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Autor: Rauch, H.E.

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Geodesics, symmetric spaces, and differential geometry in the large[1])

By H. E. RAUCH, Philadelphia

1. Introduction, symmetric manifolds, and main theorem

Modern differential geometry, in particular Riemannian geometry, the subject of this paper, is a branch of analysis arising from problems of geometry and clothed in its language. Now it is an inevitable task in studying any analytical system to view it "in the large," i. e. in conjunction with the manifold of all its parameter systems and solutions with the result that geometrical considerations of another kind, among them topology, enter the picture. Thus it is my task here to present an investigation of the effect of certain natural hypotheses about a Riemannian metric on the manifold which bears it – the outstanding question in differential geometry in the large.

In a manner which I will make precise the essential conclusion will be that for a significant class of manifolds [2]) parallelism (the holonomy group) and curvature determine the topological structure of the manifold and that they do so via the geodesics since 1) the geodesics determine the topological structure (and more) and 2) parallelism and curvature determine the geodesics; hence the present paper deals, in reality, with the effect of curvature on geodesics.

To preface the main result and as an essential preliminary let me relate the results of prior investigations dealing, first, with 2).

Consider for the moment an n-dimensional Riemannian manifold M^n (the notation introduced in this paragraph will, with minor variations, be standard throughout the paper) with metric tensor g_{ij}, all being of sufficient differentiability, say, C^3 at least. A classical method of Bonnet (see Rauch [2][3]), pp. 39–46 and references there for what follows) appli-

[1]) For a brief summary and discussion of the main results of this paper see *Rauch* [1].

[2]) Including the classical spaces of geometry and analysis, i. e., those of constant curvature, hermitian elliptic (complex projective) space, the Grassmann manifolds properly metrized, and the compact semi-simple Lie groups.

[3]) Hereinafter referred to as C. D. G.

cable to surfaces and, with straightforward generalizations, to n-dimensions gives the following information about the effect of the curvature of g_{ij} on the relative behavior of its *infinitely near* geodesics. Let σ be a fixed geodesic issuing from a fixed $P \in M^n$, and introduce the co-ordinates (of Fermi) which are locally euclidean along σ, $(z_1, \ldots, z_{n-1}, z_n)$, where $z_n = s$ is the arc-length along σ measured from P, and

$$z = (z_1, \ldots, z_{n-1})$$

is the set of cartesian coordinates of a point on the $(n-1)$-plane $N(r)$ orthogonal to σ at $Q(r), z_n = s = r$. Let $z = z(e, s)$, $z_n = s$ be the co-ordinates of a one-parameter family of geodesics issuing from P and including σ for $e = 0$ $(z(0, s) = 0)$. Then the "infinitesimal displacement vector"

$$\eta = \frac{\partial z(0, s)}{\partial e}$$

satisfies the Jacobi equations:

$$\eta_\alpha'' + R_{n\beta n\alpha}\, \eta_\beta = 0 \qquad (n \text{ not summed}) . \tag{1}$$

The prime denotes differentiation with respect to s; all Greek indices run from 1 to $n-1$; and the repeated index convention for summation is used unless contrary indication is given. $R_{n\beta n\alpha}$ in (1) are the components of the Riemann-Christoffel tensor of g_{ij} in the coordinates z along σ, i. e., they are functions of s only. If $u = (u_1, \ldots, u_{n-1}, 0)$ is any unit vector perpendicular to σ at $Q(s)$ then $R_{n\alpha n\beta} u_\alpha u_\beta$ is the Riemannian curvature of M^n at the point $Q(s)$ and the 2-section σ spanned by u and the unit tangent vector to σ. In particular for a manifold of constant curvature the equations (1) become

$$\mu_\alpha'' + K\, \mu_\alpha = 0 . \tag{2}$$

Suppose that a vector solution η of (1) vanishes at P and again for the first time at $Q(r)$. Then $Q(r)$ is the first point *conjugate* to P on σ. Loosely speaking, $Q(r)$ is the first point where a geodesic issuing from P and neighboring σ "meets" σ again. The arc of σ given by $0 \leq s \leq r$ will be *relatively minimizing*; for $s \geq r$ it will *cease* to be so. Also, a sufficiently small neighborhood of $Q(s)$, $s < r$, will be *simply covered* by the geodesics issuing from P and close to σ.

Looking at (2) one sees that if $K > 0$ then the first conjugate point is at

$$r = \frac{\pi}{\sqrt{K}}$$

and the geodesics curve in on one another; if, however, $K \leq 0$, then there is no conjugate point; and the geodesics fan out like the straight lines in ordinary space. Label the equations obtained from (1) by replacing $R_{n\alpha n\beta}$ by $\bar{R}_{n\alpha n\beta}$, $(\bar{1})$—similarly for $\tilde{R}_{n\alpha n\beta}$, $(\tilde{1})$. One obtains then what I call

Bonnet's Lemma. *Assume that*

$$\tilde{R}_{n\alpha n\beta} u_\alpha u_\beta \leq R_{n\alpha n\beta} u_\alpha u_\beta \leq \bar{R}_{n\alpha n\beta} u_\alpha u_\beta \ .$$

Then the first point conjugate to P on σ with respect to (1) (*if any*) *lies between those taken with respect to* $(\tilde{1})$ *and* $(\bar{1})$ *respectively, being closer to P than the former and farther than the latter.*

The proof of Bonnet's Lemma will be given in section 3 along with related analytical details.

Two obvious corollaries will be the key to the two theorems which form the sum total of what had been done in this direction before C.D.G. and the present paper.

Corollary 1. *Let $K(P, \gamma)$ be the Riemannian curvature of M^n at the point P for the 2-section γ, and let $Q(r)$, as above, be the first point conjugate to P on any σ.*

If

$$0 < L \leq K(P, \gamma) \leq H \quad \text{for all } P \in M^n, \text{ all } \gamma \ ,$$

then

$$\frac{\pi}{\sqrt{H}} \leq r \leq \frac{\pi}{\sqrt{L}} \ .$$

Corollary 2. *With the same notation, if for all P and γ*

$$K(P, \gamma) \leq 0 \ ,$$

then there is no point conjugate to P on any σ issuing from it.

Assume now that the M^n in question is complete with respect to its metric – every *bounded* sequence of points has a limit point.

Then Corollary 1 has as a consequence the following

Theorem (Bonnet). *A complete Riemannian M^n for which*

$$0 < K \leq K(P, \gamma) \ ,$$

for all $P \in M^n$ and all γ, is compact and is of intrinsic diameter less than or equal to $\dfrac{\pi}{\sqrt{K}}$.

Similarly Corollary 2 implies the second

Theorem (Hadamard, Cartan). *The simply connected covering $\tilde{M}^n$ of a complete M^n for which*

$$K(P, \gamma) \leq 0 ,$$

for all $P \in M^n$ and all γ, is homeomorphic to euclidean space, E^n.

The ideas used in the simple proofs of these theorems form an integral part of the paper, and so they find their place here.

An initial, highly important observation is that the universal covering $\tilde{M}^n$ of M^n may automatically be endowed with the same local differential geometry as M^n itself (e. g., the Clifford-Klein space-forms and the sphere) so that in the last analysis all conclusions will apply to the simply connected M^n. That is true of the first theorem.

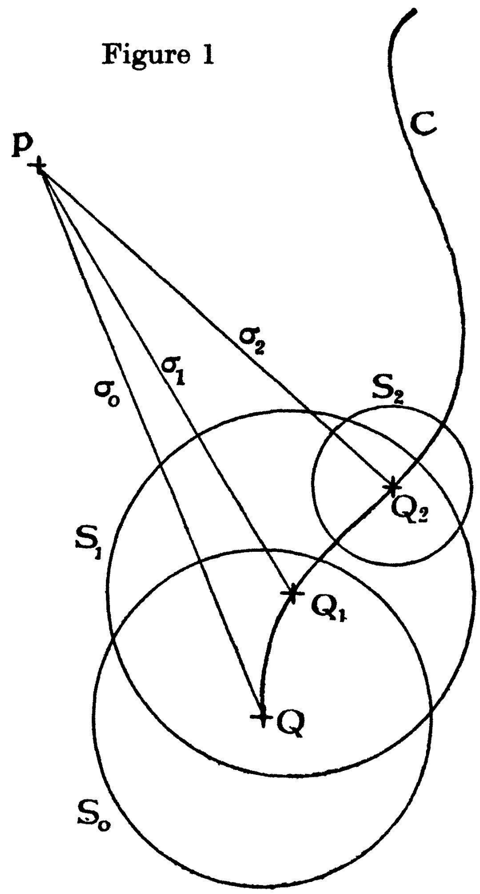
To dispose of Bonnet's theorem one must know that every point of a complete M^n may be joined to a fixed point P by an *absolutely* minimizing (shortest) geodesic arc. But according to Corollary 1 each such arc must be no longer than $\frac{\pi}{\sqrt{K}}$, otherwise it could not even be *relatively* minimizing. Therefore, every point of M^n is closer to P than $\frac{\pi}{\sqrt{K}}$ — the last part of the theorem. In particular *every* sequence is bounded and has, therefore, a limit point. In other words M^n is compact.

Concerning Cartan's theorem, the proof to follow will contain what are unnecessary elements for the immediate purpose but important for later applications. Consider the geodesics σ issuing from a fixed point $P \in M^n$. The totality of the σ make up what I call the *space of geodesics*, M^n_g, associated with M^n. A point of M^n_g is a point of M^n together with a σ on which it lies and the arc-length s measured along σ to the point in question, i. e., every point Q of M^n will give rise to a set of points $Q_g \in M^n_g$ which *cover* Q, the set being discrete if Q is not conjugate to P on any σ. A *neighborhood* N_g of $Q_g \in M^n_g$ will consist of those points in M^n_g which cover a cell-like neighborhood N of a point $Q \in M^n$ covered by Q_g. Again N_g will be a cell if Q_g is not conjugate to P.

But according to Corollary 2 there are no conjugate points. Therefore, M^n_g is a well defined manifold and becomes homeomorphic to E^n when one makes P correspond to the point O and the σ to the straight lines through O in E^n.

