

**Zeitschrift:** Commentarii Mathematici Helvetici  
**Herausgeber:** Schweizerische Mathematische Gesellschaft  
**Band:** 26 (1952)

**Artikel:** Note on a basis of P. hall for the higher commutators in free groups.  
**Autor:** Meier-Wunderli, H.  
**DOI:** <https://doi.org/10.5169/seals-21261>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 10.02.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

# Note on a basis of P. Hall for the higher commutators in free groups

By H. MEIER-WUNDERLI, Cambridge (England)

1. *Introduction.* Let  $F_n$  be the free group with  $n$  generators  $\alpha_1, \alpha_2, \dots, \alpha_n$  and  $H_1 \supset H_2 \supset \dots$  the lower central series. The dimension  $d_w$  of the Abelian factor group  $H_w/H_{w+1}$  ( $w = 1, 2, \dots$ ) is given by the Witt formula<sup>1)</sup>

$$d_w = \frac{1}{w} \sum_{t|w} \mu(t) n^{\frac{w}{t}} \quad (1)$$

which, by the Möbius inversion formula, follows from the Witt equation

$$n^w = \sum_{t|w} t d_t. \quad (2)$$

Witt's proof works in the Magnus Lie Ring of  $F_n$  and is therefore not a pure grouptheoretical one. Moreover, since the commutator relations are more complicated in the multiplicative group  $F_n$  than in the corresponding Lie Ring, it would be difficult to translate Witt's proof into the language of the group itself.

On the other hand, a basis theorem due to P. Hall tells us, that his *basic commutators*, viz. the commutators arising in his wellknown commutator collecting process<sup>2)</sup> actually form a basis for the higher commutators in free groups. But again the proof depends on formula (2)<sup>3)</sup>.

It is the object of the present note to give a simple grouptheoretical proof<sup>4)</sup> for the fundamental theorem of P. Hall and for the Witt equation (2).

<sup>1)</sup> E. Witt, Treue Darstellung Liescher Ringe. J. reine angew. Math. 177, 152 bis 160 (1937).

<sup>2)</sup> P. Hall, A contribution to the theory of groups of prime-power order. Proc. London Math. Soc., II. s. 36, 29—95 (1933).

<sup>3)</sup> For a proof working in the Lie Ring see a forthcoming paper of Marshall Hall: A basis for free Lie Rings and higher commutators in free groups. Added in proof: M. Hall has published his paper in Proc. Amer. Math. Soc., vol. 1, No 5 (1950).

<sup>4)</sup> Das Fehlen eines gruppentheoretischen Beweises dieser, für die Behandlung des Problems von Burnside so fundamentalen Tatsache hat Philip Hall mir gegenüber öfters hervorgehoben. Es sei mir erlaubt, ihm an dieser Stelle herzlich zu danken für die vielen Anregungen, die ich während eines Studienaufenthaltes an der Universität Cambridge von ihm empfangen durfte.

2. *Basic commutators.*  $P_i \equiv \alpha_i$  ( $i = 1, 2, \dots, n$ ) is the ordered set of basic commutators of weight  $w(P_i) = 1$ . Denote by  $P_1, P_2, \dots, P_r$  the ordered set of basic commutators of weight  $1 \leq w(P_j) < c$ . Define inductively a basic commutator  $P$  of weight  $c$  to be a commutator of the form  $P = [P_k, P_j]$  with the properties

- (i)  $w(P_k) + w(P_j) = c$
- (ii)  $w(P_k) \geq w(P_j) > 0$
- (iii)  $w(P_k) = w(P_j)$  implies  $P_j < P_k$   
 $w(P_k) > w(P_j)$  implies  $P_h \leq P_j$  for  $P_k = [P_i, P_h]$

and the relation of partial order “ $<$ ”

- (iv)  $P_i < P$  ( $i = 1, 2, \dots, r$ )
- (v) The order of the  $P'$ s is the alphabetical order of the pairs  $[P_k, P_j]$  starting from the right.

3. *The group  $B_c$ .* Define  $B_c$  to be the group generated by the basic commutators  $P_1, P_2, \dots, P_r$  of weight  $\leq c$  and with the defining relations  $[P_j, P_i] = P_k$  ( $1 \leq i < j \leq r$ ), where  $P_k$  stands for a basic commutator and is either a  $P_i$  or identity according to as  $w(P_k) \leq c$  or  $> c$ .

4. **Lemma.**  $B_c$  is nilpotent of class  $c$  with the  $P'_i$ s as an uniqueness basis.

