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The Foundations of Minkowskian Geometry

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The exploration of Minkowski (or finite dimensional Banach) spaces is a necessary preliminary step for any progress in the theory of Finsler spaces [1]). They are the local spaces which belong to any finitely compact space with geodesics and a minimum of differentability properties. Therefore they are natural geometric objects to study.

The fact that such a study has never been seriously undertaken seems to be due to a lack of optimism on the one hand and of the proper tools on the other. It was taken for granted that the narrow group of motions of linear spaces necessitates a large number of invariants to determine simple geometric objects. This is not so : a curve in a plane is determined by two curvatures instead of one (section 5) and more generally, a hypersurface is determined by three fundamental forms instead of two (section 7) [2]).

The new tools which are used in this paper are 1) *the theorem* (19) *on convex bodies* on which most of the present work is based, 2) an *adequate concept of area*, 3) a function which corresponds to the *sine function* in euclidean geometry, and which seems to have many more applications than similar functions defined by others. It also has the advantage over the latter that its definition does not require differentiability properties of the unit sphere.

The analytical reason for the success of these and other concepts introduced here as compared to the tools previously used is their different structure: they are *integro-differential expressions instead of pure differential expressions*. This has as a consequence that they depend continuously on the metric.

A brief synopsis of the content follows: first the basic concepts are recalled (section 1). Then the sine function is introduced (section 2).

[1]) This is the main point of the author's lecture [1] which will be useful as an introduction to the present paper for the readers not familiar with Finsler spaces.

[2]) However, the problem of determining a k-dimensional manifold in an n-dimensional space is open for $k < n - 1$ and seems to be difficult.

This function is used in section 3 to discuss *normality of linear sub-spaces* as completely as seems reasonable in Minkowski spaces. The results go in a different direction and considerably farther, than previous discussions of normality in Banach spaces. The solutions of the isoperimetric problem in Minkowski spaces are not spheres but new convex surfaces (see [2]) which are closely connected with normality of hyperplanes to straight lines (section 4). They are significant because *many euclidean results remain valid if the unitsphere is replaced by a properly normalized solution of the isoperimetric problem which we call isoperimetrix.*

Next, curvatures of curves are defined. Because important principles are involved the statement that two curvatures determine a curve in a plane is discussed in detail. One of these curvatures is obviously a Minkowski invariant. The other can be chosen in an invariant manner by using an idea of Loewner (explained in section 1), which permits to associate intrinsically a euclidean space with a given Minkowski space. It is shown (section 6) that the *theorems of Meusnier* (in its most general form) *and Euler* for the distribution of the curvatures of curves on a surface hold. We also compare the present first curvature with the previously given definition of curvature (see [3], [4], [5]) with which Menger's curvature (see [6]) coincides in linear spaces. This curvature is not used here because it may not exist even if the curve is analytic, unless the unitsphere is of class C^2 and has positive Gauss curvature[3]).

Finally a brief introduction to surface theory is given (without any attempt whatever at completeness), just enough to show that the field is promising. The main result is that the *methods of relative differential geometry* (see [7] pp. 64, 65) *become significant* for Minkowskian geometry if the isoperimetrix, instead of the unit sphere, is taken as carrier of the spherical image.

1. Basic concepts

Let $\overline{R}$ be an n-dimensional euclidean space with the euclidean metric $e(x, y)$. An n-dimensional symmetric Minkowski (or Banach) space is a convex (in Menger's sense) metrization R with distance xy of $\overline{R}$ which is invariant under the translations of R (compare [8, section 2]). Denote by $C(p, r)$ the locus $px = r$. Then $C(p, r)$ is a convex surface with p as euclidean (and Minkowskian) center. We chose a fixed point z as origin and call $C = C(z, 1)$ the Minkowskian unit sphere. Then the distance xy, of two points $x \neq y$ in R can be obtained from $\overline{R}$ and C

[3]) The reason is again that the present curvature is an integro-differential expression.

as follows: If x', y' is the diameter of C parallel to the (euclidean) line $g(x, y)$ through x and y then

$$xy = 2e(x, y)/e(x', y') \ . \tag{1}$$

The different euclidean metrizations of R for which the translations of R are also translations are related to each other by non-degenerate affine transformations[4]) and are called *associated* to R. In each euclidean space associated to R the relation (1) holds.

Any concept or theorem which is independent of the choice of the associated space has a Minkowskian meaning[5]). In particular r-dimensional linear spaces V_r and parallelpipeds P_r are Minkowskian concepts, which because of their frequent occurrence will be briefly called r-flats and r-boxes respectively.