Now I claim that M^n_g *covers* M^n in the accepted sense, or, in other words, $M^n_g = \tilde{M}^n$ (M^n_g is simply connected, being homeomorphic to E^n). Given a curve C in M^n beginning at Q, I must show that one can *develop C along the curve C_g in M^n_g*, i. e., find a locally homeomorphic image C_g in M^n_g, which begins at any assigned Q_g covering Q. The obvious means of doing that constitutes what was called the c-process in C.D.G., p. 47, because of its patent analogy with a standard argument in analytic continuation of functions of a complex variable. Let t be a parameter on C such that the points of C may be written $Q(t)$, $t \geq 0$, when $Q(0) = Q$. Given Q there will be at least one geodesic arc σ joining Q to P because M^n is complete; therefore, there will be at least one $Q_g \in M_g^n$ covering Q. Pick one and the corresponding arc σ_0 joining Q to P. Q not being conjugate to P on σ_0, a sufficiently small sphere S_0 about Q will correspond to a similar sphere in M_g^n, i. e., S_0 will be simply covered by geodesic arcs neighboring σ_0. In particular a sufficiently small arc: $Q(t)$, $0 \leq t \leq t_1$ of C will automatically be developed in M_g^n, and one obtains a second arc σ_1 joining $Q(t_1)$ to P and neighboring σ_0. Continue the process (Figure 1), obtaining σ_i and $Q(t_i) = Q_i$. In this manner, one develops C in M_g^n step by step. I have only to show that, at least when C is *rectifiable* (which is enough) *all of C can be so developed.* I need the essential.

Figure 1



Lemma 1. *The length of any σ_i obtained in developing C in M_g^n by the c-process is less than the sum of the lengths of the initial σ_0 and the arc: $Q(t_i)$, $0 \leq t \leq t_i$, of C already developed.*

The proof is an obvious consequence of the relative minimizing property of geodesic arcs without conjugate points, namely: the length of σ_l will be less than the length of σ_{l-1} plus the length of $\overline{Q(t_{l-1})Q(t_l)}$. Consequently if the Q_i converged to an interior point Q' of C, the arcs σ_i, being bounded in length by σ and the arc $\overline{QQ'}$ of C would converge to a geodesic arc σ' joining Q' to P on which Q' could not be conjugate to P. Then one could start the whole process over again. q. e. d.

One sees then that the really delicate considerations occur in the case of positive curvature. To gain some insight into this case let us examine a broad class of special examples *which, in fact, will furnish the clues to the solution of the more general problem.* Consider the remarkable class of Riemannian manifolds called *symmetric* and discovered by Elie Cartan (see Cartan [1], [2], and [3] and references there for what follows). Denoting them generically by $\boldsymbol{E}^n$, one finds that the symmetric manifolds are co-set spaces of *semi-simple* Lie groups G and they fall into two categories – compact or open according as G is compact or not. The $\boldsymbol{E}^n$ differ from other homogeneous spaces by their characteristic property of admitting an involutive transformation or symmetry. As Riemannian manifolds the $\boldsymbol{E}^n$ posses an intrinsic metric which is invariant under G and in which the symmetry appears as an isometric reflection of the geodesics through a fixed point. But most important is the fact that the open $\boldsymbol{E}^n$ all have non-positive curvature so that the simply connected ones are all homeomorphic to E^n by Cartan's theorem; while on the other hand the compact ones *which include the group spaces of the G themselves* all have non-negative curvature. *I confine myself, therefore, to the consideration of compact, simply connected* $\boldsymbol{E}^n$ [4]). The geodesics of the $\boldsymbol{E}^n$ are the orbits of one-parameter subgroups of translations of G.

The vital differential-geometric property of the symmetric $\boldsymbol{E}^n$ (which is, in fact, equivalent to the existence of the symmetry) is *Property E: the Riemannian curvature of any 2-section γ at $P \in \boldsymbol{E}^n$ is preserved by parallel displacement of (P, γ).* Parallel displacement in $\boldsymbol{E}^n$ is equivalent to translation by an element of G and an analysis of the translations together with Property E leads to *Property F: the holonomy group $\boldsymbol{H}$ of $\boldsymbol{E}^n$ coincides with the isotropy group g.* $\boldsymbol{H}$, I recall, is the group of linear transformations of the tangent space $\boldsymbol{T}_P$ at $P \in \boldsymbol{E}^n$ onto $\boldsymbol{T}_Q$ at Q (see addendum at end of paper) generated by the parallel displacement of a frame along all curves joining P to Q (in particular when $P = Q$). The same definition holds good for the holonomy group H of an arbitrary Riemannian $\boldsymbol{M}^n$ with the understanding that since $\boldsymbol{E}^n$ is assumed simply connected one is dealing with the *restricted* holonomy group, i. e. when $P = Q$ only those curves which are shrinkable are admitted. g is the subgroup of G which leaves a point P fixed, and Property F, strictly speaking, deals not with g but with the isomorphic group of linear transformations of $\boldsymbol{T}_P$ into itself generated by g which, when $\boldsymbol{E}^n = G$, is the adjoint group.

Using Properties E and F one may draw some analytical conclusions which, in turn, will lead to important statements about the geodesic structure of a given $\boldsymbol{E}^n$. Returning to the beginning of this section, one may consider a fixed $P \in \boldsymbol{E}^n$, a fixed geodesic σ issuing from P, and the co-ordinates (z, s) in which g_{ij} becomes euclidean along σ. In the co-ordi-

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[4]) It should be stated that every $\boldsymbol{E}^n$ is the direct product of irreducible factors, and the curvature alternative applies to them; however, no assumption of irreducibility is made in this paper. I do assume that all factors be compact.

nates (z, s) a necessary and sufficient condition for a vector to be propagated parallely along σ is that it have constant components. Hence the tangent vector to σ, $(0, 0, \ldots, 1)$, and the unit vector,

$$u = (u_1, \ldots, u_{n-1}, 0) \ ,$$

span a 2-section γ which is propagated parallel to itself the length of σ. Accordingly the form $R_{n\alpha n\beta} u_\alpha u_\beta$, being the curvature at the point $(0, s)$ in the direction γ, has constant coefficients by Property E. Therefore, after a preliminary orthogonal transformation of the $z's$ (the same for all s), $R_{n\alpha n\beta}$ being symmetric in α and β, the equations (1) will take the form

$$\begin{aligned} &\eta''_{\alpha_1} + K_1 \eta_{\alpha_1} = 0 && \alpha_1 = 1, \ldots, n_1 \\ &\eta''_{\alpha_m} + K_m \eta_{\alpha_m} = 0 && \alpha_m = n_{m-1} + 1, \ldots, n-1 \end{aligned} \tag{3}$$

where $K_1 > K_2 > K_3 > \cdots > K_m \geq 0$ are the *distinct* characteristic roots of $R_{n\alpha n\beta} u_\alpha u_\beta$ and the n_i are their multiplicities.

Introduce now geodesic polar coordinates $(y, s) = (y_1, \ldots, y_{n-1}, s)$ with P as pole. That means to choose an auxiliary $(n-1)$-sphere Σ about P, on it coordinates y such that $dy_\alpha dy_\alpha$ is its line-element; and to assign the coordinates (y, s) to a point in $\boldsymbol{E}^n$ which lies at the distance s out from P on the geodesic σ whose initial direction at P is specified by y on Σ.

Then the nature of g_{ij} in the coordinate systems (y, s) and (z, s) and the relationship between the latter enable one to show (C. D. G., p. 45, last paragraph) that if the line element of any $\boldsymbol{M}^n$, let alone $\boldsymbol{E}^n$, be written in polar form along σ:

$$ds^2 + a_{\alpha\beta}(s) dy_\alpha dy_\beta \ ,$$

then $a_{\alpha\beta} = \eta^\alpha \cdot \eta^\beta = \eta^\alpha_\gamma \eta^\beta_\gamma$ where the η^α are $n-1$ vector solutions of (1) such that $\eta^\alpha(0) = 0$ and $\eta^{\alpha\prime}(0) \cdot \eta^{\beta\prime}(0) = \delta_{\alpha\beta}$. In particular for $\boldsymbol{E}^n$ if one chooses

$$\eta^{\alpha_1} = \left(0, \ \ldots, \ \frac{\sin s \sqrt{K_1}}{\sqrt{K_1}} \ , \ldots, 0\right) \ ,$$

the non-zero entry being in the α_1-st place, etc. as solutions of (3), then the line-element of $\boldsymbol{E}^n$ in polar form along σ falls into blocks :

$$ds^2 + \frac{\sin^2 s \sqrt{K_1}}{K_1} (dy_1^2 + \ldots + dy_{n_1}^2) + \ldots + s^2 (dy^2_{n_{m-1}+1} + \ldots + dy^2_{n-1}), \tag{4}$$

where I have assumed that $K_m = 0$ – the usual circumstance.

Observe, however, that the characteristic roots and multiplicities are not necessarily the same for all σ issuing from P but only for those which are transformable into σ by g $(\simeq \boldsymbol{H})$. Indeed $\boldsymbol{H}$ is only transitive on Σ in the $\boldsymbol{E}^n$ of *rank* $\lambda = 1$ (Cartan [2]) which are only those of constant curvature (spheres) and the hermitian elliptic (projective) spaces defined over the complex, quaternion, and Cayley algebras. For rank $\lambda > 1$ Σ is divided into domains of transitivity Σ_σ under $\boldsymbol{H}$, and for each Σ_σ the roots K and multiplicities n_α are the same since $\boldsymbol{H}$ conserves curvature (Property F) while the roots may vary continuously from Σ_σ to $\Sigma_{\sigma'}$ (the multiplicities changing only when some roots become coincident).

In contrast to the manifolds of negative curvature the dominant role is now played by the locus of first conjugate points, henceforth designated by C, of P. Indeed, the following observations about C easily deduced from the preceding will be essential.

(I) On each σ issuing from P the first conjugate point p will fall at

$$s = \frac{\pi}{\sqrt{K_1}},$$

i. e., will be determined *only by the highest characteristic root* of the curvature form — call it K_σ hereafter. That follows immediately from (3).

(II) (4) shows clearly that each such conjugate point p is the meeting-place of ∞^{n_1} geodesics of the same length (actually of closed geodesics making up the spindle of great circles on an S^{n_1} going through the poles P and p) whose initial tangents cut out on Σ a subsphere, hereafter designated as $\Sigma_p \cdot p$ is, of course, conjugate to P on all these geodesics.

(III) It follows easily from II that C, unlike the conjugate locus in an arbitrary M^n, is a smoothly embedded locus in $\boldsymbol{E}^n$ and that the points of C sufficiently close to p together with sufficiently small arcs of the geodesics terminating therein form a cell-like neighborhood of p. In other words, *if one forms* $\boldsymbol{E}^n_g$, as in the proof of Cartan's theorem, *one may complete it to a compact manifold by the addition of* C.

