_t(q_1, q_2, q_3)(s) = \begin{cases} q_1\left(\dfrac{4s}{t+1}\right) & \text{for} \quad 0 \leq s \leq \dfrac{t+1}{4} \\[2ex] q_2(4s - t - 1) & \text{for} \quad \dfrac{t+1}{4} \leq s \leq \dfrac{t+2}{4} \\[2ex] q_3\left(\dfrac{4s - t - 2}{2 - t}\right) & \text{for} \quad \dfrac{t+2}{4} \leq s \leq 1\ . \end{cases}$$

With the help of a diagram, consisting of the unit square in a t-s-plane together with the segments $(0, \frac{1}{4}) - (1, \frac{1}{2})$ and $(0, \frac{1}{2}) - (1, \frac{3}{4})$, one checks that g has the required properties; continuity is proved as in [10, p. 475]. For any n-cube $w = w(x_1, \ldots, x_n)$ in $E \times \Omega \times \Omega$ we define an $(n+1)$-cube kw by $kw(x_1, \ldots, x_{n+1}) = (-1)^n g x_{n+1}(w(x_1, \ldots, x_n))$, Clearly $D^{(\infty)}$ goes into $D^{(\infty)}$, and we have a map of the chains, which raises dimension by 1. One verifies that

II.5.1 $$dk + kd = f_1 - f_0 ,$$

so that k provides a chain homotopy between the induced chain maps f_0 and f_1. (We note that because of property (2) k sends $A^p \otimes C(\Omega) \otimes C(\Omega)$ into A^p; from this one could prove directly that II.5.5 holds.)

Clearly the subset $\Omega \times \Omega \times \Omega$ is carried into Ω by g_t; it follows that the two maps f_{0*} and f_{1*} of $H(\Omega) \otimes H(\Omega) \otimes H(\Omega)$ into $H(\Omega)$ are identical (the chain maps are chain homotopic). This means that $*$-multiplication in $H(\Omega)$ is associative. That the point e acts as unit follows, as noted in II.1, from the fact that the two maps of Ω into itself, defined by $q \to q \cdot e$, resp. $e \cdot q$, are homotopic to the identity.

We come to the connection of $*$ with the chain equivalence [10, p. 447]

$$\psi : J = C(X) \otimes C(\Omega) \to E_0(E) .$$

We recall that $J_p = C_p(X) \otimes C(\Omega)$ corresponds to $E_0^p(E)$, and that ψ commutes with $\omega \otimes d_\Omega$ and d_0, which can be expressed by saying that we take $C(X)$ with $d_X = 0$. For a given p-cube u of X, and cubes v, w of Ω we form the two cubes $K(u, v * w)$ and $K(u, v) * w$, both of which belong to T^p, and both of which belong to $D^{(\infty)}$ if any one of u, v, w does. We get therefore two induced maps from $C(X) \otimes C(\Omega) \otimes C(\Omega)$ to $E_0(E)$, which we call κ_1 and κ_2; we can write $\kappa_1 = \psi \circ (1 \otimes \bar{\varrho})$ and $\kappa_2 = \pi_0 \circ (\psi \otimes 1)$ (see II.1 for $\bar{\varrho}$ and II.3 for π_0), which shows that κ_1 and κ_2 are chain maps (for $d_X = 0$). We note that $B \cdot K(u, v) * w = u$ and $F \cdot K(u, v) * w = v * w$, so that $\kappa_1 = \psi \circ \varphi \circ \kappa_2$. With the operator k from [10, p. 448] we form now $s = k \circ \kappa_2$. We have then $d_0 s + sd = d_0 k \kappa_2 + k \kappa_2 d = (d_0 k + kd_0) \kappa_2 = (\psi \circ \varphi - 1) \kappa_2 = \kappa_1 - \kappa_2$, so that we have a chain homotopy between κ_1 and κ_2. If x is a chain of X, and u, v are cycles of Ω, then $\kappa_1(x \otimes u \otimes v)$ and $\kappa_2(x \otimes u \otimes v)$ are homologous. We can restate this as

