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Note on a basis of P. Hall for the higher commutators in free groups

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1. *Introduction.* Let F_n be the free group with n generators $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $H_1 \supset H_2 \supset \cdots$ the lower central series. The dimension d_w of the Abelian factor group H_w/H_{w+1} $(w = 1, 2, \ldots)$ is given by the *Witt* formula [1])

$$d_w = \frac{1}{w} \sum_{t \mid w} \mu(t)\, n^{\frac{w}{t}} \tag{1}$$

which, by the Möbius inversion formula, follows from the Witt equation

$$n^w = \sum_{t \mid w} t\, d_t \ . \tag{2}$$

Witt's proof works in the Magnus Lie Ring of F_n and is therefore not a pure grouptheoretical one. Moreover, since the commutator relations are more complicated in the multiplicative group F_n than in the corresponding Lie Ring, it would be difficult to translate Witt's proof into the language of the group itself.

On the other hand, a basis theorem due to *P. Hall* tells us, that his *basic commutators*, viz. the commutators arising in his wellknown commutator collecting process [2]) actually form a basis for the higher commutators in free groups. But again the proof depends on formula (2) [3]).

It is the object of the present note to give a simple grouptheoretical proof [4]) for the fundamental theorem of P. Hall and for the Witt equation (2).

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[1]) *E. Witt,* Treue Darstellung Liescher Ringe. J. reine angew. Math. 177, 152 bis 160 (1937).

[2]) *P. Hall,* A contribution to the theory of groups of prime-power order. Proc. London Math. Soc., II. s. 36, 29—95 (1933).

[3]) For a proof working in the Lie Ring see a forthcoming paper of Marshall Hall: A basis for free Lie Rings and higher commutators in free groups. Added in proof: M. Hall has published his paper in Proc. Amer. Math. Soc., vol. 1, N⁰ 5 (1950).

[4]) Das Fehlen eines gruppentheoretischen Beweises dieser, für die Behandlung des Problemes von Burnside so fundamentalen Tatsache hat Philip Hall mir gegenüber öfters hervorgehoben. Es sei mir erlaubt, ihm an dieser Stelle herzlich zu danken für die vielen Anregungen, die ich während eines Studienaufenthaltes an der Universität Cambridge von ihm empfangen durfte.

2. *Basic commutators.* $P_i \equiv \alpha_i$ $(i = 1, 2, \ldots, n)$ is the ordered set of basic commutators of weight $w(P_i) = 1$. Denote by $P_1, P_2, \ldots, P_r$ the ordered set of basic commutators of weight $1 \leqslant w(P_j) < c$. Define inductively a basic commutator P of weight c to be a commutator of the form $P = [P_k, P_j]$ with the properties

(i) $w(P_k) + w(P_j) = c$

(ii) $w(P_k) \geqslant w(P_j) > 0$

(iii) $w(P_k) = w(P_j)$ implies $P_j < P_k$
$w(P_k) > w(P_j)$ implies $P_h \leqslant P_j$ for $P_k = [P_i, P_h]$

and the relation of partial order "$<$"

(iv) $P_i < P$ $(i = 1, 2, \ldots, r)$

(v) The order of the $P's$ is the alphabetical order of the pairs $[P_k, P_j]$ starting from the right.

3. *The group B_c.* Define B_c to be the group generated by the basic commutators $P_1, P_2, \ldots, P_r$ of weight $\leqslant c$ and with the defining relations $[P_j, P_i] = P_k$ $(1 \leqslant i < j \leqslant r)$, where P_k stands for a basic commutator and is either a P_i or identity according to as $w(P_k) \leqslant c$ or $> c$.

4. **Lemma.** *B_c is nilpotent of class c with the P_i's as an uniqueness basis.*

Proof: The basic commutators of weight c generate an Abelian subgroup A of B_c. Indeed it follows from 2 (iii) that in this case $[P_j, P_i]$ is always basic and so by 3 equal to the identity element of B_c.

Let us assume for an induction hypothesis, that the subgroup $N_{i+1,c} = \{P_{i+1}, P_{i+2}, \ldots, P_r\} \supseteq A$ has the following properties:

(i) N_{i+1} has an uniqueness basis $P_{i+1}, P_{i+2}, \ldots, P_r$

(ii) $[P_k, P_j] = 1$ whenever $w(P_k) + w(P_j) > c$ $(i + 1 \leqslant j < k \leqslant r)$

(iii) N_{i+1} is nilpotent.

We have to show that the same holds for $N_{i,c} = \{P_i, P_{i+1}, \ldots, P_r\}$. To begin with let us consider the mapping

$$\varphi_i = P_j \to T_j^i = P_i^{-1} P_j P_i = P_j P_k \qquad (j = i + 1, i + 2, \ldots, r)$$

so far as defined in 3. By 2 (iii) it is clear, that φ_i is in the first place only defined for those $P_j' s$ which are not yet commutators in N_{i+1}. From the basis and nilpotency property of N_{i+1} we conclude then, that φ_i is defined on a minimal basis of N_{i+1} and hence defined in N_{i+1}. Further, by property (ii) for N_{i+1} and the structure of the $T_j^{i'}s$ it follows, that φ_i is homomorphic. Since each T_j^i belongs to a chain

$$T_j^i = P_j P_{j_1}, \qquad T_{j_1}^i = P_{j_1} P_{j_2}, \ldots, T_{j_{n-1}}^i = P_{j_{n-1}} P_{j_n}, T_{j_n}^i = P_{j_n}$$

it is clear, that the mapping φ_i is onto. Thus φ_i is an automorphism of N_{i+1}. By Schreier's Theory then N_i has to be a cyclic extension of N_{i+1} with $\{P_i\}$ and so property (i) for N_i follows from that of N_{i+1}.