(IV) As a consequence of (III) one may show that $\boldsymbol{E}^n_g$ covers $\boldsymbol{E}^n$, but as $\boldsymbol{E}^n$ was assumed simply connected so one may conclude that $\boldsymbol{E}^n_g = \boldsymbol{E}^n$. Therefore, $\boldsymbol{E}^n$ consists of a cell, made up by the geodesic arcs issuing from P, and a singular locus C which must then contain all the topological properties of $\boldsymbol{E}^n$. In the case when $\boldsymbol{E}^n$ is itself a compact, semi-simple Lie group space C is the locus of singular elements – those which do not admit a unique canonical representation.

While (I)-(IV) are the only properties of $\boldsymbol{E}^n$ needed in this paper I should like to make two additional observations without proof. Together with the foregoing they show how clearly and intuitively the topology of the $\boldsymbol{E}^n$ is laid bare and that new ways of research are opened up by the present method.

(V) For rank $\lambda > 1$ the smallest K_m is zero, and the corresponding geodesics make up the maximal toroid containing σ which fact, in view of (I) and (IV), shows how irrelevant the toroids are in *visualizing* the internal structure of $\boldsymbol{E}^n$.

(VI) An explicit analysis of the locus C, particularly of the sub-loci of those points equivalent to any one $p \in C$ under $\boldsymbol{H}$ or some subgroup, is possible and should disclose the generating homology cycles. In particular it should account for the known structure of the compact semi-simple Lie groups (Hopf [1]).

As the simplest examples to illustrate what has preceded let me cite the sphere, S^n, bearing the metric of constant positive curvature K and the complex projective space P^n (2n real dimensions) with the hermitian elliptic metric (Study [1]). For the former $G = 0^+(n+1)$, the orthogonal group with positive determinant on $n+1$ variables; $\boldsymbol{H} = g = 0^+(n)$; $K_\sigma = K$; $n_1 = n-1$, and P and p are opposite poles joined by semi-great circles while (VI) holds good. In the latter case $G = U(n+1)$, the unitary group on $n+1$ complex variables; $\boldsymbol{H} = g = U(n)$; $K_\sigma = K > 0$, $K_2 = \frac{K}{4}$; $n_1 = 1$, $n_2 = 2n-1$; if P is the origin then p is the point at infinity on the projective line (complex) joining P to p; and the geodesics joining them all lie in that line forming an ordinary S^2 of curvature K. (VI) holds, too, the subgroup of $U(n)$ which transforms the geodesics through the origin of a projective line among themselves being the "circle group" generated by multiplying all the inhomogeneous coordinates by $e^{i\theta}$.

Such a particular, detailed, and accessible geometrical structure of an $\boldsymbol{E}^n$ would seem to be an *intrinsic and characteristic property of the peculiar integrable nature* of $\boldsymbol{E}^n$, that is to say, of the fact that $\boldsymbol{E}^n$ admits a set of infinitesimal displacements (equivalent to a set of partial differential equations) satisfying Lie's first theorem (complete integrability of said equations) with the consequence that the ordinary differential equations of the geodesics admit those very infinitesimal transformations (thus becoming integrable in Lie's sense). In other words, if the metric coefficients of $\boldsymbol{E}^n$ were slightly perturbed in such a way that the resulting metric no longer admitted a group of displacements and its geodesics no longer satisfied integrable equations then, though the new geodesics differ only slightly from the old, the particular delicate structure of C is completely destroyed; and the new manifold bearing the perturbed metric might differ radically in topology from that of $\boldsymbol{E}^n$ whose global structure was completely determined by the local rigid integrability of its metric structure ($\boldsymbol{E}^n = \boldsymbol{E}^n_g$ being, in fact, constructed by integration of the equations of its geodesics).

But, and this is the main result of the paper, if the holonomy group of the perturbed metric is no larger than $\boldsymbol{H}$ and if the variation in terms of curvature is sufficiently constrained, no matter how arbitrary the perturbation otherwise, the locus C may be reconstructed, as it were, and the sundered ends of the geodesics rejoined to it by means of additional geodesic arcs – thus reconstituting the geodesic and topological structure by means of *broken* geodesics.

Thus one sees that the symmetric manifolds, far from being isolated phenomena of a special nature, derive their structure from certain parallelism and curvature properties which when satisfied to a certain degree of approximation delimit a general class of Riemannian manifolds with the same structure. And in all probability that is the strongest statement one can make about the effect of general differential-geometric hypotheses on the topology of a general manifold.

Thus I present finally

Theorem 1. *Let $\boldsymbol{E}^n$ $(n \geq 2)$ be an n-dimensional, simply connected, symmetric Riemannian manifold of positive curvature, with holonomy group $\boldsymbol{H}$. Then there exists a constant $0 < c(\boldsymbol{E}^n) < 1$ of the following nature:* if M^n, a complete n-dimensional Riemannian manifold of class C^3 and restricted holonomy group H, is such that[5])

(a) $H \subseteq \boldsymbol{H}$.

(b) There exists for each $P \in M^n$ a transformation $h_P \in \boldsymbol{H}$ of the tangent space T_P at $P \in M^n$ onto the tangent space $\boldsymbol{T}$ at a fixed point of $\boldsymbol{E}^n$ under which

$$c(\boldsymbol{E}^n)\, \boldsymbol{K}(h_P\,\gamma) < K(P, \gamma) < \boldsymbol{K}(h_P\,\gamma) \quad \text{for all } \gamma$$

(where γ is a 2-section in T_P, $h_P\gamma$ its image in $\boldsymbol{T}$, $K(P, \gamma)$ the curvature of M^n at P in the direction γ, $\boldsymbol{K}(h_P\gamma)$ the corresponding curvature of $\boldsymbol{E}^n$); then the universal covering $\tilde{M}^n$ of M^n is homeomorphic to $\boldsymbol{E}^n$.

The main theorem of C. D. G. is the special case of Theorem 1 when $\boldsymbol{E}^n = S^n$ and $c(S^n) \sim .75$. When $\boldsymbol{E}^{2n} = P^n$ one can compute by the

[5]) Hypotheses (a) and (b), the explanatory remarks following Theorem 1, and the paragraph in section 3 where formula (9) is proved may be replaced by the simpler formulation in the paragraph on the last page added in proof.

methods of section 3 $c(P^n)$ and find that it equals about .95. One sees then that as the degree of complication increases the numerical value of $c(\boldsymbol{E}^n)$ may cease to be impressive, but the real content of the theorem is qualitative[6]).

Some clarification of the hypotheses is necessary. First of all, one may renormalize the highest curvature of $\boldsymbol{E}^n$ to be any desired positive value by multiplication of the metric by a constant. Thus the hypotheses apply to $\boldsymbol{E}^n$ itself despite the *strict inequalities* which happen to be very important. Next[7]), by h_P belonging to $\boldsymbol{H}$ one means that one can pick bases in T_P and $\boldsymbol{T}$ so that one may think of them as being identified, in which case that and (a) both become clear. With that understanding, too, the meaning of all of (b) becomes clear. In fact, in $T_P = \boldsymbol{T}$ one thinks of all the 2-sections of $\boldsymbol{E}^n$ and all those of M^n as situated with their vertices at the origin; then there exists by hypothesis an element of $\boldsymbol{H}$ taking one set into the other such that the curvatures on corresponding sections satisfy the inequality. Now it is important to observe that it does not matter at what point in $\boldsymbol{E}^n$ $\boldsymbol{T}$ is taken since $\boldsymbol{E}^n$ is homogeneous and, more important, *since every element of* $\boldsymbol{H}$ *leaves all curvature properties of* $\boldsymbol{E}^n$ *at a point invariant* h_P *need only be defined modulo left multiplication by* $\boldsymbol{H}$.

Just as in C. D. G. the central idea in the proof of theorem 1 is to deduce from its assumptions that the metric of M^n written in polar form along a given σ approximates that of $\boldsymbol{E}^n$ sufficiently closely to permit duplication of $\boldsymbol{E}^n$'s geodesic structure by means of broken geodesics. It will be seen to be sufficient to consider those $\bar{\sigma}$ (*bars will be used henceforth to distinguish entities in* M^n *which correspond to those in* $\boldsymbol{E}^n$) issuing from a fixed $\bar{P}$ in M^n whose initial directions lie on $\bar{\Sigma}_p$ where $\bar{\Sigma}_p$ is defined as follows: having picked $P \in \boldsymbol{E}^n$ and $p \in C \subset \boldsymbol{E}^n$ and a fixed σ joining them, consider Σ_p (as defined in (II)) – then the inverse image under $h_{\bar{P}}$ of Σ_p on Σ will be $\bar{\Sigma}_p$ (it being understood that $h_{\bar{P}}$, which may be taken as a map of $T_{\bar{P}}$ onto $\boldsymbol{T}_P$, induces a map of $\bar{\Sigma}$, the sphere of polar

6) Since the holonomy group H is always a subgroup of the orthogonal group which in turn is thu holonomy group of the sphere (constant curvature) the hypotheses of Theorem 1 need not be so complicated in this case. In fact when $\boldsymbol{E}^n = S^n$ one may identify any T_P with $\boldsymbol{T}$ and the identity map will fullfill the hypotheses, thus accounting for the seemingly simpler formulation of Theorem 1 in C. D. G. Also, the case $\boldsymbol{E}^{2n} = P^n$, the hermitian elliptic, projective space, implies that the "unknown" manifold M^{2n} is a so-called "Kähler" manifold, the latter being precisely a Riemannian manifold whose holonomy group is a subgroup of the unitary group.

7) Hypotheses (a) and (b), the explanatory remarks following Theorem 1, and the paragraph in section 3 where formula (9) is proved may be replaced by the simpler formulation in the paragraph on the last page added in proof.

co-ordinates about $\bar{P}$ in M^n, onto Σ the analogous sphere in $\boldsymbol{E}^n$). Observing that the same (y, s) may be used about P and $\bar{P}$ simultaneously one finds (next section) that theorem 1 may be deduced from

Theorem 2. *Under the conditions of the preceding paragraph and the hypotheses of Theorem* 1 *with the exception that the constant* $c(\boldsymbol{E}^n)$ *in* (b) *is replaced by an arbitrary* $0<c<1$ *one has*

$$ds^2 + a_{\alpha\beta}(s)\,dy_\alpha dy_\beta < ds^2 + \left(\frac{\sin^2 s\sqrt{c K_\sigma}}{c K_\sigma}\right)^{1-\sin^2\theta_0}\left(s^2\right)^{\sin^2\theta_0} dy_\alpha dy_\alpha\,,$$

where the left-hand term is the line-element of M^n *written in polar form along* $\bar{\sigma}$; $0 \leq s \leq \dfrac{\lambda\pi}{\sqrt{K_\sigma}}$, $\lambda<1$ *and* $\theta_0 = \theta_0(\lambda, c)$ *for fixed* λ *tends to* 0 *as* $c \to 1$. (α *and* β *run from* 1 *to* n_1 *and* $\bar{\Sigma}_p$ *and* Σ_p *are both given by* $y_{n_1+1} = \ldots = y_{n-1} = 0$).