The r-dimensional exterior Minkowski measure $|M|_r$ of a set M in an r-flat V_r is defined by a formula similar to (1): In a fixed associated euclidean space $\overline{R}$ let $|M|_r^L$ denote the r-dimensional exterior Lebesgue measure of M. If $U(V_r)$ is the set in which the r-flat parallel to V_r through z intersects the solid Minkowskian unitsphere $U: px \leq 1$, put

$$\sigma(V_r) = \omega^{(r)}/|U(V_r)|_r^L \ , \qquad \omega^{(r)} = \pi^{r/2}/\Gamma(r/2+1) \tag{2}$$

then

$$|M|_r = \sigma(V_r)\,|M|_r^L \ . \tag{3}$$

If $xy = e(x, y)$ then $|M|_r = |M|_r^L$ because $\omega^{(r)}$ is the volume of the unit sphere in euclidean r-space. The measure (3) is independent of the choice of the associated space, in fact, it coincides with r-dimensional Hausdorff measure (see [8, section 2]). In particular

$$|M|_n = \sigma\,|M|_n^L \ , \qquad \text{where} \qquad \sigma = \omega^{(n)}/|U|_n^L \ . \tag{3a}$$

The statement that M is a manifold of the differentiability class $C^{(s)}$ and similar ones are affine invariant and have therefore a Minkowskian meaning. The same applies to Lipschitz surfaces, which may however be

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[4]) This means more explicitly: if $\overline{R}_1$ and $\overline{R}_2$ are associated to R and $e_i(x, y)$ is the distance in $\overline{R}_i$, then an affine transformation Φ of $\overline{R}_1$ on itself exists such that $e_1(x\Phi, y\Phi) = e_2(x, y)$ for all x, y.

[5]) In particular, affine geometry and affine differential geometry are part of Minkowskian geometry, but not an interesting part, because the Minkowskian group of motions is even narrower than the euclidean.

defined in direct Minkowskian terms: Let $x(u)$, $x \subset R$, $u \in Q$, where Q is a convex region in an r-dimensional euclidean space with rectangular coordinates $u_1, \ldots, u_r$. If a constant q exists such that

$$x(u)\, x(v) \leq q\, e(u, v) \qquad \text{for all } u, v \text{ in } Q \tag{4}$$

then $x(u)$ is a Lipschitz representation of a Lipschitz surface in R. The intrinsic area $A(x(u))$ of $x(u)$ is then evaluated as follows (see [2, section 7]). Let $x_1, \ldots, x_n$ be rectangular coordinates in an associated euclidean space $\overline{R}$ to R so that $|x - y| = [\Sigma (x_i - y_i)^2]^{\frac{1}{2}} = e(x, y)$. Because of (4) $|x(u) - x(v)| \leq q' \, |u - v|$ with a suitable q', hence the partials $\partial x_i / \partial u_k$ exist almost everywhere in Q and are bounded. If the matrix $(\partial x_i / \partial u_k)$ has rank r then the vectors $\partial x / \partial u_k = (\partial x_1 / \partial u_k, \ldots, \partial x_n / \partial u_k)$ span an r-flat V_r^u. Then

$$A(x(u)) = \int_Q \sigma(V_r^u)\, \Delta(u)\, du_1 \ldots du_r \tag{5}$$

where $\Delta(u) du_1, \ldots, du_r$ is the euclidean area element (that is

$$\Delta(u) = \left[\sum_{i_1 < i_2 < \cdots < i_r} \left(\frac{\partial (x_{i_1}, \ldots, x_{i_r})}{\partial (u_1, \ldots, u_r)} \right)^2 \right]^{\frac{1}{2}}$$

and the integrand is defined as 0 where $\Delta(u) = 0$ or the matrix $(\partial x / \partial u_k)$ has rank less than r. If $x(p)$ is defined on an r-dimensional manifold of class C' which can be covered by a finite number of coordinate neighborhoods u, such that $x(p) = x(u)$ is a Lipschitz representation of a Lipschitz surface then (5) still holds. This applies in particular to closed convex surfaces[6]) in $(r+1)$-flats.

Ellipsoids go into ellipsoids under affine transformations and are therefore Minkowskian concepts. A direct Minkowskian characterization is for instance this: ellipsoids are the only compact convex bodies for which the centers of gravity of parallel plane sections lie on a straight line, (see [9, pp. 18—19, 23—24, 213]).

Since Minkowskian geometry has more invariants than the euclidean, more special ellipsoids can be distinguished than in the euclidean. One of these was discovered by Loewner and is particularly important for the sequel: Among all ellipsoids with center that contain U there is exactly one, called *Loewner ellipsoid*, which has smallest Minkowskian (or

[6]) A closed convex surface is the boundary of a compact convex set with interior points with respect to some, and then all euclidean spaces associated to R.

because of (3a)) also euclidean) volume. This ellipsoid as unitsphere of a Minkowski space defines a euclidean space $E(R)$ associated to R whose distance we denote by $E(x, y)$. This metric is *intrinsically determined by* xy through the following requirements:

a) $E(x, y)$ *is associated to* xy (6)
b) $E(x, y) \leq xy$
c) *The volume of the solid unitsphere* $E(z, x) \leq 1$ *is minimal among all metrics which satisfy a) and b).*

The distance $E(x, y)$ is easily seen to be invariant under all motion of R. Any invariant of $E(R)$ is therefore also an invariant of R.

