Zeitschrift: Commentarii Mathematici Helvetici

Herausgeber: Schweizerische Mathematische Gesellschaft

Band: 25 (1951)

Artikel: A Theorem on Orientable Surfaces in Four-Dimensional Space.

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DOI: https://doi.org/10.5169/seals-20704

A Theorem on Orientable Surfaces in Four-Dimensional Space

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1. *Introduction.* Let M be a closed oriented surface differentiably imbedded in a Euclidean space E of four dimensions. Let G denote the Grassmann manifold of oriented planes through a fixed point O of E. It is well known that G is homeomorphic to the topological product $S_1 \times S_2$ of two 2-spheres. By mapping each point P of M into the oriented plane through O parallel to the oriented tangent plane to M at P, we define a mapping t: $M \to G$. If M, S_1, S_2 denote also the fundamental cycles of the respective manifolds and t_* denotes the homomorphism induced by t, we have

$$t_*(M) \sim u_1 S_1 + u_2 S_2 \ .$$

In a recent paper[1]) Blaschke studied the situation described above by methods of differential geometry and proved that the sum $u_1 + u_2$ equals the Euler characteristic of M. He also asserted that $u_1 = u_2$. The object of this note is to give a proof of this assertion, as well as a new proof of the theorem on $u_1 + u_2$.

2. *Review of some known results on sphere bundles.* Let B be an oriented sphere bundle of d-spheres over a base space X with projection f. The relation between the homology properties of B and X are summarized in the following exact sequence[2]);

$$\cdots \to H^p(X) \xrightarrow{f^*} H^p(B) \xrightarrow{\psi} H^{p-d}(X) \xrightarrow{\smile \Omega} H^{p+1}(X) \to \cdots$$

where each H denotes a cohomology group relative to a coefficient group which is the same for all the terms of the sequence. The homomorphisms that occur in the sequence can be described briefly as follows:

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[1]) *Blaschke, W.*, Ann. Mat. Pura Appl. (4) 28, 205—209 (1949).

[2]) *Gysin, W.*, Comm. Math. Helv. 14, 61—122 (1942). — *Chern, S. S. and Spanier, E. H.*, Proc. Nat. Acad. Sci., U. S. A. 36, 248—255 (1950).

f^* is the dual homomorphism induced by the projection f; ψ is a mapping which amounts to „integrating over the fiber"; the third homomorphism is the cup product with the characteristic class Ω (with integer coefficients) of the bundle. From this sequence we see that if, for every coefficient system, the fiber $S^d \not\sim 0$ in B then the unit element 1 of the integral cohomology ring of X is in the image of ψ and $\Omega = 0$.

Let E be oriented. Over the oriented surface $M \subset E$ there are two vector bundles, the *tangent bundle* of tangent vectors and the *normal bundle* of normal vectors. By taking unit vectors we get two bundles of circles over M. According to a theorem of Seifert and Whitney[3]) the characteristic class of the normal bundle is zero. Since this theorem holds in a more general situation and can be proved in a simple way, we state and prove the theorem for the general case[4]).

Theorem. Let M be an orientable manifold imbedded in a Riemann manifold M'. If $M \sim O$ in M', then the characteristic class of the normal bundle of M in M' is zero.

Proof. Let B be a small tube around M. B is then the normal bundle of M. We will show that no fiber S of B bounds in B. Assume that $S = \partial C$ in B mod p for some p. Let D be the set of normal vectors of length $\leqq \epsilon$ having S as boundary. Then $C - D$ is a cycle mod p in M' intersecting M in exactly one point. This is impossible because $M \sim O$ in M'.

The above theorem also follows easily from results of Thom[5]).

