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**Autor:** Bott, R. / Samelson, H.  
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# On the Pontryagin product in spaces of paths

By R. BOTT and H. SAMELSON<sup>1)</sup>

## Introduction

For a topological (arcwise connected) space  $X$ , let  $E$  be the function space consisting of the paths in  $X$  which start at a certain point  $x$ , and let  $\Omega$  be the subspace of  $E$  consisting of the closed paths or loops; these spaces have been studied in particular by M. Morse [7] and J.-P. Serre [10];  $E$  is a fiber space over  $X$ . Now  $\Omega$  admits a natural multiplication: two loops in succession make a new loop (actually there is a more general operation between  $E$  and  $\Omega$ ). This multiplication gives rise to a multi-

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plication of the elements of the homology group of  $\Omega$ ; we call this the Pontryagin-multiplication, since an entirely similar concept for group spaces was introduced by Pontryagin [8] (cf. also [9]). In part II we study the relation between Leray's spectral sequence of  $E$  (in Serre's formulation [10]) and the Pontryagin product. A closely related situation, involving compact Lie groups and Leray's cohomology theory, has been considered by Leray [5, 6] and A. Borel [1]; our results are analogous to theirs. In part III we determine, as application, the Pontryagin ring of a space which is union of several spheres with a point in common. — We have to consider the homology of Cartesian product spaces. This makes necessary a modification of Serre's cubic homology theory; we present this in part I. Generally speaking we follow Serre's definitions, notation and conventions, with some minor deviations; we assume familiarity with his paper.

## I. Cubic homology

I.1. We recall briefly some definitions from [10]. Let  $X$  be a 0-connected (i. e. arcwise connected) space. A singular  $n$ -cube in  $X$  is a map  $u: I^n \rightarrow X$ , where  $I$  denotes the unit interval  $[0,1]$ , and  $I^n$  means the Cartesian product of  $n$  factors  $I$ ; 0-cubes are simply points of  $X$ . With a real  $\varepsilon$ ,  $0 \leq \varepsilon \leq 1$ , and an integer  $i$ ,  $1 \leq i \leq n$ , the operator  $\lambda_i^\varepsilon$  associates with each singular  $n$ -cube a singular  $(n-1)$ -cube by

$$(\lambda_i^\varepsilon u)(x_1, \dots, x_{n-1}) = u(x_1, \dots, x_{i-1}, \varepsilon, x_i, \dots, x_{n-1}).$$

The singular  $n$ -cubes are free generators of the (abelian) group  $Q_n(X)$ . The direct sum  $Q(X)$  of all  $Q_n$ ,  $n \geq 0$ , is mapped into itself by the operator  $d$ , which on  $n$ -cubes is defined as  $\sum_1^n (-1)^i (\lambda_1^0 - \lambda_i^1)$ . To define homology groups, one has to introduce the  $d$ -stable subgroup  $D = \sum D_n(X)$  where  $D_n(X)$  is generated by the  $n$ -cubes which are degenerate along the last coordinate; the homology of  $X$  is that of  $Q/D$ . We now introduce, for each integer  $p \geq 1$ , the subgroup  $D^{(p)}$ , generated by those cubes which are degenerate along any one of their last  $p$  coordinates (in particular  $D^{(1)} = D$ ; if  $p > n$ , this means that  $u \in Q_n$  is degenerate along some coordinate); with  $D_n^{(p)} = D^{(p)} \cap Q_n$  we have  $D^{(p)} = \sum D_n^{(p)}$ . Clearly  $D^{(1)} \subset D^{(2)} \subset \dots$ . We put  $D^{(\infty)} = \cup_p D^{(p)}$ ; clearly  $D_n^{(\infty)} = D_n^{(n)}$ . We prove

**Theorem I.1.A.:** (a) For each  $p$ , with  $1 \leq p \leq \infty$ ,  $D^{(p)}$  is a  $d$ -stable subgroup of  $Q$ ; (b) the natural map of the homology group of  $Q/D^{(p)}$

into that of  $Q/D^{(p+1)}$  and of  $Q/D^{(\infty)}$ , induced by  $D^{(p)} \subset D^{(p+1)} \subset D^{(\infty)}$ , is an isomorphism onto.

Proof: First take  $p < \infty$ . (a) is verified easily from the formula for  $d$ : If  $u$  belongs to  $D^{(p)}$ , and is degenerate along the  $r$ -th coordinate, then the terms  $\lambda_r^0 u$  and  $\lambda_r^1 u$  in  $du$  cancel; all other terms belong to  $D^{(p)}$  or even to  $D^{(p-1)}$ . For (b), the usual application of the exact sequence of the triple  $(Q, D^{(p+1)}, D^{(p)})$  [3, p. 28] shows that we have to prove that the relative homology group  $H(D^{(p+1)}, D^{(p)})$  is 0. To do this, we consider the operator  $\Delta$ , a linear map of  $Q$  into itself, raising dimension by 1, defined by  $\Delta u(x_1, \dots, x_{n+1}) = u(x_2, \dots, x_n, x_1 x_{n+1})$  for  $u \in Q_n$ ,  $n > 0$ ,  $\Delta u(x_1) = u$ , for  $u \in Q_0$ . We state the following properties of  $\Delta$ :

1.  $\Delta(D^{(p)}) \subset D^{(p)}$  for all  $p \geq 1$ .
2. Defining  $\tau = 1 - d\Delta - \Delta d$  ( $1 = \text{identity map}$ ), we have  $\tau(D^{(p+1)}) \subset D^{(p)}$ .

Property 1. is clear; if  $u$  does not depend on  $x_r$ ,  $\Delta u$  does not depend on  $x_{r+1}$ . For 2. a direct computation shows that for  $u \in Q_n$

$$\tau u = \lambda_1^0 \Delta u \pm (\lambda_{n+1}^1 \Delta u - \lambda_{n+1}^0 \Delta u) \pm (\Delta \lambda_n^1 u - \Delta \lambda_n^0 u).$$

If  $u$  belongs to  $D^{(p+1)}$ , then all the terms in  $\tau u$  belong to  $D^{(p)}$  or cancel if  $u \in D^{(1)}$ . If now  $x$  is a cycle of  $D^{(p+1)} \bmod D^{(p)}$ , i. e.  $x \in D^{(p+1)}$ ,  $dx \in D^{(p)}$ , then  $d\Delta x = x - \tau x - \Delta dx$  with  $\Delta x \in D^{(p+1)}$  and  $\tau x$  and  $\Delta dx \in D^{(p)}$ , i. e.  $x$  is  $\sim 0$  in  $D^{(p+1)} \bmod D^{(p)}$ , and  $H(D^{(p+1)}, D^{(p)})$  is 0. The assertion of I.1.A. for  $p = \infty$  follows now easily from  $D^{(\infty)} = \bigcup D^{(p)}$ . The identification of the homology groups of the various  $Q/D^{(p)}$  is natural, i. e. it commutes with the maps induced by a map of one space into another. Any one of the  $Q/D^{(p)}$  will be called the group  $C(X)$  of chains of  $X$ , and its homology group will be called the homology group  $H(X)$  of  $X$ ; we shall always use  $Q/D^{(\infty)}$ . Clearly all the chain groups are free; an application of a known theorem [3, p. 155] tells us that I.1.A. actually holds with arbitrary coefficients.

Since  $X$  is 0-connected, we can and shall, as in [10], restrict ourselves to cubes, all of whose vertices lie at a chosen point  $x_0$ . The resulting homology group is canonically isomorphic with the earlier one.