Since Loewner did not publish his result a proof will be given for the convenience of the reader:

Let E_1^* and E_2^* be two ellipsoids of minimal volume with center z that contain U. By a volume preserving affine transformation one of them, say E_1^*, can be transformed in a sphere E_1 with radius a, then E_2^* becomes another ellipsoid E_2. After a proper choice of rectangular coordinates E_2 will have an equation of the form $\sum_{i=1}^{n} x_i^2/b_i^2 = 1$, $b_i > 0$. The volumes of E_1 and E_2 are equal:

$$\omega^{(n)} a^n = \omega^{(n)} b_1 \ldots b_n \ .$$

The fact that E_1 and E_2 contain the transform $\overline{U}$ of U is expressed by the inequalities

$$\sum x_i^2 a^{-2} \leq 1 \quad \text{and} \quad \sum x_i^2 b_i^{-2} \leq 1 \quad \text{for} \quad x \subset \overline{U} \ .$$

Therefore also

$$\sum x_i^2 (a^{-2} + b_i^{-2})/2 \leq 1 \quad \text{for} \quad x \subset \overline{U} \ ,$$

so that the ellipsoid E' with the equation $\sum x_i^2 (a^{-2} + b_i^{-2})/2 = 1$ also contains $\overline{U}$. The volume of E' is because of $(a^2 + b_i^2)^{\frac{1}{2}} \geq (2ab_i)^{\frac{1}{2}}$

$$\omega^{(n)} \Pi \sqrt{2}\, a b_i (a^2 + b_i^2)^{-\frac{1}{2}} \leq \omega^{(n)} \Pi (a b_i)^{\frac{1}{2}} = \omega^{(n)} a^n \ .$$

But by the minimum property of E_1 the volume of E' must be at least $\omega^{(n)} a^n$, so that $a^2 + b_i^2 = 2ab_i$ for all i, hence $b_i = a$ or $E_1 = E_2$ and $E_1^* = E_2^*$.

2. The sine function

In R let the a-flat A and the b-flat B, $a, b > 0$, intersect in the d-flat D, $d \geq 0$ (which implies that D is not empty). The relative position of A and B as sets in an associated euclidean space can be described by a single angle only if

$$\min(a, b) = d + 1 \tag{7}$$

(compare [10] or [11]), so that the ordinary sine of the angle between A and B, which we denote by $\sin(A, B)$ is defined only in this case. Therefore *the Minkowski sine will be defined only if* (7) *holds*. In that case

$$q = a + b - d \tag{8}$$

is the dimension of the flat Q of lowest dimension that contains A and B.

Call m-box a possibly degenerate m-dimensional parallelpiped. We denote an m-box by $\overline{P}_m$ with or without superscripts and put $P^L_m = |\overline{P}|^L_m$ and $P_m = |\overline{P}_m|_m$, $P^L_0 = P_0 = 1$.

Let a, b, q, d satisfy (7) and (8), $d \geq 0$. Consider a d-flat D and an a-flat A and a b-flat B with $D \subset A \cap B$. Let $\overline{P}_d$ be a proper (that is, non-degenerate) d-box in D and $\overline{P}_a$, $\overline{P}_b$ proper a- and b-boxes in A and B respectively which contain $\overline{P}_d$ as face. Because of (7) the boxes $\overline{P}_a$ and $\overline{P}_b$ span a q-box $\overline{P}_q$ which is degenerate if and only if one of the flats A, B contains the other. Then (see [12])

$$\sin(A, B) = P^L_d P^L_q / P^L_a P^L_b \,. \tag{9}$$

This relation suggests to define the *Minkowskian sine* $sm(A, B)$ by

$$sm(A, B) = P_d P_q / P_a P_b \,. \tag{10}$$

Obviously $sm(A, B) = sm(B, A)$ and $sm(A, B) = 0$ if and only if one of the flats A, B contains the other. By (3) and (9)

$$\begin{aligned} sm(A, B) &= P^L_d P^L_q \sigma(D)\, \sigma(Q) / P^L_a P^L_b\, \sigma(A)\, \sigma(B) \\ &= \sin(A, B)\, \sigma(D)\, \sigma(Q) / \sigma(A)\, \sigma(B) \,, \end{aligned} \tag{11}$$

where $\sigma(D) = 1$ if D has dimension 0 and $\sigma(Q) = 1$ if $\overline{P}_q$ is degenerate. (11) shows that $sm(A, B)$ does not depend on the choice of the boxes $\overline{P}_d$, $\overline{P}_a$, $\overline{P}_b$.