I remark that the reverse inequality is also valid but not needed here. A more precise statement in the case $\boldsymbol{E}^n = S^n$ is Theorem 3 of C. D. G.

2. Deduction of Theorem 1 from Theorem 2

First of all, let me exhibit the $c(\boldsymbol{E}^n)$ whose existence is the real assertion of Theorem 1. That this $c(\boldsymbol{E}^n)$ will really do what is claimed for it in the subsidiary hypotheses will become clear as this section progresses. Its introduction at this point is artificial, but it clarifies the exposition. Namely, I first pick a fixed $0<\varepsilon<1$ and then choose λ and c so close to 1 (but not equal) that

$$\pi\left(\lambda\frac{\pi}{\sqrt{K_{\min}}}\right)^{\varepsilon}\left(\sin\lambda\sqrt{c}\,\pi/\sqrt{c K_{\min}}\right)^{1-\varepsilon} = \tfrac{1}{2}\pi/\sqrt{K_{\max}}\,, \qquad (5)$$

where $K_{\min} = \inf K_\sigma$[8]) and $K_{\max} = \sup K_\sigma$. Then fix λ and choose c even closer to 1 if necessary so that $\sin^2\theta_0 = \varepsilon$. That is possible by Theorem 2 ($\theta_0 \to 0$ fixed $\lambda<1$ and $c \to 1$). One sees how much room there is for juggling ε, λ, and c in order to find the best possible c particularly when the dependence of ε (i. e. θ_0 really) on c and λ is so complicated (see section 3).

To show that the $c(\boldsymbol{E}^n)$ thus produced does indeed fill the bill I need some preliminary statements, remarks, and notation.

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[8]) $K_{\min}$ is definitely greater than zero. K_σ being the maximum root, not zero, for each σ and $\boldsymbol{H}$ being compact, the infimum will also not be zero.

Let M^n henceforth be the "trial" manifold which is to be compared with E^n as per Theorem 1. Then the notation in section 1 following Theorem 1 will be retained, in particular, the use of bars over symbols to distinguish entities in M^n from their mates in E^n.

Now it will be proved in section 3 that the assumptions (a) and (b) about M^n in Theorem 1 imply that Bonnet's Lemma holds for M^n when the extreme members of the inequality are set equal respectively to

$$K_1(u_1^2 + \cdots + u_{n_1}^2) + \cdots + K_m(u_{n_{m-1}+1}^2 + \cdots + u_{n-1}^2)$$

on the right and the same quantity multiplied by $c(E^n)$ on the left of the inequality. Then as a consequence of that and of I one sees that *on a* $\bar{\sigma}$ *issuing from* $\bar{P} \in M^n$ *the first point conjugate to* $\bar{P}$ *will lie between*

$$s = \pi / \sqrt{K_{\bar{\sigma}}} \quad \textit{and} \quad \pi / \sqrt{cK_{\bar{\sigma}}}\ .$$

The first important statement which results is:

(i) Let $\bar{S}^n_{\bar{P}}(r)$ be the geodesic sphere of radius r about $\bar{P} \in M^n$, i. e. the set of geodesic arcs of length r issuing from $\bar{P}$. If $r < \pi/\sqrt{K_{\max}}$ then the italicized statement implies that one can make $\bar{S}^n_P(r)$ into a (open, bounded) manifold, denoted by the same symbol, in the same way that M^n_g was generated in the proof of Cartan's Theorem. $\bar{S}^n_{\bar{P}}(r)$ *is homeomorphic to the interior of a solid euclidean sphere while the set in* M^n *covered by it is only locally homeomorphic thereto.*

This distinction between a "space of geodesics" and the set in M^n covered by it will be *absolutely vital* in the proof of Theorem 1 and is necessitated by the fact that M^n itself may *a priori* be of such small diameter that even a short arc of geodesic will wrap itself 'round and 'round M^n making it impossible to work directly thereon. What may often appear to be a circumlocution will be justified in this light.

At this point let me also dispose of another property of $\bar{S}^n_{\bar{P}}(r)$ required in the sequel. Namely

(ii) $\bar{S}^{n-1}_{\bar{P}}(r)$, $r \leq \frac{1}{2}\pi/\sqrt{K_{\max}}$, the boundary of $\bar{S}^n_{\bar{P}}(r)$, which is homeomorphic to a euclidean S^{n-1} is locally concave toward $\bar{P}$, i. e. has *positive-definite second fundamental form* when any sufficiently small piece of it is considered as embedded in M^n. That means that if attention is focused on one geodesic radius $\bar{\sigma}$ and on the hyperplane made up of short geodesic arcs orthogonal to $\bar{\sigma}$ at its endpoint then the geodesics neighboring $\bar{\sigma}$ and joining $\bar{P}$ to the hyperplane are longer than $\bar{\sigma}$. A proof of (ii) will be given in section 3 (see C. D. G., p. 43). An important consequence of (ii) is

(iii) $\bar{S}^n_P(r)$, $r \leq \frac{1}{2}\pi/\sqrt{K_{\max}}$, is *geodesically convex* in the sense that any geodesic arc $\bar{\sigma}$ in M^n joining two points which are covered by $\bar{S}^n_P(r)$ will itself be covered and thus appear as a geodesic in $\bar{S}^n_P(r)$ joining two points covering the original points in M^n, *if $\bar{\sigma}$ is a member of a continuous one-parameter family of geodesic arcs all of whose end-points are in $\bar{S}^n_P(r)$ and at least one of which is entirely developed therein* (such as a radius of $\bar{S}^n_P(r)$).

The proof of (iii) is a trivial consequence of (ii) since if it were not true there would be a member of the one-parameter family which would be entirely developed in $\bar{S}^n_P(r)$ except for one point at which it would be tangent to $\bar{S}^{n-1}_P(r)$; but (ii) shows that is impossible.

The stage for the real substance of the proof, the comparison of M^n (satisfying (a) and (b) in Theorem 1) with $\boldsymbol{E}^n$ with an eye toward imitation of the latter's structure, will now be set by a slightly more detailed discussion of that structure than was given in section 1.

First consider $\boldsymbol{E}^n$ and a point P in it which is fixed for the remainder of the section. The first conjugate points with respect to P on the geodesics σ issuing from P form the conjugate locus C as in section 1. I observe once more that Σ, the sphere of polar coordinates about P is subject to a double "fibering" by the σ cutting it and ending in C. Namely Σ is first fibered by the domains of equivalence Σ_σ containing the initial directions of those σ which are transformable into one another by $g \in \boldsymbol{H}$. Then each Σ_σ is fibered into the spheres Σ_p comprising the initial directions of those σ meeting at one point $p \in C$. On each of these last mentioned σ mark off the point at the distance $\lambda\pi/\sqrt{K_\sigma}$ from P, where λ has the value chosen in (5). The locus of the resulting points I call $\boldsymbol{E}^{n-1}(\lambda)$, having done the same for all $p \in C$, while the totality of arcs of σ ending therein I call $\boldsymbol{E}^n(\lambda)$. In addition let $\Sigma_p(\lambda)$ be the subset of $\boldsymbol{E}^{n-1}(\lambda)$ cut out by those σ whose initial directions lie in Σ_p; and if σ_p be any particular one of the latter and γ any half great circle in Σ_p with one end at the point where σ_p cuts Σ_p, then the σ which cut Σ_p along γ will cut $\Sigma_p(\lambda)$ along an arc Γ. Concerning all these things one has the all-embracing

(iv) $\boldsymbol{E}^n(\lambda)$ is homeomorphic to a solid sphere while $\boldsymbol{E}^{n-1}(\lambda)$ is homeomorphic to an $(n-1)$-sphere and is fibered in the same way as Σ. $\boldsymbol{E}^n - \boldsymbol{E}^n(\lambda)$ is a solid "tube" T about C consisting of a "base space" C and "fibers" Δ_p which are solid "disks" of geodesic arcs of length

$$(1 - \lambda)\,\pi/\sqrt{K_\sigma}$$

issuing from each $p \in C$, these arcs being nothing other than the continuation of the σ issuing from P and going through $\Sigma_p(\lambda)$ on $E^{n-1}(\lambda)$. C is itself already a cross-section in T. All these statements follow easily from (I) – (IV) in section 1.

(v) The following closely related additional observations, seemingly irrelevant, will actually serve as a guide for completing the proof. Namely, each fiber Δ_p of T being solid, i. e., cell-like, one could, by picking a point q other than p in each Δ_p, construct another cross section (Schnittfläche) C' of C in T by a familiar process since no "obstruction" will arise. One could then replace the radii of Δ_p by geodesic arcs τ joining the new $q \in \Delta_p$ to the points of $\Sigma_p(\lambda)$ so that in place of the σ joining P to C one obtains the *broke* σ' joining P to C'. Now the τ, coming from the interior of the convex Δ_p, obviously meet the σ nicely and at larger than right angles. Therefore from $E^n(\lambda)$, the τ, and C' one can construct a *space of broken geodesics* $E^{n\prime}$, which is *a manifold* (even at the "corners") and *covers* E^n and is, therefore, *homeomorphic* to E^n.

