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A Theorem on Orientable Surfaces in Four-Dimensional Space

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1. *Introduction.* Let M be a closed oriented surface differentiably imbedded in a Euclidean space E of four dimensions. Let G denote the Grassmann manifold of oriented planes through a fixed point O of E . It is well known that G is homeomorphic to the topological product $S_1 \times S_2$ of two 2-spheres. By mapping each point P of M into the oriented plane through O parallel to the oriented tangent plane to M at P , we define a mapping $t: M \rightarrow G$. If M, S_1, S_2 denote also the fundamental cycles of the respective manifolds and t_* denotes the homomorphism induced by t , we have

$$t_*(M) \sim u_1 S_1 + u_2 S_2 .$$

In a recent paper¹⁾ Blaschke studied the situation described above by methods of differential geometry and proved that the sum $u_1 + u_2$ equals the Euler characteristic of M . He also asserted that $u_1 = u_2$. The object of this note is to give a proof of this assertion, as well as a new proof of the theorem on $u_1 + u_2$.

2. *Review of some known results on sphere bundles.* Let B be an oriented sphere bundle of d -spheres over a base space X with projection f . The relation between the homology properties of B and X are summarized in the following exact sequence²⁾;

$$\dots \rightarrow H^p(X) \xrightarrow{f^*} H^p(B) \xrightarrow{\psi} H^{p-d}(X) \xrightarrow{\cup \Omega} H^{p+1}(X) \rightarrow \dots$$

where each H denotes a cohomology group relative to a coefficient group which is the same for all the terms of the sequence. The homomorphisms that occur in the sequence can be described briefly as follows:

¹⁾ Blaschke, *W.*, Ann. Mat. Pura Appl. (4) 28, 205—209 (1949).

²⁾ Gysin, *W.*, Comm. Math. Helv. 14, 61—122 (1942). — Chern, *S. S. and Spanier, E. H.*, Proc. Nat. Acad. Sci., U. S. A. 36, 248—255 (1950).

f^* is the dual homomorphism induced by the projection f ; ψ is a mapping which amounts to „integrating over the fiber“; the third homomorphism is the cup product with the characteristic class Ω (with integer coefficients) of the bundle. From this sequence we see that if, for every coefficient system, the fiber $S^d \rightsquigarrow 0$ in B then the unit element 1 of the integral cohomology ring of X is in the image of ψ and $\Omega = 0$.

Let E be oriented. Over the oriented surface $M \subset E$ there are two vector bundles, the *tangent bundle* of tangent vectors and the *normal bundle* of normal vectors. By taking unit vectors we get two bundles of circles over M . According to a theorem of Seifert and Whitney³⁾ the characteristic class of the normal bundle is zero. Since this theorem holds in a more general situation and can be proved in a simple way, we state and prove the theorem for the general case⁴⁾.

Theorem. Let M be an orientable manifold imbedded in a Riemann manifold M' . If $M \sim O$ in M' , then the characteristic class of the normal bundle of M in M' is zero.

Proof. Let B be a small tube around M . B is then the normal bundle of M . We will show that no fiber S of B bounds in B . Assume that $S = \partial C$ in $B \bmod p$ for some p . Let D be the set of normal vectors of length $\leq \epsilon$ having S as boundary. Then $C - D$ is a cycle mod p in M' intersecting M in exactly one point. This is impossible because $M \sim O$ in M' .

The above theorem also follows easily from results of Thom⁵⁾.

3. *Plücker coordinates in G .* Let e_1, e_2, e_3, e_4 be an orthonormal base for E such that $e_1 \wedge e_2 \wedge e_3 \wedge e_4$ ⁶⁾ is the orientation of E . If R is any oriented plane of E , let f_1, f_2 be an orthonormal base in R such that $f_1 \wedge f_2$ is the orientation of R . Then

$$f_1 \wedge f_2 = a_{12} e_1 \wedge e_2 + a_{23} e_2 \wedge e_3 + a_{31} e_3 \wedge e_1 + a_{34} e_3 \wedge e_4 \\ + a_{14} e_1 \wedge e_4 + a_{24} e_2 \wedge e_4 .$$

These “Plücker coordinates” a_{ij} of R are independent of the choice of f_1, f_2 and satisfy the two relations

³⁾ Seifert, H., Math. Zeitschr. 41 (1936) 1—17. — Whitney, H., Lectures in Topology, Univ. of Mich. Press (1941) 101—141.

⁴⁾ We owe this simple description of the proof to Professor H. Hopf, who also called our attention to the problem settled in this paper.

⁵⁾ Thom, R., C. R. Paris 230, 507—508 (1950).

⁶⁾ The wedge denotes Grassmann multiplication as in Bourbaki, N., Algèbre Multilinéaire, Hermann, Paris (1948).

$$a_{12} a_{34} + a_{23} a_{14} + a_{31} a_{24} = 0 \quad (1)$$

$$\sum a_{ij}^2 = 1 . \quad (2)$$

Conversely, any set of six real numbers satisfying (1) and (2) are the Plücker coordinates of some oriented plane in E ; hence, G is homeomorphic to the subset of six space consisting of a_{ij} such that (1) and (2) hold. We introduce a linear change of coordinates by

$$\begin{aligned} x_1 &= a_{12} + a_{34} & x_2 &= a_{23} + a_{14} & x_3 &= a_{31} + a_{24} \\ y_1 &= a_{12} - a_{34} & y_2 &= a_{23} - a_{14} & y_3 &= a_{31} - a_{24} . \end{aligned}$$

Then G is homeomorphic to the subset of six space consisting of (x_i, y_j) such that $\sum x_i^2 = \sum y_j^2 = 1$.