II.5.2 $$\psi(x \otimes u) * v \sim \psi(x \otimes (u * v)) \text{ in } E_0(E), \text{ for } x \in C(X),\ u,\ v \in Z(\Omega) .$$

Going to E_1 and recalling that $*$ commutes with taking homology classes, we find

II.5.3 $\quad (x \otimes u) * v = x \otimes (u * v)$ in $E_1(E)$, for
$x \in C(X)$, u, $v \in H(\Omega)$;

here $E_1(E)$ is identified with $C(X) \otimes H(\Omega)$ by ψ_*, thus defining $(x \otimes u) * v = \psi_*^{-1}(\psi_*(x \otimes u) * v)$. The differential d_1 of E_1, according to Serre, becomes $d_X \otimes 1$ on $C(X) \otimes H(\Omega)$, and E_2 is the corresponding homology group. Let x now be a cycle of X and u, v as before. Making use of the canonical map of $H(X) \otimes H(\Omega)$ into $H(C(X) \otimes H(\Omega)) = E_2$ (with [ ] again meaning homology class), we see

II.5.4 $\quad [x]\otimes(u*v)=[x\otimes(u*v)]=[(x\otimes u)*v]=[x\otimes u]*v=([x]\otimes u)*v$.

Associativity of $H(\Omega)$ and relation II.5.3 imply the equation

$(x\otimes u)*(v*w)=((x\otimes u)*v)*w$ in $E_1(E)$, with $x \in C(X)$, u, v, $w \in H(\Omega)$;

and the $x \otimes u$ span E_1. Going to homology classes one finds then

II.5.5 $\quad z * (v * w) = (z * v) * w$ for $z \in E_r(E)$, $r \geq 1$, v, $w \in H(\Omega)$.

Similarly one proves, starting from E_1,

II.5.6 $\quad z * e = z$ for $z \in E_r(E)$, $r \geq 1$, e the unit of $H(\Omega)$.

We collect our results in the following theorem, into which we incorporate a statement about the grading of the various groups, and about coefficients; both statements are proved easily by going back to the map $\varrho: \Gamma \to C(E)$, by which all other maps are induced:

Theorem II. 5.A: Let X be a 1-connected space. The map $\gamma: E \times \Omega \to E$ induces a pairing, written $*$, of

a) $H(\Omega)$ and $H(\Omega)$ to $H(\Omega)$; in detail: $H_m(\Omega)$ and $H_n(\Omega)$ to $H_{m+n}(\Omega)$;
b) $E_r(E)$ and $H(\Omega)$ to $E_r(E)$, $r \geq 1$; $E_r^{p,q}(E)$ and $H_n(\Omega)$ to $E_r^{p,q+n}(E)$;
c) $E_0(E)$ and $C(\Omega)$ to $E_0(E)$; $E_0^{p,q}(E)$ and $C_n(\Omega)$ to $E_0^{p,q+n}(E)$

with the following properties:

1) The pairing is bilinear, and associative, i. e., for $x \in E_r$, $r \geq 1$, u, v, $w \in H(\Omega)$ the relations $(x * u) * v = x * (u * v)$, $(u * v) * w = u * (v * w)$ hold; the point e satisfies $e * v = v * e = v$ for $v \in H(\Omega)$, and $x * e = x$ for $x \in E_r(E)$, $r \geq 1$;
2) $d_r(x * v) = d_r x * v$ for $r \geq 1$; $d_0(x * v) = d_0 x * v + \omega x * dv$;
3) $*$ commutes with the identification $E_{r+1} = H(E_r)$;
4) In E_1 and E_2 one has, for $x \in C(X)$, resp. $H(X)$, u, $v \in H(\Omega)$, $(x \otimes u) * v = x \otimes (u * v)$, where E_1 is canonically identified with

$C(X) \otimes H(\Omega)$, and where $H(X) \otimes H(\Omega)$ is mapped canonically into $H(X, H(\Omega)) = E_2$;

the coefficients for $C(\Omega), H(\Omega), E_r, r \geq 0$, are taken from a commutative ring R with unit; $C(X)$ and $H(X)$ can be understood either over the integers or over R (the tensor products are taken accordingly).