One more observation:

(vi) The sphere of radius $\frac{1}{2}\pi/\sqrt{K_{\max}} - \varrho$ ($\varrho > 0$ sufficiently small) about p, S_p^n, contains $\Sigma_p(\lambda)$ in its interior. That is not obvious because $K_\sigma \leq K_{\max}$; therefore, even though $\lambda > \frac{1}{2}$ it is not clear that

$$(1-\lambda)\,\pi/\sqrt{K_\sigma} \leq \tfrac{1}{2}\pi/\sqrt{K_{\max}} \; .$$

To establish it I use the following device. The curve Γ on $\Sigma_p(\lambda)$ (see remarks before (iv)) has length

$$(\pi/\sqrt{K_\sigma})\sin([\lambda\pi/\sqrt{K_\sigma}]\,\sqrt{K_\sigma}) = (\pi/\sqrt{K_\sigma})\sin\lambda\pi \;,$$

by (4), which is less than or equal to

$$(\pi/\sqrt{K_{\min}})\sin\lambda\pi < \tfrac{1}{2}\pi/\sqrt{K_{\max}} - \varrho \;,$$

by (5). Now any such Γ by its definition starts at the point $q \in \Sigma_p(\lambda)$ where σ_p cuts the latter. Therefore, Γ may be developed by the c-process in S_q^n since, by the preceding inequality and Lemma 1 in section 1, the geodesic radii joining q to Γ obtained in the process will be shorter than $\frac{1}{2}\pi/\sqrt{K_{\max}}$. But the totality of the Γ exhaust $\Sigma_P(\lambda)$ just as their pre-images, the half great circles, in Σ_p exhaust the latter. Therefore, $\Sigma_p(\lambda)$ *may be developed* in toto *in* S_q^n; and since S_q^n by (iii) is convex ((iii) applies equally to E^n), Δ_p *which is really the geodesically convex closure* of $\Sigma_p(\lambda)$ *is also contained in* S_q^n. But the same reasoning applies also to any

$q \in \Sigma_p(\lambda)$. Therefore, Δ_p lying in every S^n_q, every fixed point in Δ_p is joined to every $q \in \Sigma_p(\lambda)$ by a geodesic arc of length $< \frac{1}{2}\pi/\sqrt{K_{\max}} - \varrho$. In particular S^n_q contains $\Sigma_p(\lambda)$. q. e. d.

Now it is a very simple matter indeed after these preliminaries to show how (iv), (v), and (vi) can be applied word for word in M^n with only technical modifications – *with the exception that the locus C which is a priori absent in M^n must be constructed precisely by the devices indicated in* (v) *and* (vi).

In fact, let $\bar{P} \in M^n$ also be fixed once and for all as was $P \in \boldsymbol{E}^n$. $\bar{\Sigma}$ is the sphere of polar coordinates about $\bar{P}$, and $\bar{\sigma}$ is any geodesic issuing from $\bar{P}$. As remarked in section 1, the linear transformation $h_{\bar{P}}$ (notation slightly changed) in (b) of Theorem 1 may chosen as a transformation of $T_{\bar{P}}$ onto $\boldsymbol{T}_P$ (which under proper choice of bases becomes an element of $\boldsymbol{H}$). Under $h_{\bar{P}}$ Σ and $\bar{\Sigma}$ correspond, and according to (b) under this correspondence any Σ_p, $p \in C \subset \boldsymbol{E}^n$, is mapped onto what was called $\bar{\Sigma}_p$. And for the $\bar{\sigma}$ whose initial directions fall on the subsphere $\bar{\Sigma}_p$ one has Theorem 2. One can then pursue the parallel with (iv), (v), and (vi). Suppose in particular that one defines $\bar{M}^n(\lambda)$ and $\bar{M}^{n-1}(\lambda)$ by marking off on each $\bar{\sigma}$ the point $\bar{q}$ at the distance $\lambda\,\pi/\sqrt{K_\sigma}$ along $\bar{\sigma}$ from $\bar{P}$, where σ (i. e., its initial direction) corresponds to $\bar{\sigma}$ under $h_{\bar{P}}^{-1}$ and K_σ is the corresponding maximum curvature in $\boldsymbol{E}^n$. Then, thanks to the italicized statement before (i), one obtains the limited analogue of (iv):

(vii) $\bar{M}^n(\lambda)$ and $\bar{M}^{n-1}(\lambda)$ are homeomorphic respectively to a solid sphere and an $(n-1)$-sphere while the sets covered by them are only locally so.

Furthermore, just as $\Sigma_p(\lambda)$ on $\boldsymbol{E}^{n-1}(\lambda)$ is homeomorphic to Σ_p on Σ and is filled out in the natural way by the curves Γ of length

$$< \tfrac{1}{2}\,\pi/\sqrt{K_{\max}} - \varrho\ ,$$

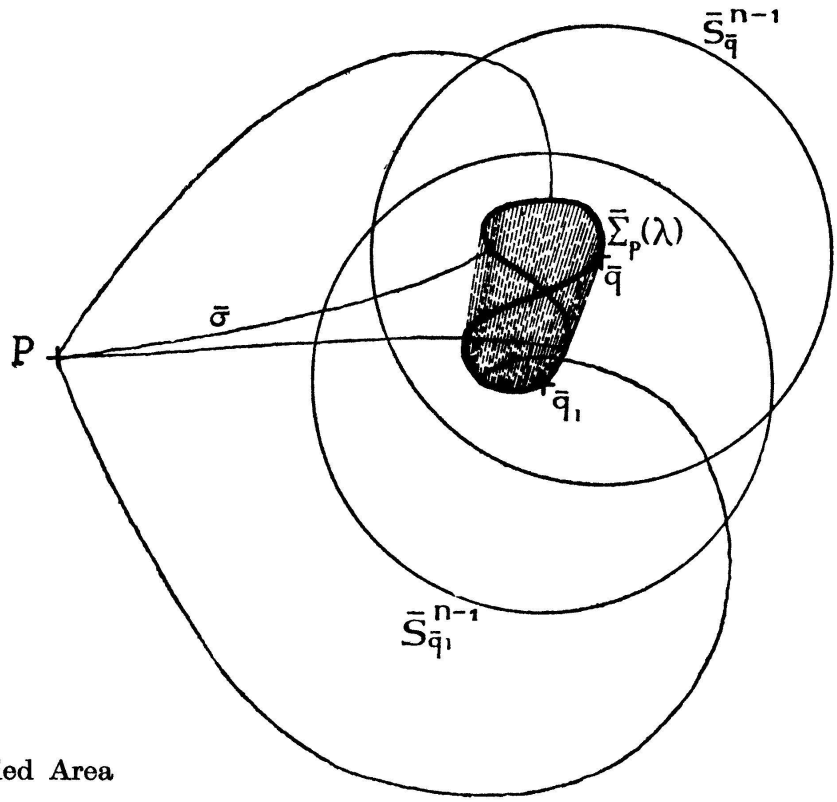
so is $\bar{\Sigma}_p(\lambda)$ on $\bar{M}^{n-1}(\lambda)$ homeomorphic to $\bar{\Sigma}_p$ on $\bar{\Sigma}$ and filled out by $\bar{\Gamma}$ *whose length by Theorem* 2 *and* (5) *is also* $< \frac{1}{2}\,\pi/\sqrt{K_{\max}} - \varrho$. Thus one has the limited analogue of (vi):

(viii) If $\bar{S}^n_{\bar{q}}$ is the geodesic sphere of radius $\frac{1}{2}\pi/\sqrt{K_{\max}} - \varrho$ about any $\bar{q} \in \bar{\Sigma}_p(\lambda)$ then the latter can be developed in $\bar{S}^n_{\bar{q}}$, starting with $\bar{q}$, by the c-process[9]). *In particular, by* (iii) *the geodesically convex closure* (in the sense indicated there), $\bar{\Delta}_p$, *of* $\bar{\Sigma}_p(\lambda)$ *will be contained in every* $\bar{S}^n_{\bar{q}}$,

[9]) The hypothesis that M^n is complete is used whenever the c-process is used.

$\bar{q} \in \bar{\Sigma}_p(\lambda)$, *so that the latter will automatically be developed in* $\bar{S}^n_{\bar{p}}$ *where* $\bar{p}$ *is any point in* $\bar{\Delta}_p$ (Figure 2).

I can now easily construct a locus $\bar{C} \subset M^n$ *which will be locally homeomorphic to* $C \subset E^n$, and the process will be precisely that described in constructing C' in (v). Namely, after decomposing C into sufficiently small simplexes I pick any vertex p and arbitrarily pick as its image point, $\bar{p}$, in the corresponding $\bar{\Delta}_p$ of M^n. I do the same for the p belonging successively to the edges, faces, etc., the continuous extension of the map always being possible because $\bar{\Delta}_p$ (being convex) is solid, so



$\bar{\Delta}_p$-Shaded Area

Figure 2

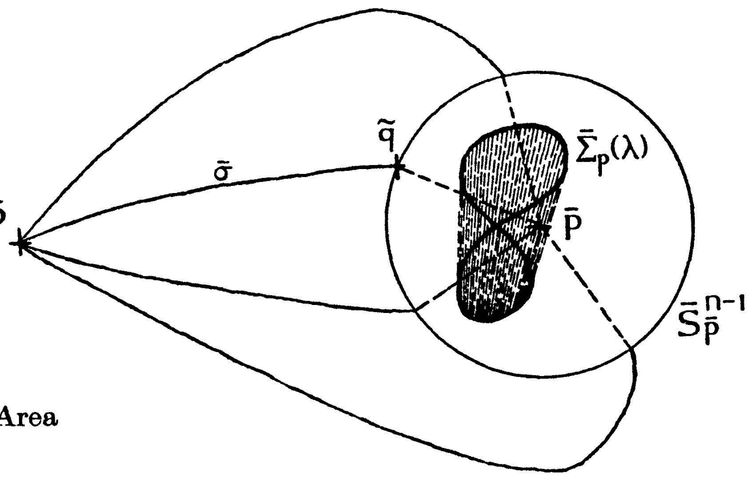
that no obstruction will arise. A little more care and the use of standard approximation theorems will ensure that the map of C onto $\bar{C}$, the set of $\bar{p}$ thus obtained, be a *local homeomorphism*. I suppress the routine and tedious details.

The construction of $\bar{M}^n$, *the space of broken geodesics which covers* M^n, will follow easily from further appeal to (viii).

Indeed, according to (viii) $\bar{\Sigma}_p(\lambda)$ is contained in $\bar{S}^n_{\bar{p}}$, the geodesic sphere of radius $\frac{1}{2}\pi/\sqrt{K_{\max}} - \varrho$, whose boundary, $\bar{S}^{n-1}_{\bar{p}}$, by (ii) has positive definite second fundamental form. Now I claim

(ix) If, in $\bar{S}^n_{\bar{p}}$, one applies the c-process to the geodesic radius $\bar{\sigma}$ of $M^{n-1}(\lambda)$ which ends in the point $\bar{q} \in \bar{\Sigma}_{\bar{p}}(\lambda)$, starting with $\bar{q}$, then one reaches $\bar{S}^{n-1}_{\bar{p}}$ before reaching $\bar{P}$ and finds that at the point $\tilde{q}$ where $\bar{\sigma}$ cuts $\bar{S}^{n-1}_{\bar{p}}$ it does so transversely, i. e., $\bar{\sigma}$ *is not tangent to* $\bar{S}^{n-1}_{\bar{p}}$ at $\tilde{q}$.