3. *Plücker coordinates in G.* Let e_1, e_2, e_3, e_4 be an orthonormal base for E such that $e_1 \wedge e_2 \wedge e_3 \wedge e_4$[6]) is the orientation of E. If R is any oriented plane of E, let f_1, f_2 be an orthonormal base in R such that $f_1 \wedge f_2$ is the orientation of R. Then

$$f_1 \wedge f_2 = a_{12}\, e_1 \wedge e_2 + a_{23}\, e_2 \wedge e_3 + a_{31}\, e_3 \wedge e_1 + a_{34}\, e_3 \wedge e_4 \\ + a_{14}\, e_1 \wedge e_4 + a_{24}\, e_2 \wedge e_4\ .$$

These "Plücker coordinates" a_{ij} of R are independent of the choice of f_1, f_2 and satisfy the two relations

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[3]) *Seifert, H.*, Math. Zeitschr. 41 (1936) 1—17. — *Whitney, H.*, Lectures in Topology, Univ. of Mich. Press (1941) 101—141.

[4]) We owe this simple description of the proof to Professor H. Hopf, who also called our attention to the problem settled in this paper.

[5]) *Thom, R.*, C. R. Paris 230, 507—508 (1950).

[6]) The wedge denotes Grassmann multiplication as in Bourbaki, N., Algèbre Multilinéare, Hermann, Paris (1948).

$$a_{12}\, a_{34} + a_{23}\, a_{14} + a_{31}\, a_{24} = 0 \tag{1}$$

$$\Sigma\, a_{ij}^2 = 1 \ . \tag{2}$$

Conversely, any set of six real numbers satisfying (1) and (2) are the Plücker coordinates of some oriented plane in E; hence, G is homeomorphic to the subset of six space consisting of a_{ij} such that (1) and (2) hold. We introduce a linear change of coordinates by

$$x_1 = a_{12} + a_{34} \qquad x_2 = a_{23} + a_{14} \qquad x_3 = a_{31} + a_{24}$$

$$y_1 = a_{12} - a_{34} \qquad y_2 = a_{23} - a_{14} \qquad y_3 = a_{31} - a_{24} \ .$$

Then G is homeomorphic to the subset of six space consisting of (x_i, y_j) such that $\Sigma\, x_i^2 = \Sigma\, y_j^2 = 1$.

Let S_1, S_2 be the unit spheres in the x-space and y-space respectively. We orient S_1 and S_2 by the orientations (x_1, x_2, x_3) and (y_1, y_2, y_3) of the x-space and y-space. Let $h\colon\ G \to S_1 \times S_2$ be the homeomorphism defined above using the Plücker coordinates.

Let α; $G \to G$ map each oriented plane R into its normal plane R', oriented so that R, R' determine the given orientation of E. We want to determine the mapping $h\,\alpha\, h^{-1}\colon\ S_1 \times S_2 \to S_1 \times S_2$. If R has Plücker coordinates a_{ij} and R' has Plücker coordinates b_{ij}, it is easy to see that the following equations are satisfied

$$\sum_k a_{ik}\, b_{jk} = 0 \qquad (i \neq j)$$

$$\Sigma\, a_{ij}\, b_{kl} = 1 \ ,$$

the last summation being taken over all even permutations of $1, 2, 3, 4$.

It follows from these that $b_{ij} = a_{kl}$, where i, j, k, l is an even permutation of $1,\ 2,\ 3,\ 4$. Therefore, we see that

$$h\,\alpha\, h^{-1}(x, y) = (x, -y)$$

where $-y$ denotes the antipodal point to y.

4. *The Theorem.* Let M be a closed oriented surface in E. Let t: $M \to G$ and n: $M \to G$ be the maps defined by taking tangent planes and normal planes respectively. It is clear that $t = \alpha\, n$ and $n = \alpha\, t$.