$H(\Omega)$, with $*$ as multiplication, will be called the Pontryagin-ring or-algebra $H_*(\Omega)$ of Ω; it has e as unit.

Remark: The theorem actually applies to a more general situation, in a slightly extended form. Let Y be a fiber space over B, with projection p; suppose a space M operates on Y and on itself, i. e. maps of $Y \times M$ into Y and of $M \times M$ into M (written as products) are given, with the following properties:

1) $p(y \cdot m) = p(y)$ for all $y \in Y$, $m \in M$;

2) there is an e in Y and an e' in M, such that $e \cdot e' = e$ and $e' \cdot e' = e'$;

3) the two maps $(y, m_1, m_2) \to (y \cdot m_1) \cdot m_2$, resp. $y \cdot (m_1 \cdot m_2)$ of $Y \times M \times M$ into Y are homotopic, with (e, e', e') stationary and with $F \times M \times M$ staying in F, where F is the fiber through e;

4) the map $q \to q \cdot e'$ of F into itself is homotopic to the identity;

5) the maps $m \to m \cdot e'$, resp. $e' \cdot m$ of M into itself are homotopic to the identity;

6) the two maps $(m_1, m_2, m_3) \to (m_1 \cdot m_2) \cdot m_3$, resp. $m_1 \cdot (m_2 \cdot m_3)$ of $M \times M \times M$ into itself are homotopic, with (e', e', e') stationary;

7) B 1-connected, M and the fibers of Y 0-connected.

$H(M)$ will then operate on $E_r(Y)$ $(1 \leq r \leq \infty)$, on $H(F)$ and on $H(Y)$; it also operates on the subgroups D^p, by which $H(Y)$ is filtered. The associativity relations etc. hold, suitably modified; the operation on E_∞ is obtained from that on $H(Y)$ by going to the factor groups. In the proof of II.5.2 e. g. one has to replace $C(X) \otimes C(\Omega) \otimes C(\Omega)$ by $C(B) \otimes C(F) \otimes C(M)$. Conditions 2) — 7), in particular 4) — 6), could of course be modified.

This situation occurs when a Lie group operates on a space, e. g. for principal bundles; this is what has been considered, in cohomology, with Leray's theory, by Leray [5, 6] and Borel [1].

II.6. We give a brief discussion of cohomology relations; the contents of this section can be considered as a translation into singular theory of results of Leray [6] and Borel [1]. For simplicity we restrict ourselves to a field k as coefficients (in the general case one would have to consider pairing of groups to their tensor product).

For a cochain a in $C^r(X)$ and a cube u in $Q_n(X)$, with $r \leq n$, one can define the $\frown$-product, which we write also as $a \cdot u$, by the formula

$$a \cdot u = \sum_H \varrho_{H,K} \cdot a(\lambda_H^1 u)\, \lambda_K^0 u \ ,$$

where K runs through the subsets of r elements of $\{1, 2, \ldots, n\}$, H is the complement of K, and $\varrho_{H,K}$ is the familiar sign determined by the number of inversions (cf. the $\smile$-product in [10, p. 441], by which, as in [2], the $\frown$-product is determined); this induces a $\frown$-product between $C^r(X)$ and $C_n(X)$ with values in $C_{n-r}(X)$. The $\smile$- and $\frown$-product, the Kronecker index $KI(a, x)$ (defined as $a(x)$ for $a \in C^n(X)$, $x \in C_n(X)$; we identify $C^n(X)$ with the k-linear forms on $C_n(X)$), the index $In(x)$ (defined as $\sum k_i$ for $x = \sum k_i\, u_i \in C_0(X)$), and the differential d (boundary or coboundary) satisfy all the usual relations [2, p. 432]; the unit 1 of $C^*(X)$ is the constant function, with value $1 \in k$.