The proof of the last part of (ix) is clear, i. e., if $\bar{P}$ *is not reached first* then it is obvious that $\bar{\sigma}$ cannot be tangent to $\bar{S}^{n-1}_{\bar{p}}$ at $\tilde{q}$ because the local convexity would imply that the arc of $\bar{\sigma}$ sufficiently close to $\tilde{q}$ would have to lie entirely outside $\bar{S}^{n}_{\bar{p}}$, contradicting the facts. The slight technical difficulty is to show that $\bar{P}$ is not attained in applying the c-process to $\bar{\sigma}$, i. e., that the entire arc of $\bar{\sigma}$ from $\bar{P}$ to $\bar{q}$ is not developed in $\bar{S}^{n}_{\bar{p}}$. *It is enough to show that is not the case for one single* $\bar{\sigma}$ by using the local



$\bar{\Delta}_p$-Shaded Area

Figure 3

convexity again in a simple argument (C. D. G., p. 52, last paragraph, and p. 53). Now even that is not self-evident for $\bar{S}^{n}_{\bar{p}}$, but it certainly is for $\bar{S}^{n}_{\bar{q}}$, one of whose geodesic radii is just a sub-arc (of length

$$\tfrac{1}{2}\,\pi/\sqrt{K_{\max}}-\varrho)$$

of a $\bar{\sigma}$, radius of $\bar{\Sigma}_{\bar{p}}(\lambda)$ (of length $\lambda\,\pi/\sqrt{K_\sigma}>\frac{1}{2}\pi/\sqrt{K_{\max}}$). Then by sliding $\bar{S}^{n}_{\bar{q}}$ continuously to $\bar{S}^{n}_{\bar{p}}$ one sees by the same local convexity that $\bar{P}$ cannot suddenly enter $\bar{S}^{n}_{\bar{p}}$ (otherwise some arc of some $\bar{\sigma}$ which was partly in and partly out of $\bar{S}^{n}_{\bar{p}}$ would also be tangent to $\bar{S}^{n-1}_{\bar{p}}$). q. e. d.

To form $\bar{M}^n$, *the space of broken geodesics covering* M^n, one now joins each $\bar{p}\in\bar{C}$ to all the corresponding $\tilde{q}$, the points where the $\bar{\sigma}$ issuing from $\bar{p}$ and ending in $\bar{\Sigma}_p(\lambda)$ cut $\bar{S}^{n-1}_{\bar{p}}$, by the geodesic radii of the latter which were obtained by developing the $\bar{\sigma}$ in $\bar{S}^{n}_{\bar{p}}$ (Figure 3). Notice that $M^{n-1}(\lambda)$ has disappeared from consideration altogether while only part of $M^n(\lambda)$ remains.

Thanks to (ix) and (v) it is then a simple matter, requiring only some continuity considerations of a routine nature and, therefore, omitted here, to verify that $\bar{M}^n$ is homeomorphic to E^n and covers M^n – which shows that the choice of $c(E^n)$ in (5) fulfills the requirements of Theorem 1. Q. E. D.

3. Proof of Theorem 2 and related analytical facts[10]

Returning to the analytical situation in the opening paragraphs of the introduction, one sees that some words about Bonnet's lemma are in order. The proof thereof is a simple consequence of some standard facts about solutions of the Jacobi equations (1) (see for example, Morse, Calculus of Variations in the Large, Chapter I). In fact, if $\eta(s)$ is a vector solution of (1) such that $\eta(0) = \eta(s_1) = 0$ then, after a simple integration by parts, one finds that

$$I(s_1) \equiv \int_0^{s_1} \Omega(\eta, \eta')\, ds = 0$$

and (not so easily) *conversely*, where

$$\Omega(\eta, \eta') \equiv \eta_\alpha \eta_\alpha - R_{n\alpha n\beta}\, \eta_\alpha \eta_\beta \;.$$

$I(s_1)$ is the second variation of the length integral in M^n for curves connecting P and $Q(s_1)$ on σ. Now define

$$\bar{I}(r) \equiv \int_0^r \bar{\Omega}(\bar{\eta}, \bar{\eta}')\, ds \quad \text{and} \quad \tilde{I}(r) \equiv \int_0^r \tilde{\Omega}(\tilde{\eta}, \tilde{\eta}')\, ds$$

where $\bar{\Omega}(\bar{\eta}, \bar{\eta}') = \bar{\eta}'_\alpha \bar{\eta}'_\alpha - \bar{R}_{n\alpha n\beta}\, \bar{\eta}_\alpha \bar{\eta}_\beta$ and similar notation holds for $\tilde{I}(r)$, $\bar{\eta}$ being a solution of $(\bar{1})$, $\tilde{\eta}$ of $(\tilde{1})$. Observe that the curvature inequality of Bonnet's lemma implies

$$\bar{\Omega}(\eta, \eta') \leq \Omega(\eta, \eta') \leq \tilde{\Omega}(\eta, \eta') \;, \tag{6}$$

just the reverse thereof, where the *same argument* appears in all terms. Observing that if the interval $0 \leq s \leq r$ contains no point conjugate to P $(s = 0)$ on σ with respect to (1), $(\bar{1})$, or $(\tilde{1})$ then $\eta(s)$, $\bar{\eta}(s)$, and $\tilde{\eta}(s)$ are relatively minimizing for their respective integrals one finds

$$I(r) = \int_0^r \Omega(\eta, \eta')\, ds \leq \int_0^r \Omega(\tilde{\eta}, \tilde{\eta}')\, ds \leq \int_0^r \tilde{\Omega}(\tilde{\eta}, \tilde{\eta}')\, ds = \tilde{I}(r) \tag{7}$$

[10]) The differentiability hypotheses are used in appropriate places throughout this section without explicit mention.

by (6) where $\eta(0) = \tilde{\eta}(0)$ and $\eta(r) = \tilde{\eta}(r)$. A similar reasoning establishes the analogous

$$\bar{I}(r) \leq I(r) \tag{8}$$

where the left-hand side is evaluated for an $\bar{\eta}$ with the same end conditions as the preceding η and $\tilde{\eta}$. Bonnet's lemma then follows immediately from (7), (8), and the above remark to the effect that $I(r) = 0$ implies that $Q(r)$ is conjugate to P on σ.

In order to make the applications of the lemma necessary in the initial paragraphs of section 2 one must show that (6) and hence (7) and (8) are valid when $\bar{I}(r)$ is replaced by $J(r) = \int_0^r \Omega_1(\mu, \mu')ds$ and $\bar{I}(r)$ by $J_c(r) = \int_0^r \Omega_c(\nu, \nu')\,ds$, where

$$\Omega_1(\mu, \mu') = \mu'_\alpha \mu'_\alpha - (K_1\{\mu_1^2 + \cdots + \mu_{n_1}^2\} + \cdots + K_m\{\mu^2_{n_{m-1}+1} + \cdots + \mu^2_{n-1}\})$$

$$\Omega_c(\nu, \nu') = \nu'_\alpha \nu'_\alpha - c(K_1\{\nu_1^2 + \cdots + \nu_{n_1}^2\} + \cdots + K_m\{\nu^2_{n_{m-1}+1} + \cdots + \nu^2_{n-1}\})$$

and μ und ν are solutions respectively of (3) and of (3) modified by multiplying the roots K_i by c.

In other words, I must deduce from the assumptions (a) and (b) about M^n made in Theorem 1 that

$$\begin{aligned} &c\,\{K_1(u_1^2 + \cdots + u_{n_1}^2) + \cdots + K_m(u^2_{n_{m-1}+1} + \cdots + u^2_{n-1})\} < R_{n\alpha n\beta}u_\alpha u_\beta \\ &< K_1(u_1^2 + \cdots + u_{n_1}^2) + \cdots + K_m(u^2_{n_{m-1}+1} + \cdots + u^2_{n-1}) \end{aligned} \tag{9}$$

for all $s \geq 0$ (remember that $R_{n\alpha n\beta}$ is a function of the arc-length along σ)[11]), where $u = (u_1, \ldots, u_{n-1}, 0)$ is a unit vector with constant coefficients in the Fermi coordinates along $\bar{\sigma}$ (therefore parallelly propagated).

To prove (9)[12]), I observe that $\bar{\sigma}$ may be thought of as a geodesic simultaneously in M^n, E^n. In particular $\bar{P}$ may be thought of as common to all, as well as the set of Fermi coordinates z in $\bar{N}(s)$ in each of which a basis has been chosen so that a parallelly propagated vector has constant components (the metrics are all euclidean and, therefore, osculating along $\bar{\sigma}$). To utilize (b), I observe that bases may be chosen in $T_{\bar{P}}$ and $\boldsymbol{T}_P$ $(P = \bar{P})$ so that $h_{\bar{P}}$ whose existence is postulated in (b) will be the identity in $\bar{N}(0)$ *with the result that* (9) *is automatically true for* $s = 0$.

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[11]) I now change notation to conform with section 2.

[12]) Hypotheses (a) and (b), the explanatory remarks following Theorem 1, and the paragraph in section 3 where formula (9) is proved may be replaced by the simpler formulation in paragraph on the last page added in proof.