To prove (ii) we have to show that $[P_j, P_i] = 1$ whenever $w(P_j) + w(P_i) > c$. If $[P_j, P_i]$ is basic, this is already so. If not, then it follows from 2 (iii), that P_j is defined in N_{i+1} by an equation of the form $P_j = [P_{j_1}, P_{i_1}]$. Transforming this equation by P_i we have $P_j [P_j, P_i] = [P_{j_1} \cdot Q, P_{i_1} \cdot R]$, where Q and R denote products of basic commutators whose weights are $\geqslant w(P_{j_1}) + w(P_i)$ and $\geqslant w(P_{i_1}) + w(P_i)$ resp. and so using (ii) for N_{i+1} the righthandside turns out to be just P_j. Thus (ii) is true for N_i.

Property (i) and (ii) for N_i then tell us, that A is normal in N_i and contained in the centre. So we have $N_{i,c}/A = N_{i,c-1}$. Applying the same argument to $N_{i,c-1}, N_{i,c-2}, \ldots$ it follows at once that (iii) holds for N_i.

The induction hypothesis has therefore been shown to work and the lemma is proved.

5. **Lemma.** *A commutator of weight w is mod H_{w+1} expressible as a product of basic commutators.*

Proof: If a, b, c are elements of $H_\alpha, H_\beta, H_\gamma$ resp. we have mod $H_{\alpha+\beta+\gamma+1}$ the congruences

$$\begin{aligned}
[a, bc] &\equiv [a, b]\,[a, c] \\
[bc, a] &\equiv [b, a]\,[c, a] \\
[a, b, c] &\equiv [a, c, b]\,[b, c, a]^{-1}\ .
\end{aligned}$$ [5])

Since the assertion of the lemma holds for $w = 1$ we may proceed by induction and assume, that each non basic commutator of weight $1 < c < w$ is mod H_{c+1} expressible as a product of basic ones.

Let $[y, z]$ be a non basic commutator of weight w. As a consequence

[5]) See Witt in [1]).

of the induction hypothesis and the first two rules above we may further assume y and z as being basic and y before z in the ordering. For $y = [u, v]$ we then have $u > v > z$. By the third rule above we may write

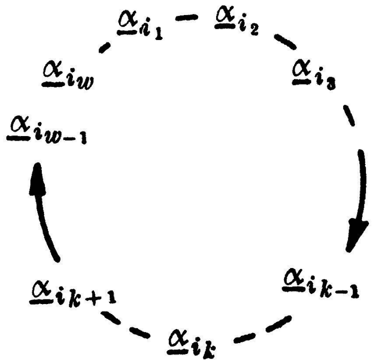
$$[y, z] = [u, v, z] \equiv [u, z, v]\,[v, z, u]^{-1} \bmod H_{w+1} \, .$$

Again by the induction hypothesis and the first two rules above we see, that each factor of the righthand side can be written as a product of commutators of the form $[r, v]$ and $[t, u]$ resp., where r, t, v and u are basic and later then z in the ordering. Since the number of basic commutators with weight $< w$ is finite, it is obvious that the discribed process continued long enough leads to the required factorisation q. e. d.

6. As a consequence of the Lemma's we have proved the main result

Theorem (Hall). *The ordered set of basic commutators of weight* $\leqslant w$ *forms an uniqueness basis for the free nilpotent group of class* w.

7. **The Witt equation.** To prove (2) let us consider formal cyclic words of lenght w in the $\alpha_1, \alpha_2, \ldots, \alpha_n$, viz.

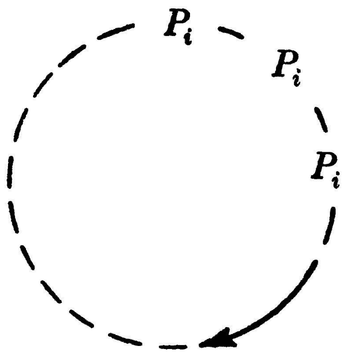


(3)

Define a sequence $\alpha_{i_h} \alpha_{i_{h+1}} \alpha_{i_{h+2}} \cdots \alpha_{i_{h+l}}$ of (3) to be a chain C of length $l + 1$ $(l \geqslant 0)$ if and only if in the sense of the ordering rule of the basic commutators we have $\alpha_{i_h} > \alpha_{i_{h+1}} \leqslant \alpha_{i_{h+2}} \leqslant \cdots \leqslant \alpha_{i_{h+l}} \leqslant \alpha_{i_{h+l+1}} > \alpha_{i_{h+l+2}}$.

If we bracket each chain C whose length is > 1 with square brackets so that it stands for a repeated commutator (without shifting the factors), then it follows at once from the definition 2, that this can be done in one and only one way so as to yield a basic commutator. So, if we replace (3) by this basic commutator cycle, then by means of the partial order of the basic commutators we may again have chains C'. Once more by 2 it is obvious, that each C' can be bracketed in one and only one way so as to yield a sequence of basic commutators in order.

So we may form in various stages chains $C, C', C'' \ldots$ until, after a finite number of steps, we shall arrive to a unique cycle consisting of one or of equal basic commutators, viz.



$$w\,(P_i) = t \mid w \; . \tag{4}$$

If we define two cycles (3) to be the same, if and only if they look the same, then it follows from the above construction and 2, that the number of all possible cycles (3) of type (4) is exactly $t d_t$.

The total number of all cyclic words (3), viz. n^w can therefore be expressed in the form

$$n^w = \sum_{t \mid w} t\, d_t$$

and this proves the Witt equation (2).

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