Let Y be a fiber space over B, with projection p and fiber F. One verifies that the $\frown$-product pairs $A^{*p,q}$ and $A^{p',q'}$ to $A^{p'-p,q'-q}$. It follows easily that, in addition to the $\smile$-product in E_r^* (as in [10]), there is an induced $\frown$-product, pairing $E_r^{*p,q}$ and $E_r^{p',q'}$ to $E_r^{p'-p,q'-q}$, a KI (between $E_r^{*p,q}$ and $E_r^{p,q}$, 0 otherwise), an In (for $E_r^{0,0}$), that all the usual relations are satisfied (the differential is d_r for $r < \infty$, 0 for $r = \infty$), and that the operations are compatible with the identification $E_{r+1} = H(E_r)$ etc.; since we have field coefficients, $E_r^{*p,q}$ is identified, by way of KI, with the space $Hom(E_r^{p,q})$ of k-linear forms on $E_r^{p,q}$; all this is a straightforward generalization of the corresponding facts for $H(Y)$ and $H^*(Y)$. For E_1 the $\frown$-product translates into the natural $\frown$-product between $C^p(B, H^q(F))$ and $C_{p'}(B, H_{q'}(F))$ relative to the $\frown$-product pairing of the coefficients $H^q(F)$ and $H_{q'}(F)$, except that a factor $(-1)^{p(q'-q)}$ has to be added (cf. [10, p. 454]); corresponding statements hold for E_2. If e. g. the Betti numbers of F are finite, then the $\smile$- and $\frown$-products become the canonical products in $H^*(B) \otimes H^*(F)$ and $H(B) \otimes H(F)$. Similar statements hold for KI.

If X and Y are any two spaces, one can define $\smile$, $\frown$, KI, In in the chain complex $C(X) \otimes C(Y)$ and its cochain complex (using e. g. the fact that the map μ is a chain equivalence (I.3)); the relations $\mu^*(f \smile g) = \mu^* f \smile \mu^* g$ and $\mu_*(\mu^* f \frown x) = f \frown \mu_* x$ hold. There is an imbedding of $C^*(X) \otimes C^*(Y)$ into the cochains of $C(X) \otimes C(Y)$, sending $a \otimes b$ into $\mu^*(\pi_X^* a \smile \pi_Y^* b)$; here π_X and π_Y are the projections of $X \times Y$ onto X and Y. If e. g. Y has finite Betti numbers, this induces an isomorphism of $H^*(X) \otimes H^*(Y)$ (skew product) with the cohomology algebra of $C(X) \otimes C(Y)$ (cf. [10, p. 458 and 473]), since then, by way of KI, $H^*(X) \otimes H^*(Y)$ is the space of all linear functions on $H(X) \otimes H(Y)$.

We turn now to the situation of II.2, so that E is the space of paths, ending at x_0, in the 1-connected space X, with fiber Ω. The cochain-algebra Γ^* of $\Gamma = C(E) \otimes C(\Omega)$ is filtered in the standard fashion, Γ^{*p} = annihilator of Γ^{p-1}; reasoning as above one sees that there are induced $\smile$, $\frown$, KI, In in the spectral sequence. The terms $E_r(\Gamma)$ can now be identified with $E_r(E) \otimes H(\Omega)$ (for $r \geq 1$) [in more detail: $E_r^{p,n}(\Gamma) = \sum_{s+t=n} E_r^{p,s}(E) \otimes H_t(\Omega)$], since we have field coefficients; as always $E_0^{*p,q}(\Gamma)$ is identified with $Hom(E_0^{p,q}(\Gamma))$. We assume now that $H(\Omega)$ has finite Betti numbers. We can then, as above, identify $E_1^*(\Gamma)$ with $E_1^*(E) \otimes H^*(\Omega)$, and, inductively, $E_r^*(\Gamma)$ with $E_r^*(E) \otimes H^*(\Omega)$, with the operations $\smile$, $\frown$, KI, In, d_r going into the canonical operations for the tensor products (d_r becomes $d_r \otimes 1$). The map $\varrho : \Gamma \to C(E)$ induces now maps π_r, π_r^* of $E_r(E) \otimes H(\Omega)$ into $E_r(E)$, resp. of $E_r^*(E)$ into $E_r^*(E) \otimes H^*(\Omega)$, which satisfy the various compatibility relations; in particular they commute with the differentials, and satisfy $KI(\pi_r^* a, x) = KI(a, \pi_r x)$.