Over G there is a bundle of circles obtained by considering as the fiber over an oriented plane through O the unit circle in that plane. Let Ω denote the characteristic class of this bundle and let Ω_t, Ω_n denote the characteristic classes of the tangent and normal bundles of M. Then

$$t^*\, \Omega = \Omega_t \ , \qquad n^*\, \Omega = \Omega_n \ .$$

The bundle of circles over G defined above is the Stiefel manifold V of ordered pairs of orthogonal unit vectors through O in E and is easily seen to be homeomorphic to $S^2 \times S^3$. The following section of Gysin's sequence

$$H^1(V) \xrightarrow{\psi} H^0(G) \xrightarrow{\cup \Omega} H^2(G) \xrightarrow{f^*} H^2(V) \xrightarrow{\psi} H^1(G)$$

shows that Ω is a generator of the kernel of f^* in $H^2(V)$, since $H^1(V)$ and $H^1(G)$ are trivial. To find the kernel of f^* we determine the homomorphism

$$f_* : \ H_2(V) \rightarrow H_2(G)$$

of the second homology groups.

A generating 2-cycle in V is $S^2 \times e_4$. The points z of S^2 can be represented as vectors of the form $z_1 e_1 + z_2 e_2 + z_3 e_3$. Then

$$f\,(\sum_{i=1}^{3} z_i\, e_i,\, e_4) = \Sigma\, z_i (e_i \wedge e_4)$$

and so

$$h\, f\, (\Sigma\, z_i\, e_i,\, e_4) = (z,\, -z)\ .$$

Therefore, we see that $f_* (S^2 \times e_4) = S_1 - S_2$. If S_1^*, S_2^* denote cohomology classes dual to the homology classes S_1, S_2, then the kernel of f^* consists of all elements of the form $u\,(S_1^* + S_2^*)$ where u is an integer. Orient S_1 and S_2 so that $\Omega = S_1^* + S_2^*$. Orient M so that $\Omega_t \cdot M = \chi_M$ $=$ Euler characteristic of M. Then

$$\Omega_t = t^* (S_1^* + S_2^*) = t^* S_1^* + t^* S_2^*$$

and

$$\Omega_n = n^* (S_1^* + S_2^*) = t^* \alpha^* (S_1^* + S_2^*) = t^* (S_1^* - S_2^*) = t^* S_1^* - t^* S_2^*\ .$$

Since $\Omega_n = 0$, we see that

$$(t^* S_1^*) \cdot M = (t^* S_2^*) \cdot M = (\tfrac{1}{2})\, \chi_M\ .$$

We summarize the above results in the theorem:

Let M be a closed orientable surface in four space E. Let G be the Grassmann manifold of oriented planes through O in E and let t: $M \rightarrow G$ be the map into oriented planes through O parallel to the tangent planes of M. Since G is homeomorphic to $S_1 \times S_2$, we have $t_*(M) = u_1 S_1 + u_2 S_2$. Then S_1, S_2 and M can be oriented so that $u_1 = u_2 = (\tfrac{1}{2})\, \chi_M$ where χ_M *is the Euler characteristic of M.*

5. *Remarks.* The above theorem expresses relations between differential topological invariants of surfaces imbedded in Euclidean space

and suggests a more general problem. To describe the general situation let $M^k \subset E^{k+l}$ be a manifold of dimension k differentiably imbedded in a Euclidean space of $k + l$ dimensions. Let $G(k, l)$ be the Grassmann manifold of k-dimensional linear spaces through a point O and $G(l, k)$ that of l-dimensional linear spaces through O. There is a natural homeomorphism

$$\alpha : \; G(k, l) \to G(l, k) \; .$$

Using tangent planes and normal planes to M we define mappings

$$t : \; M \to G(k, l) \; , \qquad n : \; M \to G(l, k)$$

such that

$$t = \alpha^{-1} n \; , \qquad n = \alpha t \; .$$

The general problem is to study the relation between the homomorphisms

$$\left.\begin{array}{l} t^* \; ; \; H^p(G(k, l)) \to H^p(M) \\ n^* \; ; \; H^p(G(l, k)) \to H^p(M) \end{array}\right\} \; p = 0, 1, \ldots$$

We hope to study this question on a later occasion.

(Received 31th Mars 1951.)