Let S_1, S_2 be the unit spheres in the x -space and y -space respectively. We orient S_1 and S_2 by the orientations (x_1, x_2, x_3) and (y_1, y_2, y_3) of the x -space and y -space. Let $h: G \rightarrow S_1 \times S_2$ be the homeomorphism defined above using the Plücker coordinates.

Let $\alpha: G \rightarrow G$ map each oriented plane R into its normal plane R' , oriented so that R, R' determine the given orientation of E . We want to determine the mapping $h \alpha h^{-1}: S_1 \times S_2 \rightarrow S_1 \times S_2$. If R has Plücker coordinates a_{ij} and R' has Plücker coordinates b_{ij} , it is easy to see that the following equations are satisfied

$$\begin{aligned} \sum_k a_{ik} b_{jk} &= 0 & (i \neq j) \\ \sum a_{ij} b_{kl} &= 1 , \end{aligned}$$

the last summation being taken over all even permutations of 1, 2, 3, 4.

It follows from these that $b_{ij} = a_{kl}$, where i, j, k, l is an even permutation of 1, 2, 3, 4. Therefore, we see that

$$h \alpha h^{-1}(x, y) = (x, -y)$$

where $-y$ denotes the antipodal point to y .

4. *The Theorem.* Let M be a closed oriented surface in E . Let $t: M \rightarrow G$ and $n: M \rightarrow G$ be the maps defined by taking tangent planes and normal planes respectively. It is clear that $t = \alpha n$ and $n = \alpha t$.

Over G there is a bundle of circles obtained by considering as the fiber over an oriented plane through O the unit circle in that plane. Let Ω denote the characteristic class of this bundle and let Ω_t, Ω_n denote the characteristic classes of the tangent and normal bundles of M . Then

$$t^* \Omega = \Omega_t , \quad n^* \Omega = \Omega_n .$$

The bundle of circles over G defined above is the Stiefel manifold V of ordered pairs of orthogonal unit vectors through O in E and is easily seen to be homeomorphic to $S^2 \times S^3$. The following section of Gysin's sequence

$$H^1(V) \xrightarrow{\psi} H^0(G) \xrightarrow{\cup \Omega} H^2(G) \xrightarrow{f^*} H^2(V) \xrightarrow{\psi} H^1(G)$$

shows that Ω is a generator of the kernel of f^* in $H^2(V)$, since $H^1(V)$ and $H^1(G)$ are trivial. To find the kernel of f^* we determine the homomorphism

$$f_* : H_2(V) \rightarrow H_2(G)$$

of the second homology groups.

A generating 2-cycle in V is $S^2 \times e_4$. The points z of S^2 can be represented as vectors of the form $z_1 e_1 + z_2 e_2 + z_3 e_3$. Then

$$f \left(\sum_{i=1}^3 z_i e_i, e_4 \right) = \sum z_i (e_i \wedge e_4)$$

and so

$$h f (\sum z_i e_i, e_4) = (z, -z) .$$

Therefore, we see that $f_*(S^2 \times e_4) = S_1 - S_2$. If S_1^*, S_2^* denote cohomology classes dual to the homology classes S_1, S_2 , then the kernel of f^* consists of all elements of the form $u (S_1^* + S_2^*)$ where u is an integer. Orient S_1 and S_2 so that $\Omega = S_1^* + S_2^*$. Orient M so that $\Omega_t \cdot M = \chi_M =$ Euler characteristic of M . Then

$$\Omega_t = t^* (S_1^* + S_2^*) = t^* S_1^* + t^* S_2^*$$

and

$$\Omega_n = n^* (S_1^* + S_2^*) = t^* \alpha^* (S_1^* + S_2^*) = t^* (S_1^* - S_2^*) = t^* S_1^* - t^* S_2^* .$$

Since $\Omega_n = 0$, we see that

$$(t^* S_1^*) \cdot M = (t^* S_2^*) \cdot M = \left(\frac{1}{2}\right) \chi_M .$$

We summarize the above results in the theorem:

Let M be a closed orientable surface in four space E . Let G be the Grassmann manifold of oriented planes through O in E and let $t: M \rightarrow G$ be the map into oriented planes through O parallel to the tangent planes of M . Since G is homeomorphic to $S_1 \times S_2$, we have $t_(M) = u_1 S_1 + u_2 S_2$. Then S_1, S_2 and M can be oriented so that $u_1 = u_2 = \left(\frac{1}{2}\right) \chi_M$ where χ_M is the Euler characteristic of M .*

5. *Remarks.* The above theorem expresses relations between differential topological invariants of surfaces imbedded in Euclidean space

and suggests a more general problem. To describe the general situation let $M^k \subset E^{k+l}$ be a manifold of dimension k differentiably imbedded in a Euclidean space of $k + l$ dimensions. Let $G(k, l)$ be the Grassmann manifold of k -dimensional linear spaces through a point O and $G(l, k)$ that of l -dimensional linear spaces through O . There is a natural homeomorphism

$$\alpha: G(k, l) \rightarrow G(l, k) .$$

Using tangent planes and normal planes to M we define mappings

$$t: M \rightarrow G(k, l) , \quad n: M \rightarrow G(l, k)$$

such that

$$t = \alpha^{-1} n , \quad n = \alpha t .$$

The general problem is to study the relation between the homomorphisms

$$\left. \begin{array}{l} t^* ; H^p (G(k, l)) \rightarrow H^p(M) \\ n^* ; H^p (G(l, k)) \rightarrow H^p(M) \end{array} \right\} p = 0, 1, \dots$$

We hope to study this question on a later occasion.

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