The relation 4. of Th. II.5.A becomes now

$$\pi_2(x \otimes x' \otimes x'') = x \otimes (x' * x'') \qquad \text{for} \qquad x \in H(X),\ x', x'' \in H(\Omega).$$

By duality, i. e. by the invariance of the KI just mentioned, one derives from this that for $a \in H^*(X)$, $b \in H^*(\Omega)$ one has $\pi_2^*(a \otimes b) = a \otimes \bar{\varrho}^*(b)$ where $\bar{\varrho}^*(b)$ is the image of b under the map $\bar{\varrho}^* : H^*(\Omega) \to H^*(\Omega) \otimes H^*(\Omega)$ induced by the multiplication $\gamma : \Omega \times \Omega \to \Omega$ (II.1); this is the exact analog of results of Borel [1]. As well known, the element $\bar{\varrho}^*(b)$ has the form $1 \otimes b + b \otimes 1 + \Sigma c_i \otimes c_i'$; with $0 < \dim c_i < \dim b$, corresponding to the fact that e is unit for $H_*(\Omega)$ [10; p. 476].

The operation of $H(\Omega)$ on $E_r(E)$, by $*$-multiplication, induces by duality an operation on $E_r^*(E)$; we establish some relations for this operation, which are essentially equivalent to some of the relations given by Leray in [6]. For $x \in H_n(\Omega)$ and $a \in E_r^{*p,q}(E)$ we define an element $a * x \in E_r^{*p,q-n}(E)$ by requiring the equation

$$KI(a * x, y) = KI(a, y * x) \; (= KI(\pi_r^* a, y \otimes x))$$

to hold for all $y \in E_r^{p,q-n}(E)$. One sees easily that $(a * x) * x' = a * (x' * x)$ and $(d_r a) * x = d_r(a * x)$. Similarly we can let $H(\Omega)$ operate on $H^*(\Omega)$; relation 4. of Th. II.5.A implies then that in E_1 and E_2 we have

$a \otimes (b * x) = (a \otimes b) * x$, for $a \in C^*(X)$, resp. $H^*(X)$, $b \in H^*(\Omega)$, $x \in H(\Omega)$.

To study relations between $\cup$- and $\cap$-product, we note first the following equations, with $a, b \in E_r^*(E)$, $y \in E_r(E)$, $x \in H(\Omega)$, $\dim a = r$, $\dim b = s$, $\dim x = t$:

$$KI((ab) * x, y) = KI(\pi_r^*(a) \cdot \pi_r^*(b), y \otimes x) \quad \text{and}$$
$$KI((a * x)b, y) = (-1)^{st} KI(\pi_r^*(a)(b \otimes 1), y \otimes x)$$

(for the second equation note: left hand side $= KI(a * x, b \cap y) = KI(a, (b \cap y) * x) = KI(\pi_r^* a, (b \cap y) \otimes x) = (-1)^{st} KI(\pi_r^* a, (b \otimes 1) \cap (y \otimes x)) =$ right hand side). Secondly, the relation $z * e = z$ for $z \in E_r(E)$, e the unit of $H_*(\Omega)$, implies by duality that, for $a \in E_r^*(E)$, the element $\pi_r^*(a)$ of $E_r^*(E) \otimes H^*(\Omega)$ has the form $a \otimes 1 + R_a$, where all terms $a' \otimes a''$ in R_a have $\dim a'' > 0$. It follows easily that the element $\Delta(ab)$, defined as $\pi_r^*(ab) - \pi_r^*(a)(b \otimes 1) - (a \otimes 1)\pi_r^*(b)$, equals $R_a R_b - ab \otimes 1$. Suppose now that x is a primitive or minimal element of $H(\Omega)$ in the sense of Hopf [4], i. e. $b \cap x = 0$ for $0 < \dim b < \dim x$, and $\dim x > 0$. It follows that $KI(\Delta(ab), y \otimes x) = 0$ for all y. Combined with the above relations for the KI, this is easily seen to imply