I.2 Let  $P$  be a fiber space over the space  $B$ , with projection  $p$ , in the sense of Serre [10]. We recall that  $p$  determines a filtration of  $Q(P)$ : a cube  $u$  in  $Q(P)$  is of filtration  $\leq r$  if the cube  $p \circ u$  in  $Q(B)$  depends only on its first  $r$  coordinates; such cubes generate the subgroup



$Tr = Tr(P)$ . The group of chains  $C(P)$  is filtered by the canonical images  $Ar(P) = Tr(P) + D^{(\infty)}/D^{(\infty)}$  of the  $Tr(P)$ . This sets up the spectral sequence of  $P$ , consisting of the groups  $E_0, E_1, E_2, \dots$  and the associated differentials  $d_0, d_1, d_2, \dots$ . The terms  $E_0, E_1, E_2$ , and the differentials  $d_0, d_1$  have been determined explicitly by Serre, with the group  $Q(P)/D^{(1)}$  as chain group. The results remain exactly the same for the new definition  $Q(P)/D^{(\infty)}$  of the chain group; the necessary changes in the reasoning are the following:

On p. 447 of [10], property 2) becomes

2') If  $u \in D^{(\infty)}$ , then either  $Bu$  or  $Fu \in D^{(\infty)}$ ; this permits construction of the map  $\varphi$ . The construction  $K$  of p. 448 of [10] can be made in such a fashion that condition 3) is replaced by

3') If either  $u$  or  $v \in D^{(\infty)}$ , then  $K(u, v) \in D^{(\infty)}$ . The construction  $S$  of lemma 5, p. 448 *ibid.* can be made such that condition 5) becomes

5') If  $u \in D^{(\infty)}$ , then  $Su \in D^{(\infty)}$ . One has to define  $K$  and  $S$  in the degenerate cases such that if  $u$  and  $v$ , resp.  $w$ , do not depend on certain coordinates, then  $K(u, v)$ , resp.  $Sw$ , do not depend on the corresponding coordinates. This is possible since the maps  $g$  of the sets  $A$  [10, p. 461, 462] do not depend on these coordinates and one can extend  $g$  to  $X$  by first collapsing  $A$  along these coordinates. The operation  $K$  defines then a map  $\psi$  of  $C(B) \otimes C(F)$  into  $E_0$ , which commutes with the appropriate differentials; the operation  $S$  maps  $E_0$  into itself and provides a chain homotopy of  $\psi \circ \varphi$  with 1. The only change required for the determination of  $d_1$  is the substitution of " $\epsilon D^{(\infty)}$ " for "dégénérés" on line 10, p. 450 in [10].

I.3. We consider the Cartesian product  $X \times Y$  of two 0-connected spaces  $X$  and  $Y$ . With an  $m$ -cube  $u$  in  $X$  and an  $n$ -cube  $v$  in  $Y$  we associate the  $(m+n)$ -cube  $u \times v$  in  $X \times Y$ , defined by

I.3.1  $u \times v(x_1, \dots, x_{m+n}) = (u(x_1, \dots, x_m), v(x_{m+1}, \dots, x_{m+n}))$ . This induces a map  $\mu$  of the tensor product  $Q(X) \otimes Q(Y)$  into  $Q(X \times Y)$ ; we write also  $x \times y$  for  $\mu(x \otimes y)$ . In the tensor product we consider the usual differential  $d = d \otimes 1 + \omega \otimes d$ , with  $\omega(x) = (-1)^m x$  for  $x \in Q_m(X)$ ; we have then  $d^2 = 0$ , and  $\mu$  commutes with  $d$ . If either  $x \in D^{(\infty)}(X)$  or  $y \in D^{(\infty)}(Y)$ , then clearly  $x \times y \in D^{(\infty)}(X \times Y)$ ; by passage to the quotient groups we get therefore [3, p. 159] a map, also called  $\mu$ , of  $C(X) \otimes C(Y)$  into  $C(X \times Y)$ , which commutes with  $d$ , and induces a map  $\mu_*$  of the homology groups.

**Theorem I.3. A:** The map  $\mu: C(X) \otimes C(Y) \rightarrow C(X \times Y)$  is a chain equivalence. We give a proof, following a suggestion of J.-P. Serre: We

filter  $C(X) \otimes C(Y)$  by the subgroups  $C^p = \sum_{i \leq p} C_i(X) \otimes C(Y)$ ; this defines the spectral sequence  $(E'_r)$ . We consider  $X \times Y$  as fiber space over  $X$ , with respect to the natural projection; this defines the subgroups  $A^p$  and the spectral sequence  $(E_r)$ , as in I.2. (The operations  $K$  and  $S$  are of course quite elementary now. We can put  $K(u, v) = u \times v$ . An  $n$ -cube  $u$  of  $X \times Y$  is a pair  $(u_1, u_2)$  of  $n$ -cubes of  $X$ , resp.  $Y$ ; if  $u_1$  depends on its first  $p$  coordinates only, we put  $S^p u = (u', u'')$  with  $u'(x_1, \dots, x_p, t, y_1, \dots, y_q) = u_1(x_1, \dots, x_p, y_1, \dots, y_q)$  and  $u''(x_1, \dots, x_p, 1 - t, y_1, \dots, y_q) = u_2(tx_1, \dots, tx_p, y_1, \dots, y_q)$ . Clearly  $\mu$  maps  $C^p$  into  $A^p$ , and we get an induced map  $\mu_r: E'_r \rightarrow E_r$  of the spectral sequences. Now  $\mu_0$  is nothing else but the map  $\psi$  of [10, p. 448] and is therefore a chain equivalence. It follows that all  $\mu_r, r \geq 1$ , and  $\mu^*$  are isomorphisms onto [1, p. 122]. But then  $\mu$  is a chain equivalence by [3, theorem 13.3, p. 154].

In the usual way  $\mu$  induces also a map of  $H(X) \otimes H(Y)$  into  $H(X \times Y)$ , for coefficients in a commutative ring  $R$  with unit. If  $R$  is a principal ideal ring, then I.3.A and the algebraic Künneth formula imply that the "Künneth formula for singular homology" holds:  $\mu_*$  imbeds  $H(X) \otimes H(Y)$  isomorphically into  $H(X \times Y)$  as direct summand, and the factor group is  $\text{Tor}[H(X), H(Y)]$  (cf. [3, p. 161]); as regards dimension, we have

$$H_n(X \times Y) = \sum_{p+q=n} H_p(X) \otimes H_q(Y) \oplus \sum_{r+s=n-1} \text{Tor}[H_r(X), H_s(Y)].$$

For completeness sake we sketch the known algebraic reasoning: Let  $K$  and  $L$  be two free chain groups; denote by  $Z$  and  $B$  the cycles and boundaries of  $L$ , and by  $W$  a subgroup of  $L$  supplementary to  $Z$ . From the homology sequence of the pair  $(K \otimes (B + W), K \otimes B)$  one finds that  $H(K \otimes (B + W)) = 0$ . From the sequence of the pair  $(K \otimes (Z + W), K \otimes (B + W))$  one finds that  $H(K \otimes L)$  is isomorphic with  $H(K \otimes H(L))$  (note  $Z + W = L$  and  $Z + W/B + W = H(L)$ ). To the latter group one applies the universal coefficient theorem [3, p. 161].