*Proof:* The basic commutators of weight  $c$  generate an Abelian subgroup  $A$  of  $B_c$ . Indeed it follows from 2 (iii) that in this case  $[P_j, P_i]$  is always basic and so by 3 equal to the identity element of  $B_c$ .

Let us assume for an induction hypothesis, that the subgroup  $N_{i+1,c} = \{P_{i+1}, P_{i+2}, \dots, P_r\} \supseteq A$  has the following properties:

- (i)  $N_{i+1}$  has an uniqueness basis  $P_{i+1}, P_{i+2}, \dots, P_r$
- (ii)  $[P_k, P_j] = 1$  whenever  $w(P_k) + w(P_j) > c$  ( $i + 1 \leq j < k \leq r$ )
- (iii)  $N_{i+1}$  is nilpotent.

We have to show that the same holds for  $N_{i,c} = \{P_i, P_{i+1}, \dots, P_r\}$ . To begin with let us consider the mapping

$$\varphi_i = P_j \rightarrow T_j^i = P_i^{-1} P_j P_i = P_j P_k \quad (j = i + 1, i + 2, \dots, r)$$

so far as defined in 3. By 2 (iii) it is clear, that  $\varphi_i$  is in the first place

only defined for those  $P_j'$ s which are not yet commutators in  $N_{i+1}$ . From the basis and nilpotency property of  $N_{i+1}$  we conclude then, that  $\varphi_i$  is defined on a minimal basis of  $N_{i+1}$  and hence defined in  $N_{i+1}$ . Further, by property (ii) for  $N_{i+1}$  and the structure of the  $T_j^i$ 's it follows, that  $\varphi_i$  is homomorphic. Since each  $T_j^i$  belongs to a chain

$$T_j^i = P_j P_{j_1}, \quad T_{j_1}^i = P_{j_1} P_{j_2}, \dots, T_{j_{n-1}}^i = P_{j_{n-1}} P_{j_n}, T_{j_n}^i = P_{j_n}$$

it is clear, that the mapping  $\varphi_i$  is onto. Thus  $\varphi_i$  is an automorphism of  $N_{i+1}$ . By Schreier's Theory then  $N_i$  has to be a cyclic extension of  $N_{i+1}$  with  $\{P_i\}$  and so property (i) for  $N_i$  follows from that of  $N_{i+1}$ .

To prove (ii) we have to show that  $[P_j, P_i] = 1$  whenever  $w(P_j) + w(P_i) > c$ . If  $[P_j, P_i]$  is basic, this is already so. If not, then it follows from 2 (iii), that  $P_j$  is defined in  $N_{i+1}$  by an equation of the form  $P_j = [P_{j_1}, P_{i_1}]$ . Transforming this equation by  $P_i$  we have  $P_j[P_j, P_i] = [P_{j_1} \cdot Q, P_{i_1} \cdot R]$ , where  $Q$  and  $R$  denote products of basic commutators whose weights are  $\geq w(P_{j_1}) + w(P_i)$  and  $\geq w(P_{i_1}) + w(P_i)$  resp. and so using (ii) for  $N_{i+1}$  the righthandside turns out to be just  $P_j$ . Thus (ii) is true for  $N_i$ .

Property (i) and (ii) for  $N_i$  then tell us, that  $A$  is normal in  $N_i$  and contained in the centre. So we have  $N_{i,c}/A = N_{i,c-1}$ . Applying the same argument to  $N_{i,c-1}, N_{i,c-2}, \dots$  it follows at once that (iii) holds for  $N_i$ .

The induction hypothesis has therefore been shown to work and the lemma is proved.

**5. Lemma.** *A commutator of weight  $w$  is mod  $H_{w+1}$  expressible as a product of basic commutators.*

*Proof:* If  $a, b, c$  are elements of  $H_\alpha, H_\beta, H_\gamma$  resp. we have mod  $H_{\alpha+\beta+\gamma+1}$  the congruences

$$\begin{aligned} [a, bc] &\equiv [a, b] [a, c] \\ [bc, a] &\equiv [b, a] [c, a] \\ [a, b, c] &\equiv [a, c, b] [b, c, a]^{-1} \quad .^5) \end{aligned}$$

Since the assertion of the lemma holds for  $w = 1$  we may proceed by induction and assume, that each non basic commutator of weight  $1 < c < w$  is mod  $H_{c+1}$  expressible as a product of basic ones.