$$(ab) * x = (-1)^{st}(a * x)b + a(b * x) ;$$

this can be stated as follows: If x is minimal and $\dim x$ is even, then the operation $a \to a * x$ is a derivation, if $\dim x$ is odd, it is a "right" antiderivation (if Ω would operate on E on the left, we would get here the customary antiderivation).

All the above statements apply of course, suitably interpreted, to the case discussed in the remark at the end of II.5.

III. Applications

1. If (Y, B, F, p) is a fiber bundle (with bundle Y, base space B, fiber F, projection p) in the sense of Serre [10], (B always 1-connected), then an element x of $H_p(B)$ $(p > 0)$ is called transgressive, if $d_2 x = \cdots = d_{p-1} x = 0$. This is to be understood by using the following conventions: e represents the generator of H_0 of any space, given by a point; we identify $H(X)$ and $H(F)$ with their canonical images $H(X) \otimes e$ and $e \otimes H(F)$ in E_2 (this is compatible with the $*$-operation, as theorem II.5.A shows), so that e. g. the x in $d_2 x$ stand for $x \otimes e$. Since $d_2 x = 0$, i. e. x is a cycle of E_2, it determines an element of E_3, namely its homology

class, which we denote by x again (with the analogous convention for any cycle of any E_r), etc. The element $d_p x$ is then an element of a certain factor group of $H_{p-1}(F)$.

Let E, as in II, denote the space of paths in the space X, ending at x_0, considered as fiber space over X with fiber Ω.

Theorem III. 1. A: Suppose $H(X)$, coefficients in the principal ideal ring R, is R-free, and all elements of $H(X)$ are transgressive (in E). Then $H(\Omega)$ contains a subgroup T, isomorphic image of $H_+(X)$ (= elements of $H(X)$ of positive dimension) under a map which lowers dimension by 1; and the Pontryagin ring $H_*(\Omega)$ is the free associative algebra generated by T, with e as unit.

Proof: We start from the relation $E_2 = H(X) \otimes H(\Omega)$ (which holds since $H(X)$ is free). Thm. II.5.A implies then that E_2 is "totally transgressive" in the sense that for any element z of E_2, which lies in $E_2^{p,q}$, all differentials $d_2 z, \ldots, d_{p-1} z$ vanish $(d_2(x \otimes v) = d_2(x \otimes e) * v)$ $= d_2(x * v) = (d_2 x) * v = 0$, $[x * v] = [x] * v = x * v$ in E_3, etc.). Our argument will be based on the fact that $H(E)$ and therefore E_∞ are trivial, since E is contractible, and that therefore E_2 has to be "extinguished" by application of the d_r. We choose a base $\Sigma = \{x_i\}$, i running through an index set J, of $H_+(X)$, consisting of homogeneous-dimensional elements; for each x_i we choose an element $\bar{x}_i$ of $H_+(\Omega)$, such that $d_p x_i = \bar{x}_i$, with $p = \dim x_i$, and consider the collection M of elements $\bar{x}_{i_1 i_2 \ldots i_k} = \bar{x}_{i_1} * \bar{x}_{i_2} * \cdots * \bar{x}_{i_k}$, $k \geqq 1$, $i_j \in J$ for $j = 1, \ldots, k$. We claim that these elements are independent, and that together with e they form a basis for $H(\Omega)$; this will obviously prove the theorem, with T generated by the $\bar{x}_i$. We call k the length and $\dim \bar{x}_{i_1}$ the height of $\bar{x}_{i_1 i_2 \ldots i_k}$. Suppose there were linear relations between the elements of M; consider the relations in which the maximum length of the elements occurring is as small as possible, say k. Choose such a relation $r \equiv \Sigma c_\alpha z_\alpha = 0$, with $z_\alpha \in M$. It can be written as a sum $r' + r''$ of two parts, where the first part contains all elements of maximum height, say p. We write each z_α as $\bar{x}_\alpha * \bar{z}_\alpha$ with $\bar{z}_\alpha \in M$ or $= e$, in the obvious fashion, and form the elements $y_\alpha = x_\alpha \otimes \bar{z}_\alpha$ of E_2, and put $y = \Sigma c_\alpha y_\alpha = y' + y''$, where y' and y'' correspond to r' and r''. We go now to E_{p+1}.