There are obvious associativity relations in the case of products of more than two spaces, and other elementary relations. With two maps  $f: P \rightarrow X, g: Q \rightarrow Y$  are associated the maps  $f \times g: P \times Q \rightarrow X \times Y$ ,  $f \otimes g: C(P) \otimes C(Q) \rightarrow C(X) \otimes C(Y)$ ,  $f_* \otimes g_*: H(P) \otimes H(Q) \rightarrow H(X) \otimes H(Y)$ , with the relation  $\mu_* \circ f_* \otimes g_* = (f \times g)_* \circ \mu_*$ . If  $h$  and  $k$  are cochains on  $X$  and  $Y$ , and  $h'$  and  $k'$  are their images under the natural projections of  $X \times Y$  onto  $X$  and  $Y$ , then we have the relation

$$(h' \cup g')(u \times v) = h(u) \cdot g(v).$$

## II. The Pontryagin product

II.1. Let  $X$  be 1-connected, i. e. arcwise connected and simply connected (the latter assumption is made in order to avoid local coefficients in the spectral sequence); choose a point  $x_0$ ; all vertices of all singular cubes are to be at  $x_0$ . We consider now the spaces  $E$  and  $\Omega$  of paths and loops in  $X$ , with the compact-open topology, as in [10], with the modification that we require all paths to end at  $x_0$ , i. e.  $f(1) = x_0$  for the path  $f: I \rightarrow X$ , and that the projection  $p: E \rightarrow X$  is defined as the starting point of the path,  $p(f) = f(0)$ ; the reason is that we want  $\Omega$  to operate on the right on  $E$ . This operation of  $\Omega$  on  $E$  is the map  $\gamma$  of  $E \times \Omega$  into  $E$ , defined by associating with the pair of paths  $(x, y)$ ,  $x \in E$ ,  $y \in \Omega$  the path  $\gamma(x, y)$ , also written  $x \cdot y$ , defined by

$$\begin{aligned} \text{II.1.1} \quad x \cdot y(t) &= x(2t), & 0 \leq t \leq \frac{1}{2} \\ x \cdot y(t) &= y(2t - 1), & \frac{1}{2} \leq t \leq 1. \end{aligned}$$

$\gamma$  is obviously a continuous map; it is related to the projection  $p$  by

$$\text{II.1.2} \quad p(x \cdot y) = p(x),$$

( $x \cdot y$  and  $x$  start at the same point).

Vertices of cubes in  $X$  are to be at  $x_0$ , in  $E$  (and in  $\Omega$ ) at  $e$ , defined by  $e(t) = x_0$ ,  $0 \leq t \leq 1$ , the constant path. The composition of  $\mu: Q(E) \otimes Q(\Omega) \rightarrow Q(E \times \Omega)$  defined in I.2, with  $\gamma: Q(E \times \Omega) \rightarrow Q(E)$  determines a map  $\varrho: Q(E) \otimes Q(\Omega) \rightarrow Q(E)$ , which commutes with  $d$ , related chain maps  $\varrho$  of the chain groups and the map  $\varrho_* = \gamma_* \circ \mu_*$  of the homology groups. For  $\varrho(u \otimes v)$  resp.  $\varrho_*(z \otimes w)$  we write also  $u * v$ , resp.  $z * w$ , and call this the Pontryagin multiplication. The map  $\gamma$  has the property that it maps the subset  $\Omega \times \Omega$  into the subset  $\Omega$ ;  $\gamma$ , restricted in this fashion, will be denoted by  $\bar{\gamma}$ ; we have a corresponding  $\bar{\varrho}$  and  $\bar{\varrho}_*$ ; but we continue to use the symbols  $\cdot$  and  $*$ . The point  $e \in E$  satisfies  $e \cdot e = e$ . The map  $r_e: E \rightarrow E$ , defined by  $r_e(x) = x \cdot e$ , is homotopic to the identity, with  $e$  stationary (cf. [5, p. 475]). This is also true for  $r_e: \Omega \rightarrow \Omega$  and for  $l_e: \Omega \rightarrow \Omega$  by  $l_e(x) = e \cdot x$ . The two maps  $(x, y, z) \rightarrow (x \cdot y) \cdot z$ , resp.  $x \cdot (y \cdot z)$  of  $\Omega^3 \rightarrow \Omega$  are not identical, but homotopic (cf. II.5); this means that Pontryagin multiplication in  $H(\Omega)$  is associative; we speak then of the Pontryagin ring (better-algebra); the 0-homology class, defined and denoted by  $e$ , is unit for this ring.

II.2. The basic fact concerning  $E$  is that relative to the map  $p$  it is a fiber space over  $X$ , with fiber  $\Omega$  [10]. We recall that  $Q(E)$  is filtered (cf. I.2), giving rise to a filtration of  $\mathcal{C}(E)$  and to a spectral sequence; we will have to make use of the explicit form of  $E_0, E_1, E_2$ , as deter-

mined by Serre. Now II.1.2 has the obvious consequence that for  $u \in Q_m(E)$ ,  $v \in Q_n(\Omega)$ , one has

$$\text{II.2.1. } p(u(x_1, \dots, x_m) \cdot v(x_{m+1}, \dots, x_{m+n})) = p(u(x_1, \dots, x_m)) .$$

This implies that the filtration of  $u * v$  equals that of  $u$ . We filter  $Q(E) \otimes Q(\Omega)$  by the subgroups  $T^r(E) \otimes Q(\Omega)$  (these are actually subgroups since the  $T^r$  are direct summands of  $Q(E)$ ); we have then  $\varrho(T^r(E) \otimes Q(\Omega)) \subset T^r(E)$ . We consider now  $C(E) \otimes C(\Omega)$ , which group we denote by  $\Gamma$ , with the standard differential  $d = d_E \otimes 1 + \omega \otimes d_\Omega$ , and filter it by the subgroups  $A^p(\Gamma) = A^p(E) \otimes C(\Omega)$  (they are well defined since the  $A^p(E)$  are direct summands of  $C(E)$ ). The map  $\varrho$  gives then a map of  $A^p(\Gamma)$  into  $A^p(E)$ , and induces therefore in standard algebraic fashion a map  $\varrho$  of the spectral sequence of  $\Gamma$  into that of  $E$ , i. e. maps  $\varrho_r$  of  $E_r(\Gamma)$  into  $E_r(E)$ , which commute with the differentials  $d_r$ . Our main task will be to study these maps for  $r = 0, 1, 2$  in terms of Serre's description of  $E_0(E)$ ,  $E_1(E)$ ,  $E_2(E)$ .

II.3. Since  $A^{p-1}(E)$  is a direct summand of  $A^p(E)$ , we have a canonical isomorphism of  $E_0(\Gamma)$  and  $E_0(E) \otimes C(\Omega)$ , with respect to the differentials  $d_0$  and  $d_0 \otimes 1 + \omega \otimes d$ ; this can be interpreted, via the map  $\varrho_0: E_0(\Gamma) \rightarrow E_0(E)$ , as an operation of  $C(\Omega)$  on  $E_0(E)$ . Going to the homology groups, we have a canonical imbedding of  $E_1(E) \otimes H(\Omega)$  into  $E_1(\Gamma) = H(E_0(E) \otimes C(\Omega))$  by the Künneth theorem ( $E_0(E)$  and  $C(\Omega)$  are free groups), with  $d_1 \otimes 1 + \omega \otimes d$  going into  $d_1$  (actually the term  $\omega \otimes d$  can be dropped since we are dealing with  $H(\Omega)$ ). Similarly  $E_2(\Gamma)$  will contain the canonical image of  $E_2(E) \otimes H(\Omega)$ , but will contain additional terms from two sources: from the other summand of  $E_1$ , and from the universal coefficient theorem for  $E_1(E) \otimes H(\Omega)$ . If the coefficients are not integers, but taken from an arbitrary commutative ring, the situation is even more complicated. We therefore restrict ourselves to considering canonical maps of  $E_r(E) \otimes H(\Omega)$  into  $E_r(\Gamma)$  and via  $\varrho_r$ , into  $E_r(E)$ , and interpret this as an operation of  $H(\Omega)$  on  $E_r(E)$ , as follows:

Let  $Z$  be the group of cycles, and  $B$  the group of boundaries of  $C(\Omega)$ . The identity map  $C(E) \otimes C(\Omega) \rightarrow \Gamma$  induces obviously maps  $\nu_r: C_r^p(E) \otimes Z \rightarrow C_r^p(\Gamma)$ ,  $r \geq 1$ ; one verifies that  $C_r^p(E) \otimes B$ ,  $C_{r-1}^{p-1} \otimes Z$ ,  $B_{r-1}^p \otimes Z$  are mapped respectively into  $C_{r-1}^{p-1}(\Gamma) + B_{r-1}^p(\Gamma)$ ,  $C_{r-1}^{p-1}(\Gamma)$ ,  $B_{r-1}^p(\Gamma)$ , and that therefore, passing to the quotients, one gets an induced map  $\nu_r: E_r(E) \otimes H(\Omega) \rightarrow E_r(\Gamma)$ ; the relation  $\nu_r \circ (d_r \otimes 1) = d_r \circ \nu_r$  is clear (for  $r=0$  this is modified to  $\nu_0: C_0^p(E) \otimes C(\Omega) \rightarrow C_0^p(\Gamma)$ , inducing