Let  $[y, z]$  be a non basic commutator of weight  $w$ . As a consequence

---

<sup>5)</sup> See Witt in <sup>1)</sup>.

of the induction hypothesis and the first two rules above we may further assume  $y$  and  $z$  as being basic and  $y$  before  $z$  in the ordering. For  $y = [u, v]$  we then have  $u > v > z$ . By the third rule above we may write

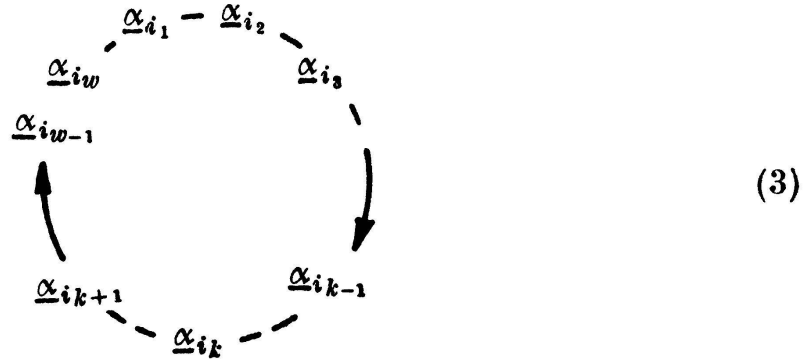
$$[y, z] = [u, v, z] \equiv [u, z, v] [v, z, u]^{-1} \text{ mod } H_{w+1}.$$

Again by the induction hypothesis and the first two rules above we see, that each factor of the righthand side can be written as a product of commutators of the form  $[r, v]$  and  $[t, u]$  resp., where  $r, t, v$  and  $u$  are basic and later then  $z$  in the ordering. Since the number of basic commutators with weight  $< w$  is finite, it is obvious that the discribed process continued long enough leads to the required factorisation q. e. d.

6. As a consequence of the Lemma's we have proved the main result

**Theorem (Hall).** *The ordered set of basic commutators of weight  $\leq w$  forms an uniqueness basis for the free nilpotent group of class  $w$ .*

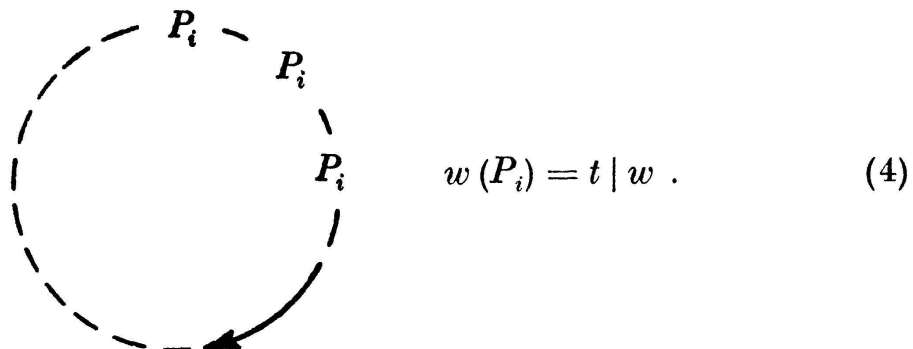
7. **The Witt equation.** To prove (2) let us consider formal cyclic words of length  $w$  in the  $\alpha_1, \alpha_2, \dots, \alpha_n$ , viz.



Define a sequence  $\alpha_{i_h} \alpha_{i_{h+1}} \alpha_{i_{h+2}} \dots \alpha_{i_{h+l}}$  of (3) to be a chain  $C$  of length  $l + 1$  ( $l \geq 0$ ) if and only if in the sense of the ordering rule of the basic commutators we have  $\alpha_{i_h} > \alpha_{i_{h+1}} \leq \alpha_{i_{h+2}} \leq \dots \leq \alpha_{i_{h+l}} \leq \alpha_{i_{h+l+1}} > \alpha_{i_{h+l+2}}$ .

If we bracket each chain  $C$  whose length is  $> 1$  with square brackets so that it stands for a repeated commutator (without shifting the factors), then it follows at once from the definition 2, that this can be done in one and only one way so as to yield a basic commutator. So, if we replace (3) by this basic commutator cycle, then by means of the partial order of the basic commutators we may again have chains  $C'$ . Once more by 2 it is obvious, that each  $C'$  can be bracketed in one and only one way so as to yield a sequence of basic commutators in order.

So we may form in various stages chains  $C, C', C'' \dots$  until, after a finite number of steps, we shall arrive to a unique cycle consisting of one or of equal basic commutators, viz.



$$w(P_i) = t | w \quad (4)$$

If we define two cycles (3) to be the same, if and only if they look the same, then it follows from the above construction and 2, that the number of all possible cycles (3) of type (4) is exactly  $td_t$ .

The total number of all cyclic words (3), viz.  $n^w$  can therefore be expressed in the form

$$n^w = \sum_{t|w} t d_t$$

and this proves the Witt equation (2).

(Received December 23, 1950.)