There r'' has become 0, since the elements $x_\alpha \otimes \bar{z}_\alpha$ in y'' map onto the $\bar{x}_\alpha * \bar{z}_\alpha$ in r'' under the appropriate differentials $d_q, q \leqq p$; r' is then also 0 in E_{p+1}. Similarly y' maps onto r' by d_{p+1}; we have therefore $d_{p+1}(y') = 0$. By total transgressivity y' is not image under any

d_r; it follows that $y' = 0$, since otherwise E_∞ would contain a non-trivial element. The $\bar{z}_\alpha$, occurring in y', are of length $< k$, and the elements of length $< k$ form a free subgroup of $H(\Omega)$, by our minimality assumption. Since $H(X)$ is free, it follows that the elements $x_\alpha \otimes \bar{z}_\alpha$, occurring in y', are independent in E_2, and, again by total transgressivity, in E_{p+1}; this contradicts the fact that y' is 0 in E_{p+1}, and our assertion concerning the independence of the elements of M is proved.

Suppose there were elements in $H_+(\Omega)$, which are not linear combinations of elements of M; let n be the smallest dimension in which this happens. All elements of $E_2^{n-q+1,q}$, for $q < n$, are then of the form $\Sigma\, x_\alpha \otimes t_\alpha$, with $\dim t_\alpha = q < n$; the t_α are therefore generated by M and e. Considering now the action of $d_2, d_3, \ldots, d_{n+1}$, one sees that $E_{n+2}^{0,n}$ is a quotient group of $H_n(\Omega)$ by a subgroup which is contained in the subgroup generated by M (note that $d_p(x \otimes t)$, $\dim x = p$, is congruent to $\bar{x} * t$ modulo the images of $d_2, \ldots, d_{p-1}$). But $E_{n+2}^{0,n}$ must be 0, *q. e. d.* We have shown that the element $\bar{x}_{i_1 \ldots i_k}$ of M and e form a basis for $H(\Omega)$; the statement about the ring structure is immediate from the relation $\bar{x}_{i_1 \ldots i_k} * \bar{x}_{j_1 \ldots j_l} = \bar{x}_{i_1 \ldots i_k j_1 \ldots j_l}$.

Since spherical cycles are transgressive [10, p. 452] the theorem III.1.A applies when all cycles of X are spherical. We can therefore state the corollary:

III. 1. B: The Pontryagin ring of the space Ω of loops in the space $S^{n_1} \vee S^{n_2} \vee \cdots \vee S^{n_k}$ (union of k spheres of dimensions $n_i > 1$, all attached at one point) is the free associative algebra on k generators of dimensions $n_i - 1$.

For the case of a single sphere this can be read off from Morse's results [7]. We describe a somewhat more general case in which theorem III.1.A is applicable.

Theorem III. 1. C: Let $\hat{X}$ be the join of a 0-connected space X with the 0-sphere S^0; then in the spectral sequence of $E(\hat{X})$ all elements of $H_+(\hat{X})$ are transgressive.