$\nu_0: E_0(E) \otimes C(\Omega) \rightarrow E_0(E)$ , with  $d_0 \circ \nu_0 = \nu_0 \circ (d_0 \otimes 1 + \omega \otimes d)$ . We form now  $\varrho_r \circ \nu_r = \pi_r$ , and have the induced maps

$$\begin{aligned}\pi_0: E_0(E) \otimes C(\Omega) &\rightarrow E_0(E) \\ \pi_r: E_r(E) \otimes H(\Omega) &\rightarrow E_r(E), \quad r \geq 1,\end{aligned}$$

which commute with the  $d_r$ . If we write  $z * v$  for  $\pi_r(z \otimes v)$  ( $r \geq 0$ ), we can express this by the formula

$$\begin{aligned}d_r(z * v) &= d_r z * v \quad \text{for } r \geq 1, z \in E_r(E), v \in H(\Omega), \quad \text{while} \\ d_0(c * v) &= d_0 c * v + \omega c * dv \quad \text{for } c \in E_0(E), v \in C(\Omega).\end{aligned}$$

Moreover the operation  $*$  commutes with the passage to homology groups; if  $[ ]$  denotes homology class, then we have  $[z * v] = [z] * v$ , if  $z$  is a cycle of  $E_r(E)$ ,  $r \geq 1$ ,  $v \in H(\Omega)$ ; for  $r = 0$  this becomes  $[z * v] = [z] * [v]$ , if  $z$  is a cycle of  $E_0(E)$  and  $v \in Z$ . This follows from the fact that the operation  $*$  is derived from the original operation  $\otimes$  in  $C(E) \otimes C(\Omega)$  by passage to quotient groups; in more detail: if  $\bar{z}$  is a chain in  $C_r^p(E)$ , representing  $z \in E_r(E)$ , and if  $\bar{u}$  is a cycle of  $\Omega$ , representing  $u \in H(\Omega)$ , then the chain  $\bar{z} * \bar{u}$  represents  $z * u$  in  $E_r(E)$ .

II.5. In order to study associativity relations, we consider the two maps  $f_0, f_1$  of  $E \times \Omega \times \Omega$  into  $E$ , defined by  $f_0(q_1, q_2, q_3) = (q_1 \cdot q_2) \cdot q_3$ , resp.  $f_1(q_1, q_2, q_3) = q_1 \cdot (q_2 \cdot q_3)$ . Of course the two maps are not equal, but they are homotopy-associative with stationary projection and even with  $(e, e, e)$  stationary, i. e. there exists a homotopy  $g$  ( $g_t$ ,  $0 \leq t \leq 1$ ), such that:

- (1)  $g_0 = f_0, g_1 = f_1$ ,
- (2)  $p \circ g_t(q_1, q_2, q_3) = p(q_1)$  for all  $q_1, q_2, q_3, t$ ,
- (3)  $g_t(e, e, e) = e$  for all  $t$ ,

defined as follows:  $q_1, q_2, q_3$  are paths in  $X$ , i. e. maps of  $[0, 1]$  into  $X$ ; the homotopy to be constructed is just a shift in parametrization. We put

$$\begin{aligned}q_1\left(\frac{4s}{t+1}\right) & \quad \text{for } 0 \leq s \leq \frac{t+1}{4} \\ g_t(q_1, q_2, q_3)(s) &= q_2(4s - t - 1) \quad \text{for } \frac{t+1}{4} \leq s \leq \frac{t+2}{4} \\ q_3\left(\frac{4s - t - 2}{2 - t}\right) & \quad \text{for } \frac{t+2}{4} \leq s \leq 1.\end{aligned}$$

With the help of a diagram, consisting of the unit square in a  $t$ - $s$ -plane

together with the segments  $(0, \frac{1}{4}) - (1, \frac{1}{2})$  and  $(0, \frac{1}{2}) - (1, \frac{3}{4})$ , one checks that  $g$  has the required properties; continuity is proved as in [10, p. 475]. For any  $n$ -cube  $w = w(x_1, \dots, x_n)$  in  $E \times \Omega \times \Omega$  we define an  $(n+1)$ -cube  $kw$  by  $kw(x_1, \dots, x_{n+1}) = (-1)^n g_{x_{n+1}}(w(x_1, \dots, x_n))$ . Clearly  $D^{(\infty)}$  goes into  $D^{(\infty)}$ , and we have a map of the chains, which raises dimension by 1. One verifies that

$$\text{II.5.1} \quad dk + kd = f_1 - f_0,$$

so that  $k$  provides a chain homotopy between the induced chain maps  $f_0$  and  $f_1$ . (We note that because of property (2)  $k$  sends  $A^p \otimes C(\Omega) \otimes C(\Omega)$  into  $A^p$ ; from this one could prove directly that II.5.5 holds.)

Clearly the subset  $\Omega \times \Omega \times \Omega$  is carried into  $\Omega$  by  $g_t$ ; it follows that the two maps  $f_{0*}$  and  $f_{1*}$  of  $H(\Omega) \otimes H(\Omega) \otimes H(\Omega)$  into  $H(\Omega)$  are identical (the chain maps are chain homotopic). This means that  $*$ -multiplication in  $H(\Omega)$  is associative. That the point  $e$  acts as unit follows, as noted in II.1, from the fact that the two maps of  $\Omega$  into itself, defined by  $q \rightarrow q \cdot e$ , resp.  $e \cdot q$ , are homotopic to the identity.

We come to the connection of  $*$  with the chain equivalence [10, p. 447]

$$\psi: J = C(X) \otimes C(\Omega) \rightarrow E_0(E).$$

We recall that  $J_p = C_p(X) \otimes C(\Omega)$  corresponds to  $E_0^p(E)$ , and that  $\psi$  commutes with  $\omega \otimes d_\Omega$  and  $d_0$ , which can be expressed by saying that we take  $C(X)$  with  $d_X = 0$ . For a given  $p$ -cube  $u$  of  $X$ , and cubes  $v, w$  of  $\Omega$  we form the two cubes  $K(u, v * w)$  and  $K(u, v) * w$ , both of which belong to  $T^p$ , and both of which belong to  $D^{(\infty)}$  if any one of  $u, v, w$  does. We get therefore two induced maps from  $C(X) \otimes C(\Omega) \otimes C(\Omega)$  to  $E_0(E)$ , which we call  $\kappa_1$  and  $\kappa_2$ ; we can write  $\kappa_1 = \psi \circ (1 \otimes \bar{\varrho})$  and  $\kappa_2 = \pi_0 \circ (\psi \otimes 1)$  (see II.1 for  $\bar{\varrho}$  and II.3 for  $\pi_0$ ), which shows that  $\kappa_1$  and  $\kappa_2$  are chain maps (for  $d_X = 0$ ). We note that  $B \cdot K(u, v) * w = u$  and  $F \cdot K(u, v) * w = v * w$ , so that  $\kappa_1 = \psi \circ \varphi \circ \kappa_2$ . With the operator  $k$  from [10, p. 448] we form now  $s = k \circ \kappa_2$ . We have then  $d_0 s + s d = d_0 k \kappa_2 + k \kappa_2 d = (d_0 k + k d_0) \kappa_2 = (\psi \circ \varphi - 1) \kappa_2 = \kappa_1 - \kappa_2$ , so that we have a chain homotopy between  $\kappa_1$  and  $\kappa_2$ . If  $x$  is a chain of  $X$ , and  $u, v$  are cycles of  $\Omega$ , then  $\kappa_1(x \otimes u \otimes v)$  and  $\kappa_2(x \otimes u \otimes v)$  are homologous. We can restate this as

$$\text{II.5.2} \quad \psi(x \otimes u) * v \sim \psi(x \otimes (u * v)) \text{ in } E_0(E), \text{ for } \\ x \in C(X), u, v \in Z(\Omega).$$