For every $s > 0$ choose a new basis in $\bar{N}(s)$ so that the linear transformation $h(s)$ of $\bar{N}(0)$ into $\bar{N}(s)$ induced by the parallel displacement of a frame of vectors (having *constant* components in the *old* bases) along $\bar{\sigma}$ in *the metric of* $\boldsymbol{E}^n$ becomes a member of $\boldsymbol{H}$ (see remarks after Theorem 1). Similarly choose still another basis in $\bar{N}(s)$ so that the corresponding $h(s)$ *generated by the parallelism due to* M^n becomes a member of H. Now by assumption (a) $H \subset \boldsymbol{H}$ *so that* $h(s)k^{-1}(s)$ (viewed as a transformation of the tangent space $T_{\bar{Q}}$ of M^n ($\bar{N}(s)$ intersecting $\bar{\sigma}$ at $\bar{Q}$), spanned by $\bar{N}(s)$ and the unit tangent vector of $\bar{\sigma}$, onto the corresponding $\boldsymbol{T}_Q$ when referred to the two bases referred to (after expanding each of them by the addition of one vector in $\bar{N}(s)$) *becomes a member of* $\boldsymbol{H}$; and by the remarks following Theorem 1 in section 1 corresponding 2-sections under *any* $h \in \boldsymbol{H}$ of $T_{\bar{Q}}$ onto $\boldsymbol{T}_Q = T_{\bar{Q}}$ will possess curvatures satisfying the inequality in (b). In particular, the 2-section γ spanned by u and $(0, \ldots, 0, 1)$, the unit tangent vector to $\bar{\sigma}$, being displaced parallelly with respect to both M^n and $\boldsymbol{E}^n$, the numerical vectors obtained by referring them to one basis in $T_{\bar{Q}}$ will be transformed into the numerical vectors obtained by referring them to the other basis in $T_{\bar{Q}} = \boldsymbol{T}_Q$ by hk^{-1}, i. e., *the resulting* 2-*sections will possess curvatures satisfying* (b); *but that is precisely* (9).

Having established (6), (7), and (8) in the desired context, let me, before going on to Theorem 2 and as a preliminary to it, establish (ii) in section 2. I first recall the following analytical details from C. D. G., p. 41–44 (changing notation slightly). Consider the locus $\bar{S}^{n-1}(r)$ of end points of geodesic arcs of fixed length r issuing from $\bar{P}$ and neighboring $\bar{\sigma}$. The points on $\bar{S}^{n-1}(r)$ sufficiently close to $\bar{\sigma}$ will be uniquely represented by their Fermi co-ordinates z in the hyperplane $\bar{N}(r)$ which is at once tangent to $\bar{S}^{n-1}(r)$ and orthogonal to $\bar{\sigma}$ at $\bar{Q}(r)$. If $\bar{Q}(r)$ is not conjugate to $\bar{P}$ for $0 \leq r \leq s$, then there will be a unique η, solution of (1), such that $\eta(0) = 0$, $\eta(r) = z$. In that case

$$I(r) = \int_0^r \Omega(\eta, \eta')\, ds = f_{\alpha\beta}(r)\, z_\alpha z_\beta = f_{\alpha\beta}(r)\, \eta_\alpha(r)\, \eta_\beta(r) \tag{10}$$

(cf. C. D. G., p. 43) where $f_{\alpha\beta}(r)\, z_\alpha z_\beta$ is the *second fundamental form* of $\bar{S}^{n-1}(r)$. Correspondingly one has

$$J(r) \equiv \int_0^r \Omega_1(\mu, \mu')\, ds = \sqrt{K_1} \cot r \sqrt{K_1}(z_1^2 + \cdots + z_{n_1}^2) \tag{11}$$
$$+ \cdots + \sqrt{K_m} \cot r \sqrt{K_m}(z_{n_{m-1}+1}^2 + \cdots + z_{n_1}^2)$$

after explicit calculation, when μ is a solution of (3) such that $\mu(0) = 0$, $\mu(r) = z$. $J_c(r)$ has the same expression modified by the introduction of the factor c under each radical and the use of a solution ν of the suitably modified (3).

Observe now that if one momentarily sets all $K_i = K_{\max}$ in $J(r)$ and replaces $\tilde{I}(r)$ by the former in (8) then (ii) follows immediately.

Finally, I come to the main task of this section, the proof of Theorem 2. To avoid repetition I refer back to the pertinent lines of section 1 for notation. Since, as remarked there, $a_{\alpha\beta}(s) = \eta^\alpha(s)\cdot\eta^\beta(s) \equiv \eta^\alpha_\gamma \eta^\beta_\gamma$ where the η^α are $n-1$ vector solutions of (1) satisfying $\eta^\alpha(0)\cdot\eta^\beta(0) = \delta_{\alpha\beta}$, $\eta^{\alpha\prime}(0)\cdot\eta^{\beta\prime}(0) = \delta_{\alpha\beta}$, in particular, for the purposes of Theorem 2 it will be sufficient to show

$$\eta(s)\cdot\eta(s) < (s^2)^{\sin^2\theta_0}(\sin^2 s\sqrt{cK_\sigma}/cK_\sigma)^{1-\sin^2\theta_0}\ , \quad 0 \le s \le \lambda\,\pi/\sqrt{K_\sigma}\ , \tag{12}$$

where η is a solution of (1) such that $\eta(0) = 0$, $\eta'(0)\cdot\eta'(0) = 1$, and the last $(n-1) - n_1$ components of η are zero at $s = 0$ (this last signifying that the initial direction of the tangent of η lies in $\bar{\Sigma}_p$). Indeed, setting $\eta = \eta^\alpha dy_\alpha/\sqrt{dy_\alpha dy_\alpha}$ one finds

$$\begin{aligned}\eta\cdot\eta &= \eta^\alpha\cdot\eta^\beta dy_\alpha dy_\beta/dy_\alpha dy_\alpha\\ &= a_{\alpha\beta}dy_\alpha dy_\beta/dy_\alpha dy_\alpha < (s^2)^{\sin^2\theta_0}(\sin^2 s\sqrt{cK_\sigma}/cK_\sigma)^{1-\sin^2\theta_0}\end{aligned}$$

which, on multiplying by the denominator, *is the conclusion of Theorem* 2 (α and β run only from 1 to n).

If the principal directions of the quadratic form $R_{n\alpha n\beta}(s)$ were propagated parallelly along $\bar{\sigma}$, one could separate the variables in (1) by an orthogonal transformation with *constant* coefficients and then apply the ordinary Sturm comparison theorem to each variable separately, obtaining thereby the inequality (12) without the exponent. However, that is not the case in general, and even Morse's n-dimensional version of the Sturm comparison theorem is of no use in this situation. I have, therefore, been forced to devise a new set of differential equations leading to a new type of comparison theorem – one which compares "normal displacements" from $\bar{\sigma}$ rather than distances along $\bar{\sigma}$ from $\bar{P}$ to first conjugate points.

The equations in question will be derived easily from (10) and from *Hamilton's principle of varying action*, not to be confused with the more familiar principle of *least action* – more popularly associated with Hamilton's name. I make reference here, once and for all, to Webster [1], pp. 131–135, particularly formula (96).

The principle being merely a formal apparatus (in reality a mere integration by parts) which is interpreted physically, it is only necessary to change notation in order to use it here. Thus the time t becomes s; the generalized position coordinates q_i become η_α; the kinetic energy T becomes (up to a constant factor) $\eta'_\alpha \eta'_\alpha$; the potential energy W becomes $R_{n\alpha n\beta}\eta_\alpha\eta_\beta$; the Lagrangian $T - W = L$ becomes $\Omega(\eta, \eta')$; and the action integral S becomes $I(r)$.

At this point, however, it is more useful to make correspond to the q_i not the η_α but the polar co-ordinates in $\overline{N}(r)$ such that $z_1 = \varrho \cos \theta$, $z_2 = \varrho \sin \theta, \ldots$, etc. (the others being irrelevant). In particular, when $\eta_\alpha = z_\alpha$, ϱ^2 *is precisely the quantity to be estimated in* (12), namely, the squared length of the normal displacement vector η. θ is the colatitude of $\eta(r)$ measured from the pole of the sphere $\overline{\Sigma}^{n-2}$ of polar co-ordinates in $\overline{N}(r)$ with center at $\overline{Q}(r)$. I pick that pole so that in the limit as $r \to 0$, $\theta \to 0$ for the particular $\eta(r)$ in question. Thus θ will measure how much η rotates out of its original position (the whole spindle of geodesics issuing from $\overline{\sigma}$ and neighboring $\overline{\sigma}$ will "twist" in general because the principal directions of $R_{n\alpha n\beta}$ do so).

Henceforth, those particular values of the variables which describe η will be written as usual while the unrestricted co-ordinates will be barred.

With the new q_i, $T = \eta'_\alpha\eta'_\alpha$ becomes $\varrho'^2 + \varrho^2\theta'^2 + \cdots$. The canonical momenta defined by $p_i = \partial T/\partial \dot{q}_i$ become $2\varrho'$, $2\varrho^2\theta'$, etc.

Hamilton's principle states that $p_i = \partial S/\partial q_i$. Translating into the new terminology and recalling that

$$S = I(r) = f_{\alpha\beta}(r)\, z_\alpha z_\beta \equiv \overline{\varrho}^2(f_1 \cos^2 \overline{\theta} + f_2 \sin \overline{\theta} \cos \overline{\theta} + f_3 \sin^2 \overline{\theta})$$

one obtains

$$\varrho' = \varrho(f_1 \cos^2 \theta + f_2 \sin \theta \cos \theta + f_3 \sin^2 \theta) \tag{13}$$

$$\begin{aligned} 2\theta' &= -f_1(2 \sin \theta \cos \theta) + f_2(\cos^2 \theta - \sin^2 \theta) + f_3(2 \sin \theta \cos \theta) \\ &= (f_3 - f_1) \sin 2\theta + f_2 \cos 2\theta \end{aligned} \tag{14}$$

where

$$f_1 = f_{11},\ f_2 = \sum_{\alpha=2}^{n-1} f_{1\alpha} \frac{\eta_1\eta_\alpha}{\varrho^2}, \text{ and } f_3 = \sum_{\alpha,\beta=2}^{n-1} f_{\alpha\beta} \frac{\eta_\alpha\eta_\beta}{\varrho^2}.$$

But from (7) and (8) with proper change of notation and from (11) and its mate one obtains

$$\bar{\varrho}^2 \left[\sqrt{K_1} \cot r \sqrt{K_1} \left(\cos^2 \bar{\theta} + \frac{z_2^2 + \ldots + z_{n_1}^2}{\bar{\varrho}^2} \right) + \ldots \right.$$
$$\left. + \sqrt{K_m} \cot r \sqrt{K_m} \left(\frac{z_{n_{m-1}+1}^2 + \ldots + z_{n-1}^2}{\bar{\varrho}^2} \right) \right]$$
$$< f_{\alpha\beta} z_\alpha z_\beta = \bar{\varrho}^2 (f_1 \cos^2 \bar{\theta} + f_2 \sin \bar{\theta} \cos \bar{\theta} + f_3 \sin^2 \bar{\theta}) \tag{15}$$
$$< \bar{\varrho}^2 \left[\sqrt{c K_1} \cot r \sqrt{c K_1} \left(\cos^2 \bar{\theta} + \frac{z_2^2 + \ldots + z_{n_1}^2}{\bar{\varrho}^2} \right) + \ldots \right.$$
$$\left. + \sqrt{c K_m} \cot r \sqrt{c K_m} \left(\frac{z_{n_{m-1}+1}^2 + \ldots + z_{n-1}^2}{\bar{\varrho}^2} \right) \right]$$