Proof: We consider $\hat{X}$ as the product $X \times I$ with all of $X \times 0$, resp. $X \times 1$, identified to a point x_0, resp. x_1. $\hat{X}$ is clearly 1-connected. Define

$$X^0 = \{(x,t) : x \in X,\ 0 < t \leq \tfrac{1}{2}\} \cup \{x_0\},\ X^1 = \{(x,t) : x \in X,\ \tfrac{1}{2} \leq t < 1\} \cup \{x_1\},$$

and identify each $x \in X$ with $(x, \tfrac{1}{2}) \in X$. The inclusion map of the pair (X^0, X) into $(\hat{X}, X^1)$ is homotopic (as map of pairs) to the map f defined by $f(x_0) = x_0$, $f(x, t) = (x, 2t)$ for $0 < t < \frac{1}{2}$, $f(x) = x_1$, for $x \in X$; f can be considered as a map of (X^0, X) into $(\hat{X}, x_1)$. Using the fact that X^0 and X^1 are contractible spaces, and using the excision $(\hat{X}, X^1) \supset (X^0, X)$, one shows easily that the map f^* is an isomorphism of $H(X^0, X)$ and $H(\hat{X}, x_1)$. Since X^0 is contractible, we can, by an application of the covering homotopy theorem, find a map $g: X^0 \to E$, such that $p \circ g = f$. If we let Ω be the fiber of E over x_1 (we might choose x_1 as base point for E), then g maps the subset X of X^0 into Ω, and so defines a map g of the pair (X^0, X) into (E, Ω). It follows now that $f_* = p_* \circ g_*$ (as map of $H(X^0, X)$ into $H(\hat{X}, x_1)$ and that p_* maps $H(E, \Omega)$ onto $H(\hat{X}, x_1)$; by the geometric definition of transgression [10, p. 452], this proves the assertion of the theorem.

III.2. Let (Y, B, F, p) be a fiber space, with F operating on Y as described at the end of II.5., and suppose B is a homology-k-sphere. Then one has the Wang sequence [10, p. 471]:

$$\to H_i(F) \to H_i(E) \to H_{i-k}(F) \overset{\theta}{\to} H_{i-1}(F) \to \cdots .$$

The map θ in this sequence is obtained as follows: Let s be a generator for $H_k(B)$; map $H_{i-k}(F)$ into $E_2^{k, i-k}$ by sending x into $s \otimes x = g(x)$; then $\theta = d_k \circ g$. Put now $d_k s = v$; then by theorem II.5.1 we have

$\theta(x) = d_k(s \otimes x) = d_k((s \otimes e) * x) = (d_k s) * x = v * x$. We have:

The map $\theta: H_{i-k}(F) \to H_{i-1}(F)$ in the Wang (homology-) sequence is given by $\theta(x) = v * x$, where v is the characteristic element determined by $d_k s = v$. In case of a principal bundle over an actual sphere S^k, the element v is the (spherical) homology class determined by the characteristic element of the bundle in the sense of Steenrod [11, pp. 97, 180], with the sign reversed.

Let a be any element of $H^p(F)$, with $0 < p < k - 1$; clearly all $d_r\, a$ vanish. In $E_k (= E_2)$ we have $a \frown s \in E_k^{k, -p} = 0$. It follows that

$$0 = d_k(a \frown s) = (-1)^{p+k} d_k a \frown s + a \frown d_k s = a \frown v;$$

in other words, v is a minimal element of $H(F)$.

Added in proof: A recent paper by T. Kudo (Homological structure of fibre bundles, J. Osaka City Univ. 2 (1952, 101–140) contains a construction similar to that of II.5. As a common generalization one could consider the situation of the «remark» in II.5, with Y and M paired not to Y, but to another fiber space Y' (with base B', fiber F', projection p'), and the projections commuting with a given map $f: B \to B'$, i. e. satisfying $p'(y \cdot m) = f \circ p(y)$; there is then an induced pairing of F and M to F'. Results and proofs are analogous to the earlier ones; for instance in E_2 one has $(x \otimes y) * z = f_*(x) \otimes (y * z)$, for $x \in H(B)$, $y \in H(F)$, $z \in H(M)$.

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