Going to  $E_1$  and recalling that  $*$  commutes with taking homology classes, we find



$$\text{II.5.3} \quad (x \otimes u) * v = x \otimes (u * v) \text{ in } E_1(E), \text{ for} \\ x \in C(X), u, v \in H(\Omega);$$

here  $E_1(E)$  is identified with  $C(X) \otimes H(\Omega)$  by  $\psi_*$ , thus defining  $(x \otimes u) * v = \psi_*^{-1}(\psi_*(x \otimes u) * v)$ . The differential  $d_1$  of  $E_1$ , according to Serre, becomes  $d_X \otimes 1$  on  $C(X) \otimes H(\Omega)$ , and  $E_2$  is the corresponding homology group. Let  $x$  now be a cycle of  $X$  and  $u, v$  as before. Making use of the canonical map of  $H(X) \otimes H(\Omega)$  into  $H(C(X) \otimes H(\Omega)) = E_2$  (with  $[ ]$  again meaning homology class), we see

$$\text{II.5.4} \quad [x] \otimes (u * v) = [x \otimes (u * v)] = [(x \otimes u) * v] = [x \otimes u] * v = ([x] \otimes u) * v.$$

Associativity of  $H(\Omega)$  and relation II.5.3 imply the equation

$$(x \otimes u) * (v * w) = ((x \otimes u) * v) * w \text{ in } E_1(E), \text{ with } x \in C(X), u, v, w \in H(\Omega);$$

and the  $x \otimes u$  span  $E_1$ . Going to homology classes one finds then

$$\text{II.5.5} \quad z * (v * w) = (z * v) * w \text{ for } z \in E_r(E), r \geq 1, v, w \in H(\Omega).$$

Similarly one proves, starting from  $E_1$ ,

$$\text{II.5.6} \quad z * e = z \text{ for } z \in E_r(E), r \geq 1, e \text{ the unit of } H(\Omega).$$

We collect our results in the following theorem, into which we incorporate a statement about the grading of the various groups, and about coefficients; both statements are proved easily by going back to the map  $\varrho: \Gamma \rightarrow C(E)$ , by which all other maps are induced:

**Theorem II. 5.A:** Let  $X$  be a 1-connected space. The map  $\gamma: E \times \Omega \rightarrow E$  induces a pairing, written  $*$ , of

- a)  $H(\Omega)$  and  $H(\Omega)$  to  $H(\Omega)$ ; in detail:  $H_m(\Omega)$  and  $H_n(\Omega)$  to  $H_{m+n}(\Omega)$ ;
- b)  $E_r(E)$  and  $H(\Omega)$  to  $E_r(E)$ ,  $r \geq 1$ ;  $E_r^{p,q}(E)$  and  $H_n(\Omega)$  to  $E_r^{p,q+n}(E)$ ;
- c)  $E_0(E)$  and  $C(\Omega)$  to  $E_0(E)$ ;  $E_0^{p,q}(E)$  and  $C_n(\Omega)$  to  $E_0^{p,q+n}(E)$

with the following properties:

- 1) The pairing is bilinear, and associative, i. e., for  $x \in E_r$ ,  $r \geq 1$ ,  $u, v, w \in H(\Omega)$  the relations  $(x * u) * v = x * (u * v)$ ,  $(u * v) * w = u * (v * w)$  hold; the point  $e$  satisfies  $e * v = v * e = v$  for  $v \in H(\Omega)$ , and  $x * e = x$  for  $x \in E_r(E)$ ,  $r \geq 1$ ;
- 2)  $d_r(x * v) = d_r x * v$  for  $r \geq 1$ ;  $d_0(x * v) = d_0 x * v + \omega x * dv$ ;
- 3)  $*$  commutes with the identification  $E_{r+1} = H(E_r)$ ;
- 4) In  $E_1$  and  $E_2$  one has, for  $x \in C(X)$ , resp.  $H(X)$ ,  $u, v \in H(\Omega)$ ,  $(x \otimes u) * v = x \otimes (u * v)$ , where  $E_1$  is canonically identified with

$C(X) \otimes H(\Omega)$ , and where  $H(X) \otimes H(\Omega)$  is mapped canonically into  $H(X, H(\Omega)) = E_2$ ;

the coefficients for  $C(\Omega)$ ,  $H(\Omega)$ ,  $E_r$ ,  $r \geq 0$ , are taken from a commutative ring  $R$  with unit;  $C(X)$  and  $H(X)$  can be understood either over the integers or over  $R$  (the tensor products are taken accordingly).

$H(\Omega)$ , with  $*$  as multiplication, will be called the Pontryagin-ring or-algebra  $H_*(\Omega)$  of  $\Omega$ ; it has  $e$  as unit.

*Remark:* The theorem actually applies to a more general situation, in a slightly extended form. Let  $Y$  be a fiber space over  $B$ , with projection  $p$ ; suppose a space  $M$  operates on  $Y$  and on itself, i. e. maps of  $Y \times M$  into  $Y$  and of  $M \times M$  into  $M$  (written as products) are given, with the following properties:

- 1)  $p(y \cdot m) = p(y)$  for all  $y \in Y$ ,  $m \in M$ ;
- 2) there is an  $e$  in  $Y$  and an  $e'$  in  $M$ , such that  $e \cdot e' = e$  and  $e' \cdot e' = e'$ ;
- 3) the two maps  $(y, m_1, m_2) \rightarrow (y \cdot m_1) \cdot m_2$ , resp.  $y \cdot (m_1 \cdot m_2)$  of  $Y \times M \times M$  into  $Y$  are homotopic, with  $(e, e', e')$  stationary and with  $F \times M \times M$  staying in  $F$ , where  $F$  is the fiber through  $e$ ;
- 4) the map  $q \rightarrow q \cdot e'$  of  $F$  into itself is homotopic to the identity;
- 5) the maps  $m \rightarrow m \cdot e'$ , resp.  $e' \cdot m$  of  $M$  into itself are homotopic to the identity;
- 6) the two maps  $(m_1, m_2, m_3) \rightarrow (m_1 \cdot m_2) \cdot m_3$ , resp.  $m_1 \cdot (m_2 \cdot m_3)$  of  $M \times M \times M$  into itself are homotopic, with  $(e', e', e')$  stationary;
- 7)  $B$  1-connected,  $M$  and the fibers of  $Y$  0-connected.

$H(M)$  will then operate on  $E_r(Y)$  ( $1 \leq r \leq \infty$ ), on  $H(F)$  and on  $H(Y)$ ; it also operates on the subgroups  $D^p$ , by which  $H(Y)$  is filtered. The associativity relations etc. hold, suitably modified; the operation on  $E_\infty$  is obtained from that on  $H(Y)$  by going to the factor groups. In the proof of II.5.2 e. g. one has to replace  $C(X) \otimes C(\Omega) \otimes C(\Omega)$  by  $C(B) \otimes C(F) \otimes C(M)$ . Conditions 2) — 7), in particular 4) — 6), could of course be modified.

This situation occurs when a Lie group operates on a space, e. g. for principal bundles; this is what has been considered, in cohomology, with Leray's theory, by Leray [5, 6] and Borel [1].

II.6. We give a brief discussion of cohomology relations; the contents of this section can be considered as a translation into singular theory of results of Leray [6] and Borel [1]. For simplicity we restrict ourselves



to a field  $k$  as coefficients (in the general case one would have to consider pairing of groups to their tensor product).