On applying the right-hand inequality of (15) to (13) and observing, first, that

$$\sqrt{c K_i} \cot r \sqrt{c K_i} \leq \frac{1}{r}$$

and, second, that $z_2^2 + \cdots + z_{n-l}^2 = \bar{\varrho}^2 \sin^2 \bar{\theta}$, one obtains

$$\frac{\varrho'}{\varrho} < (\sqrt{c K_1} \cot r \sqrt{c K_1}) \cos^2 \theta + \frac{1}{r} \sin^2 \theta$$
$$= \sqrt{c K_\sigma} \cot r \sqrt{c K_\sigma} + \sin^2 \theta \left(\frac{1}{r} - \sqrt{c K_\sigma} \cot r \sqrt{c K_\sigma} \right) \tag{16}$$
$$\leq \sqrt{c K_\sigma} \cot r \sqrt{c K_\sigma} + \sin^2 \theta_0 \left(\frac{1}{r} - \sqrt{c K_\sigma} \cot r \sqrt{c K_\sigma} \right)$$

where it must be recalled that $K_1 = K_\sigma$ and the last inequality holds because the last bracket is non-negative, θ_0 being

$$\sup \theta(r) \ , \quad 0 \leq r \leq \lambda \pi / \sqrt{K_\sigma} \ .$$

Integrating both sides of (16) from r_1 to r_2, $0 < r_1 < r_2 < \lambda\pi/\sqrt{K_\sigma}$, one has after exponentiating

$$\frac{\varrho(r_2)}{\varrho(r_1)} < \frac{\sin r_2 \sqrt{c K_\sigma}}{\sin r_1 \sqrt{c K_\sigma}} \left(\frac{r_2 \sin r_1 \sqrt{c K_\sigma}}{r_1 \sin r_2 \sqrt{c K_\sigma}} \right)^{\sin^2 \theta_0} . \tag{17}$$

Now since $\varrho^2 = \eta \cdot \eta$ one sees that $\varrho'(0) = 1 (\eta'(0) \cdot \eta'(0) = 1$ by assumption) so that

$$\lim_{r_1 \to 0} \frac{\varrho(r_1)}{\sin r_1 \sqrt{c K_\sigma}} = \frac{1}{\sqrt{c K_\sigma}} ; \ \lim_{r_1 \to 0} \frac{\sin r_1 \sqrt{c K_\sigma}}{r_1} = \sqrt{c K_\sigma} ;$$

therefore, letting $r_1 \to 0$ in (17) one obtains, after squaring, (12) where it remains to be shown that $\theta_0 \to 0$ as $c \to 1$.

To demonstrate this last I turn to (14), and to estimate the coefficients, $f_3 - f_1$, f_2, thereof I refer to (15). Setting $\bar{\theta}$ successively equal to 0 and $\frac{\pi}{2}$ one obtains

$$\sqrt{K_\sigma} \cot r \sqrt{K_\sigma} < f_1 < \sqrt{cK_\sigma} \cot r \sqrt{cK_\sigma}$$
$$\Theta < f_3 < \Theta_c \,, \tag{18}$$

where Θ and Θ_c are the left- and right-hand sides respectively of (15) without the $\cos^2 \bar{\theta}$ term.

For $f_3 - f_1$ one deduces

$$f_3 - f_1 < \Theta - \sqrt{K_\sigma} \cot r \sqrt{K_\sigma} \,. \tag{19}$$

Setting $\bar{\theta} = \frac{\pi}{4}$ on the other hand one obtains

$$\begin{aligned} f_2 &< \sqrt{cK_\sigma} \cot r \sqrt{cK_\sigma} + \Theta_c - f_1 - f_3 \\ &< \sqrt{cK_\sigma} \cot r \sqrt{cK_\sigma} - \sqrt{K_\sigma} \cot r \sqrt{K_\sigma} + \Theta_c - \Theta \end{aligned} \tag{20}$$

by (18). Calling the right-hand side of

$$(19)\ \psi_c \text{ and } \Theta - \sqrt{cK_\sigma} \cot r \sqrt{cK_\sigma}, \psi \,,$$

applying (19) and (20) to (14) and factoring, one has

$$2\theta' < [\sin 2\theta + (1 - \psi/\psi_c) \cos 2\theta]\, \psi_c \,. \tag{21}$$

Calling

$$b \equiv \sup_{0 \le r \le \lambda \pi / \sqrt{K_\sigma}} (1 - \psi/\psi_c), \qquad 1 > b > 0$$

and adding the term $b\psi_c$ to the right-hand side of (21) one obtains

$$2\theta' < [\sin 2\theta + b(1 + \cos 2\theta)]\, \psi_c \,.$$

Dividing by the bracket and integrating (Peirce, Short Table of Integrals) from r_1 to r_2 one has

$$\log \{b + \tan \theta(r_2)\} - \log \{b + \tan \theta(r_1)\} < \int_{r_1}^{r_2} \psi_c dr \,.$$

Estimating ψ_c as in (16) one has

$$\frac{b + \tan \theta(r_2)}{b + \tan \theta(r_1)} < \frac{r_2}{r_1} \frac{\sin r_1 \sqrt{cK_\sigma}}{\sin r_2 \sqrt{cK_\sigma}} \,.$$

Manipulating and letting $r_1 \to 0(\theta(r_1) \to 0)$ one has finally

$$\tan \theta (r_2) < b \left(\frac{r_2 \sqrt{c K_\sigma}}{\sin r_2 \sqrt{c K_\sigma}} - 1 \right) \leq b \left(\frac{\pi \lambda \sqrt{c}}{\sin \lambda \sqrt{c}\, \pi} - 1 \right) . \tag{22}$$

Since λ was fixed in (5) of section 2 and $b \to 0$ (that is easily seen) one sees that $\theta_0 \to 0$ as $c \to 1$ which establishes Theorem 2 and hence Theorem 1.

Added in Proof. Hypotheses (a) and (b) of Theorem 1 and the further reasoning attendant thereon can be materially simplified by means of a suggestion kindly made by Professor Hopf. To avoid the awkward business of choosing bases in the tangent spaces at two different points, T_P and T_Q, so that the linear transformation, h, between them generated by parallel displacement becomes a member of $\boldsymbol{H}$ one simply observes that h is such that $h^{-1} \boldsymbol{H} h \subseteq \boldsymbol{H}$. (a) and (b) can then be replaced by:

For each $P \in M^n$ there exists a linear transformation h_P of T_P onto $\boldsymbol{T}$ such that

$$h_P H h_P^{-1} \subseteq \boldsymbol{H} \tag{A}$$

etc.

The second paragraph after Theorem 1 then becomes unnecessary; however, it is again important to observe that h_P may be replaced by any other h'_P having the property (A).

The proof of formula (9) in section 2 now runs as follows: If I can show that the linear transformation, $t_{\bar{Q}}$ of $T_{\bar{Q}}$ onto $\boldsymbol{T}_Q$, which assigns to the vector $u = (u_1, \ldots, u_{n-1}, 0)$ in $T_{\bar{Q}}$ the vector with the same numerical components (consistent bases have already been chosen) in $\boldsymbol{T}_Q$, has property (A); then by the above observation the curvature hypothesis of Theorem 1 applies and (9) is an immediate consequence. But

$$t_{\bar{Q}} = k\, h_{\bar{P}}\, g^{-1}$$

where g is the transformation of $T_{\bar{P}}$ onto $T_{\bar{Q}}$ obtained by parallel displacement along $\bar{\sigma}$ and which also takes u into a vector with the same components because of the euclidean coordinate system along $\bar{\sigma}$; k is the analogous transformation along σ in $\boldsymbol{E}^n$; and $h_{\bar{P}}$ is the transformation of $T_{\bar{P}}$ onto $\boldsymbol{T}_P$ which already exists by the hypothesis of Theorem 1 and which also may be assumed to have the constant component property by identifying $T_{\bar{P}}$ with $\boldsymbol{T}_P$. Now

$$t_{\bar{Q}} H\, t_{\bar{Q}}^{-1} = k\, h_{\bar{P}}\, g^{-1} H\, g\, h_{\bar{P}}^{-1}\, k^{-1} \subseteq k\, h_{\bar{P}} H\, h_{\bar{P}}^{-1}\, k^{-1} \subseteq k \boldsymbol{H} k^{-1} \subseteq \boldsymbol{H},$$

and that is property (A) for $t_{\bar{Q}}$.

REFERENCES

[1] *E. Cartan:* Sur une classe remarquable d'espaces de Riemann, Bull. Soc. Math. France, t. 54 (1926), pp. 214–264; t. 55 (1927), pp. 114–134. Oeuvres Complètes, Pte. I, vol. 2.

[2] —, Sur certaines formes Riemanniennes remarquables, etc., Ann. Ecole Norm. Sup., t. 44 (Serie 3) (1927), pp. 345–467. Oeuvres Complètes, Pte. I, vol. 2.

[3] —, Les espaces Riemanniens symmétriques, Verh. Int. Math. Kong. Zürich 1932. Oeuvres Complètes, Pte. I, vol. 2.

[1] *H. Hopf:* Über die Topologie der Gruppen-Mannigfaltigkeiten und ihre Verallgemeinerungen, Ann. Math. 42 (1941), pp. 22–52.

[1] *H. E. Rauch:* On differential geometry in the large, Proc. Nat. Acad. Sci. vol. 39 (1953), pp. 440–443.

[2] —, A contribution to differential geometry in the large, Ann. Math. 54 (1951), pp. 38–55.

[1] *E. Study:* Kürzeste Wege im komplexen Gebiet, Math. Ann. 60 (1905), pp. 321 bis 378.

[1] *A. G. Webster:* Dynamics, Leipzig 1925, New York 1949.

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On the Pontryagin product in spaces of paths

By R. Bott and H. Samelson [1])

Introduction

For a topological (arcwise connected) space X, let E be the function space consisting of the paths in X which start at a certain point x, and let Ω be the subspace of E consisting of the closed paths or loops; these spaces have been studied in particular by M. Morse [7] and J.-P. Serre [10]; E is a fiber space over X. Now Ω admits a natural multiplication: two loops in succession make a new loop (actually there is a more general operation between E and Ω). This multiplication gives rise to a multi-

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