For a cochain  $a$  in  $C^r(X)$  and a cube  $u$  in  $Q_n(X)$ , with  $r \leq n$ , one can define the  $\frown$ -product, which we write also as  $a \cdot u$ , by the formula

$$a \cdot u = \sum_H \varrho_{H,K} \cdot a(\lambda_H^1 u) \lambda_K^0 u ,$$

where  $K$  runs through the subsets of  $r$  elements of  $\{1, 2, \dots, n\}$ ,  $H$  is the complement of  $K$ , and  $\varrho_{H,K}$  is the familiar sign determined by the number of inversions (cf. the  $\smile$ -product in [10, p. 441], by which, as in [2], the  $\frown$ -product is determined); this induces a  $\frown$ -product between  $C^r(X)$  and  $C_n(X)$  with values in  $C_{n-r}(X)$ . The  $\smile$ - and  $\frown$ -product, the Kronecker index  $KI(a, x)$  (defined as  $a(x)$  for  $a \in C^n(X)$ ,  $x \in C_n(X)$ ; we identify  $C^n(X)$  with the  $k$ -linear forms on  $C_n(X)$ ), the index  $In(x)$  (defined as  $\sum k_i$  for  $x = \sum k_i u_i \in C_0(X)$ ), and the differential  $d$  (boundary or coboundary) satisfy all the usual relations [2, p. 432]; the unit 1 of  $C^*(X)$  is the constant function, with value  $1 \in k$ .

Let  $Y$  be a fiber space over  $B$ , with projection  $p$  and fiber  $F$ . One verifies that the  $\frown$ -product pairs  $A^{*p,q}$  and  $A^{p',q'}$  to  $A^{p'-p, q'-q}$ . It follows easily that, in addition to the  $\smile$ -product in  $E_r^*$  (as in [10]), there is an induced  $\frown$ -product, pairing  $E_r^{*p,q}$  and  $E_r^{p',q'}$  to  $E_r^{p'-p, q'-q}$ , a  $KI$  (between  $E_r^{*p,q}$  and  $E_r^{p,q}$ , 0 otherwise), an  $In$  (for  $E_r^{0,0}$ ), that all the usual relations are satisfied (the differential is  $d_r$  for  $r < \infty$ , 0 for  $r = \infty$ ), and that the operations are compatible with the identification  $E_{r+1} = H(E_r)$  etc.; since we have field coefficients,  $E_r^{*p,q}$  is identified, by way of  $KI$ , with the space  $Hom(E_r^{p,q})$  of  $k$ -linear forms on  $E_r^{p,q}$ ; all this is a straightforward generalization of the corresponding facts for  $H(Y)$  and  $H^*(Y)$ . For  $E_1$  the  $\frown$ -product translates into the natural  $\frown$ -product between  $C^p(B, H^q(F))$  and  $C_{p'}(B, H_{q'}(F))$  relative to the  $\frown$ -product pairing of the coefficients  $H^q(F)$  and  $H_{q'}(F)$ , except that a factor  $(-1)^{p(q'-q)}$  has to be added (cf. [10, p. 454]); corresponding statements hold for  $E_2$ . If e. g. the Betti numbers of  $F$  are finite, then the  $\smile$ - and  $\frown$ -products become the canonical products in  $H^*(B) \otimes H^*(F)$  and  $H(B) \otimes H(F)$ . Similar statements hold for  $KI$ .

If  $X$  and  $Y$  are any two spaces, one can define  $\smile, \frown, KI, In$  in the chain complex  $C(X) \otimes C(Y)$  and its cochain complex (using e. g. the fact that the map  $\mu$  is a chain equivalence (I.3)); the relations  $\mu^*(f \smile g) = \mu^*f \smile \mu^*g$  and  $\mu_*(\mu^*f \frown x) = f \frown \mu_*x$  hold. There is an imbedding of  $C^*(X) \otimes C^*(Y)$  into the cochains of  $C(X) \otimes C(Y)$ , sending  $a \otimes b$  into  $\mu^*(\pi_X^* a \smile \pi_Y^* b)$ ; here  $\pi_X$  and  $\pi_Y$  are the projections of  $X \times Y$  onto  $X$  and  $Y$ . If e. g.  $Y$  has finite Betti numbers, this induces an isomorphism

of  $H^*(X) \otimes H^*(Y)$  (skew product) with the cohomology algebra of  $C(X) \otimes C(Y)$  (cf. [10, p. 458 and 473]), since then, by way of  $KI$ ,  $H^*(X) \otimes H^*(Y)$  is the space of all linear functions on  $H(X) \otimes H(Y)$ .

We turn now to the situation of II.2, so that  $E$  is the space of paths, ending at  $x_0$ , in the 1-connected space  $X$ , with fiber  $\Omega$ . The cochain-algebra  $\Gamma^*$  of  $\Gamma = C(E) \otimes C(\Omega)$  is filtered in the standard fashion,  $\Gamma^{*p} = \text{annihilator of } \Gamma^{p-1}$ ; reasoning as above one sees that there are induced  $\cup, \cap, KI, In$  in the spectral sequence. The terms  $E_r(\Gamma)$  can now be identified with  $E_r(E) \otimes H(\Omega)$  (for  $r \geq 1$ ) [in more detail:  $E_r^{p,n}(\Gamma) = \sum_{s+t=n} E_r^{p,s}(E) \otimes H_t(\Omega)$ ], since we have field coefficients; as always

$E_0^{*p,q}(\Gamma)$  is identified with  $\text{Hom}(E_0^{p,q}(\Gamma))$ . We assume now that  $H(\Omega)$  has finite Betti numbers. We can then, as above, identify  $E_1^*(\Gamma)$  with  $E_1^*(E) \otimes H^*(\Omega)$ , and, inductively,  $E_r^*(\Gamma)$  with  $E_r^*(E) \otimes H^*(\Omega)$ , with the operations  $\cup, \cap, KI, In, d_r$  going into the canonical operations for the tensor products ( $d_r$  becomes  $d_r \otimes 1$ ). The map  $\varrho: \Gamma \rightarrow C(E)$  induces now maps  $\pi_r, \pi_r^*$  of  $E_r(E) \otimes H(\Omega)$  into  $E_r(E)$ , resp. of  $E_r^*(E)$  into  $E_r^*(E) \otimes H^*(\Omega)$ , which satisfy the various compatibility relations; in particular they commute with the differentials, and satisfy  $KI(\pi_r^* a, x) = KI(a, \pi_r x)$ .

The relation 4. of Th. II.5.A becomes now

$$\pi_2(x \otimes x' \otimes x'') = x \otimes (x' * x'') \quad \text{for } x \in H(X), x', x'' \in H(\Omega).$$

By duality, i. e. by the invariance of the  $KI$  just mentioned, one derives from this that for  $a \in H^*(X), b \in H^*(\Omega)$  one has  $\pi_2^*(a \otimes b) = a \otimes \bar{\varrho}^*(b)$  where  $\bar{\varrho}^*(b)$  is the image of  $b$  under the map  $\bar{\varrho}^*: H^*(\Omega) \rightarrow H^*(\Omega) \otimes H^*(\Omega)$  induced by the multiplication  $\gamma: \Omega \times \Omega \rightarrow \Omega$  (II.1); this is the exact analog of results of Borel [1]. As well known, the element  $\bar{\varrho}^*(b)$  has the form  $1 \otimes b + b \otimes 1 + \sum c_i \otimes c'_i$ ; with  $0 < \dim c_i < \dim b$ , corresponding to the fact that  $e$  is unit for  $H_*(\Omega)$  [10; p. 476].

The operation of  $H(\Omega)$  on  $E_r(E)$ , by  $*$ -multiplication, induces by duality an operation on  $E_r^*(E)$ ; we establish some relations for this operation, which are essentially equivalent to some of the relations given by Leray in [6]. For  $x \in H_n(\Omega)$  and  $a \in E_r^{*p,q}(E)$  we define an element  $a * x \in E_r^{*p,q-n}(E)$  by requiring the equation

$$KI(a * x, y) = KI(a, y * x) (= KI(\pi_r^* a, y \otimes x))$$

to hold for all  $y \in E_r^{p,q-n}(E)$ . One sees easily that  $(a * x) * x' = a * (x' * x)$  and  $(d_r a) * x = d_r(a * x)$ . Similarly we can let  $H(\Omega)$  operate on  $H^*(\Omega)$ ; relation 4. of Th. II.5.A implies then that in  $E_1$  and  $E_2$  we have

$a \otimes (b * x) = (a \otimes b) * x$ , for  $a \in C^*(X)$ , resp.  $H^*(X)$ ,  $b \in H^*(\Omega)$ ,  $x \in H(\Omega)$ .

To study relations between  $\cup$ - and  $\cap$ -product, we note first the following equations, with  $a, b \in E_r^*(E)$ ,  $y \in E_r(E)$ ,  $x \in H(\Omega)$ ,  $\dim a = r$ ,  $\dim b = s$ ,  $\dim x = t$ :

$$\begin{aligned} KI((ab) * x, y) &= KI(\pi_r^*(a) \cdot \pi_r^*(b), y \otimes x) \quad \text{and} \\ KI((a * x)b, y) &= (-1)^{st} KI(\pi_r^*(a)(b \otimes 1), y \otimes x) \end{aligned}$$

(for the second equation note: left hand side  $= KI(a * x, b \cap y) = KI(a, (b \cap y) * x) = KI(\pi_r^* a, (b \cap y) \otimes x) = (-1)^{st} KI(\pi_r^* a, (b \otimes 1) \cap (y \otimes x)) =$  right hand side). Secondly, the relation  $z * e = z$  for  $z \in E_r(E)$ ,  $e$  the unit of  $H_*(\Omega)$ , implies by duality that, for  $a \in E_r^*(E)$ , the element  $\pi_r^*(a)$  of  $E_r^*(E) \otimes H^*(\Omega)$  has the form  $a \otimes 1 + R_a$ , where all terms  $a' \otimes a''$  in  $R_a$  have  $\dim a'' > 0$ . It follows easily that the element  $\Delta(ab)$ , defined as  $\pi_r^*(ab) - \pi_r^*(a)(b \otimes 1) - (a \otimes 1)\pi_r^*(b)$ , equals  $R_a R_b - ab \otimes 1$ . Suppose now that  $x$  is a primitive or minimal element of  $H(\Omega)$  in the sense of Hopf [4], i. e.  $b \cap x = 0$  for  $0 < \dim b < \dim x$ , and  $\dim x > 0$ . It follows that  $KI(\Delta(ab), y \otimes x) = 0$  for all  $y$ . Combined with the above relations for the  $KI$ , this is easily seen to imply

$$(ab) * x = (-1)^{st}(a * x)b + a(b * x);$$

this can be stated as follows: If  $x$  is minimal and  $\dim x$  is even, then the operation  $a \rightarrow a * x$  is a derivation, if  $\dim x$  is odd, it is a "right" antiderivation (if  $\Omega$  would operate on  $E$  on the left, we would get here the customary antiderivation).

All the above statements apply of course, suitably interpreted, to the case discussed in the remark at the end of II.5.

### III. Applications

1. If  $(Y, B, F, p)$  is a fiber bundle (with bundle  $Y$ , base space  $B$ , fiber  $F$ , projection  $p$ ) in the sense of Serre [10], ( $B$  always 1-connected), then an element  $x$  of  $H_p(B)$  ( $p > 0$ ) is called transgressive, if  $d_2 x = \dots = d_{p-1} x = 0$ . This is to be understood by using the following conventions:  $e$  represents the generator of  $H_0$  of any space, given by a point; we identify  $H(X)$  and  $H(F)$  with their canonical images  $H(X) \otimes e$  and  $e \otimes H(F)$  in  $E_2$  (this is compatible with the  $*$ -operation, as theorem II.5.A shows), so that e. g. the  $x$  in  $d_2 x$  stand for  $x \otimes e$ . Since  $d_2 x = 0$ , i. e.  $x$  is a cycle of  $E_2$ , it determines an element of  $E_3$ , namely its homology

class, which we denote by  $x$  again (with the analogous convention for any cycle of any  $E_r$ ), etc. The element  $d_p x$  is then an element of a certain factor group of  $H_{p-1}(F)$ .

Let  $E$ , as in II, denote the space of paths in the space  $X$ , ending at  $x_0$ , considered as fiber space over  $X$  with fiber  $\Omega$ .

**Theorem III. 1. A:** Suppose  $H(X)$ , coefficients in the principal ideal ring  $R$ , is  $R$ -free, and all elements of  $H(X)$  are transgressive (in  $E$ ). Then  $H(\Omega)$  contains a subgroup  $T$ , isomorphic image of  $H_+(X)$  (= elements of  $H(X)$  of positive dimension) under a map which lowers dimension by 1; and the Pontryagin ring  $H_*(\Omega)$  is the free associative algebra generated by  $T$ , with  $e$  as unit.

**Proof:** We start from the relation  $E_2 = H(X) \otimes H(\Omega)$  (which holds since  $H(X)$  is free). Thm. II.5.A implies then that  $E_2$  is "totally transgressive" in the sense that for any element  $z$  of  $E_2$ , which lies in  $E_2^{p,q}$ , all differentials  $d_2 z, \dots, d_{p-1} z$  vanish ( $d_2(x \otimes v) = d_2(x \otimes e) * v = d_2(x * v) = (d_2 x) * v = 0$ ,  $[x * v] = [x] * v = x * v$  in  $E_3$ , etc.). Our argument will be based on the fact that  $H(E)$  and therefore  $E_\infty$  are trivial, since  $E$  is contractible, and that therefore  $E_2$  has to be "extinguished" by application of the  $d_r$ . We choose a base  $\Sigma = \{x_i\}$ ,  $i$  running through an index set  $J$ , of  $H_+(X)$ , consisting of homogeneous-dimensional elements; for each  $x_i$  we choose an element  $\bar{x}_i$  of  $H_+(\Omega)$ , such that  $d_p x_i = \bar{x}_i$ , with  $p = \dim x_i$ , and consider the collection  $M$  of elements  $\bar{x}_{i_1 i_2 \dots i_k} = \bar{x}_{i_1} * \bar{x}_{i_2} * \dots * \bar{x}_{i_k}$ ,  $k \geq 1$ ,  $i_j \in J$  for  $j = 1, \dots, k$ . We claim that these elements are independent, and that together with  $e$  they form a basis for  $H(\Omega)$ ; this will obviously prove the theorem, with  $T$  generated by the  $\bar{x}_i$ . We call  $k$  the length and  $\dim \bar{x}_{i_1}$  the height of  $\bar{x}_{i_1 i_2 \dots i_k}$ . Suppose there were linear relations between the elements of  $M$ ; consider the relations in which the maximum length of the elements occurring is as small as possible, say  $k$ . Choose such a relation  $r \equiv \sum c_\alpha z_\alpha = 0$ , with  $z_\alpha \in M$ . It can be written as a sum  $r' + r''$  of two parts, where the first part contains all elements of maximum height, say  $p$ . We write each  $z_\alpha$  as  $\bar{x}_\alpha * \bar{z}_\alpha$  with  $\bar{z}_\alpha \in M$  or  $= e$ , in the obvious fashion, and form the elements  $y_\alpha = x_\alpha \otimes \bar{z}_\alpha$  of  $E_2$ , and put  $y = \sum c_\alpha y_\alpha = y' + y''$ , where  $y'$  and  $y''$  correspond to  $r'$  and  $r''$ . We go now to  $E_{p+1}$ .

There  $r''$  has become 0, since the elements  $x_\alpha \otimes \bar{z}_\alpha$  in  $y''$  map onto the  $\bar{x}_\alpha * \bar{z}_\alpha$  in  $r''$  under the appropriate differentials  $d_q$ ,  $q \leq p$ ;  $r'$  is then also 0 in  $E_{p+1}$ . Similarly  $y'$  maps onto  $r'$  by  $d_{p+1}$ ; we have therefore  $d_{p+1}(y') = 0$ . By total transgressivity  $y'$  is not image under any

$d_r$ ; it follows that  $y' = 0$ , since otherwise  $E_\infty$  would contain a non-trivial element. The  $\bar{z}_\alpha$ , occurring in  $y'$ , are of length  $< k$ , and the elements of length  $< k$  form a free subgroup of  $H(\Omega)$ , by our minimality assumption. Since  $H(X)$  is free, it follows that the elements  $x_\alpha \otimes \bar{z}_\alpha$ , occurring in  $y'$ , are independent in  $E_2$ , and, again by total transgressivity, in  $E_{p+1}$ ; this contradicts the fact that  $y'$  is 0 in  $E_{p+1}$ , and our assertion concerning the independence of the elements of  $M$  is proved.

Suppose there were elements in  $H_+(\Omega)$ , which are not linear combinations of elements of  $M$ ; let  $n$  be the smallest dimension in which this happens. All elements of  $E_2^{n-q+1, q}$ , for  $q < n$ , are then of the form  $\Sigma x_\alpha \otimes t_\alpha$ , with  $\dim t_\alpha = q < n$ ; the  $t_\alpha$  are therefore generated by  $M$  and  $e$ . Considering now the action of  $d_2, d_3, \dots, d_{n+1}$ , one sees that  $E_{n+2}^{0, n}$  is a quotient group of  $H_n(\Omega)$  by a subgroup which is contained in the subgroup generated by  $M$  (note that  $d_p(x \otimes t)$ ,  $\dim x = p$ , is congruent to  $\bar{x} * t$  modulo the images of  $d_2, \dots, d_{p-1}$ ). But  $E_{n+2}^{0, n}$  must be 0, *q. e. d.* We have shown that the element  $\bar{x}_{i_1 \dots i_k}$  of  $M$  and  $e$  form a basis for  $H(\Omega)$ ; the statement about the ring structure is immediate from the relation  $\bar{x}_{i_1 \dots i_k} * \bar{x}_{j_1 \dots j_l} = \bar{x}_{i_1 \dots i_k j_1 \dots j_l}$ .

Since spherical cycles are transgressive [10, p. 452] the theorem III.1.A applies when all cycles of  $X$  are spherical. We can therefore state the corollary:

**III. 1. B:** The Pontryagin ring of the space  $\Omega$  of loops in the space  $S^{n_1} \vee S^{n_2} \vee \dots \vee S^{n_k}$  (union of  $k$  spheres of dimensions  $n_i > 1$ , all attached at one point) is the free associative algebra on  $k$  generators of dimensions  $n_i - 1$ .

For the case of a single sphere this can be read off from Morse's results [7]. We describe a somewhat more general case in which theorem III.1.A is applicable.

**Theorem III. 1. C:** Let  $\hat{X}$  be the join of a 0-connected space  $X$  with the 0-sphere  $S^0$ ; then in the spectral sequence of  $E(\hat{X})$  all elements of  $H_+(\hat{X})$  are transgressive.

**Proof:** We consider  $\hat{X}$  as the product  $X \times I$  with all of  $X \times 0$ , resp.  $X \times 1$ , identified to a point  $x_0$ , resp.  $x_1$ .  $\hat{X}$  is clearly 1-connected. Define

$$X^0 = \{(x, t) : x \in X, 0 < t \leq \frac{1}{2}\} \cup \{x_0\}, X^1 = \{(x, t) : x \in X, \frac{1}{2} \leq t < 1\} \cup \{x_1\},$$

and identify each  $x \in X$  with  $(x, \frac{1}{2}) \in X$ . The inclusion map of the pair  $(X^0, X)$  into  $(\hat{X}, X^1)$  is homotopic (as map of pairs) to the map  $f$  defined

by  $f(x_0) = x_0$ ,  $f(x, t) = (x, 2t)$  for  $0 < t < \frac{1}{2}$ ,  $f(x) = x_1$ , for  $x \in X$ ;  $f$  can be considered as a map of  $(X^0, X)$  into  $(\hat{X}, x_1)$ . Using the fact that  $X^0$  and  $X^1$  are contractible spaces, and using the excision  $(\hat{X}, X^1) \supset (X^0, X)$ , one shows easily that the map  $f^*$  is an isomorphism of  $H(X^0, X)$  and  $H(\hat{X}, x_1)$ . Since  $X^0$  is contractible, we can, by an application of the covering homotopy theorem, find a map  $g: X^0 \rightarrow E$ , such that  $p \circ g = f$ . If we let  $\Omega$  be the fiber of  $E$  over  $x_1$  (we might choose  $x_1$  as base point for  $E$ ), then  $g$  maps the subset  $X$  of  $X^0$  into  $\Omega$ , and so defines a map  $g$  of the pair  $(X^0, X)$  into  $(E, \Omega)$ . It follows now that  $f_* = p_* \circ g_*$  (as map of  $H(X^0, X)$  into  $H(\hat{X}, x_1)$ ) and that  $p_*$  maps  $H(E, \Omega)$  onto  $H(\hat{X}, x_1)$ ; by the geometric definition of transgression [10, p. 452], this proves the assertion of the theorem.

III.2. Let  $(Y, B, F, p)$  be a fiber space, with  $F$  operating on  $Y$  as described at the end of II.5., and suppose  $B$  is a homology- $k$ -sphere. Then one has the Wang sequence [10, p. 471]:

$$\rightarrow H_i(F) \rightarrow H_i(E) \rightarrow H_{i-k}(F) \xrightarrow{\theta} H_{i-1}(F) \rightarrow \dots$$

The map  $\theta$  in this sequence is obtained as follows: Let  $s$  be a generator for  $H_k(B)$ ; map  $H_{i-k}(F)$  into  $E_2^{k, i-k}$  by sending  $x$  into  $s \otimes x = g(x)$ ; then  $\theta = d_k \circ g$ . Put now  $d_k s = v$ ; then by theorem II.5.1 we have

$$\theta(x) = d_k(s \otimes x) = d_k((s \otimes e) * x) = (d_k s) * x = v * x. \quad \text{We have:}$$

The map  $\theta: H_{i-k}(F) \rightarrow H_{i-1}(F)$  in the Wang (homology-) sequence is given by  $\theta(x) = v * x$ , where  $v$  is the characteristic element determined by  $d_k s = v$ . In case of a principal bundle over an actual sphere  $S^k$ , the element  $v$  is the (spherical) homology class determined by the characteristic element of the bundle in the sense of Steenrod [11, pp. 97, 180], with the sign reversed.

Let  $a$  be any element of  $H^p(F)$ , with  $0 < p < k - 1$ ; clearly all  $d_r a$  vanish. In  $E_k(= E_2)$  we have  $a \cap s \in E_k^{k, -p} = 0$ . It follows that

$$0 = d_k(a \cap s) = (-1)^{p+k} d_k a \cap s + a \cap d_k s = a \cap v;$$

in other words,  $v$  is a minimal element of  $H(F)$ .

Added in proof: A recent paper by T. Kudo (Homological structure of fibre bundles, J. Osaka City Univ. 2 (1952, 101-140) contains a construction similar to that of II.5. As a common generalization one could consider the situation of the «remark» in II.5, with  $Y$  and  $M$  paired not to  $Y$ , but to another fiber space  $Y'$  (with base  $B'$ , fiber  $F'$ , projec-



tion  $p'$ ), and the projections commuting with a given map  $f: B \rightarrow B'$ , i. e. satisfying  $p'(y \cdot m) = f \circ p(y)$ ; there is then an induced pairing of  $F$  and  $M$  to  $F'$ . Results and proofs are analogous to the earlier ones; for instance in  $E_2$  one has  $(x \otimes y) * z = f_*(x) \otimes (y * z)$ , for  $x \in H(B)$ ,  $y \in H(F)$ ,  $z \in H(M)$ .

Institute for Advanced Study and University of Michigan.

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