It must be emphasized that $sm(A, B)$ *cannot be interpreted as function of a real number*, the „angle formed by a A and B". The absence of rotations does not permit to combine such an interpretation with the geometric properties which this sinefunction has (or which one would require of any useful sinefunction).

In the simplest case, $d = 0$, $a = b = 1$ the definition (10) yields analogs to many elementary results of trigonometry, of which we mention only a few: If A, B, C are the lines which carry the sides of lengths A', B', C' respectively of a triangle with Minkowski area Δ then $sm(A,B) = 2\Delta/A'B'$, so that the *law of sines*

$$sm(A, B) : sm(B, C) : sm(C, A) = C' : A' : B' \tag{12}$$

holds. If D is a line through the intersection x of A and B and a point y of the opposite side, then the sum of the areas of the triangle into which D decomposes Δ equals Δ, which yields with $D' = xy$ the often useful relation

$$sm(A, B)/D' = sm(A, D)/B' + sm(B, D)/A' \, . \tag{13}$$

Different trigonometric functions have been considered by others, of which the most important is *Finsler's cosine function* (see [13]). Unfortunately it is defined only if C is differentiable: Let A^+ and B^+ be two oriented straight lines through p and let a follow p on A^+. If the tangent plane T to $C(p, pa)$ at a intersects B^+ at b, then the Finsler cosine of A^+ and B^+ is defined by

$$cm(A^+, B^+) = \begin{cases} pa/pb & \text{if } b \text{ follows } p \text{ on } B^+ \\ -\,pa/pb & \text{if } b \text{ precedes } p \text{ on } B^+ \\ 0 & \text{if } T \text{ is parallel to } B^+ \end{cases} \tag{14}$$

Notice that in general $cm(A^+, B^+) \neq cm(B^+, A^+)$. As in ordinary trigonometry cm is closely related to sm[7]), in fact the relations are quite similar. If B^+ is so oriented that b follows p and $G = g(a, b)$, then expressing the area of the triangle p, a, b in two different ways and (14) yield

$$pb \cdot ab \cdot sm(B, G) = pa \cdot ab \cdot sm(A, G) = pb \cdot ab \cdot cm(A^+, B^+) sm(A, G) \, .$$

Anticipating a notation defined by (18) we put $sm(A, G) = \alpha(G)$ and see that

$$cm(A^+, B^+) = sm(B, G)/\alpha(G) \tag{15}$$

which corresponds to $\cos\alpha = \sin(\pi/2 - \alpha)$.

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[7]) It is a priori certain that a relation of the form $s\,m^2 + c\,m^2 = 1$ cannot exist for two functions which are both geometrically interesting in Minkowski spaces. For $\sin^2 + \cos^2 = 1$ expresses Pythagoras' Theorem which does not hold in Minkowski spaces.

With the notations which led to (13) assume that the line C lies in the tangent plane to $C(x, D')$ at y, then (13) and (14) yield

$$sm(A,B) = sm(A,D)cm(D^+,B^+) + sm(B,D)cm(D^+,A^+) \qquad (16)$$

where A^+, B^+, D^+ are the orientations of A, B, D in which points of C follow x.

The definitions of both sm and cm can in an obvious way be extended to the third and fourth quadrants. Then the addition formula (16) can be proved quite generally, but this will not be needed here.

3. Normality

A first application of the function sm will be a thorough discussion of normality. A line G in a Minkowski space R is normal to an r-flat V at f if $G \cap V = f$ and every point $x \in G$ has f as foot on V, that is $xf \leq xy$ for $x \in G$ and $y \in V$. If G is normal to V at f, then V cannot contain interior points of $C(x, xf)$. Hence V lies on a supporting hyperplane H to $C(x, xf)$ at f. Because the spheres $C(p, r)$ are homothetic, the converse follows: if V lies in H, then G is normal to V at f, so that G is normal to H and to any flat in H through f. There will be only one normal at f to H or to V in the flat (G, V) spanned by G and V if C (and hence $C(x, xf)$) is strictly convex. For a given line G and a given point p on G a hyperplane H exists to which G is normal, and H is unique when C is differentiable [8]).

The difficulty of defining normality of a flat of dimension greater than one to other flats derives from the fact in Minkowskian geometry H is (with the previous notation) in general not the locus of all lines through f normal to G. Hence a hyperplane normal to a line is for instance not defined. The present section will provide such a definition, its importance will become appearant in the following sections. For the reasons just indicated it does not seem reasonable to attempt defining normality for intersecting flats of arbitrary dimensions. We restrict ourselves to flats A, B for which $sm(A,B)$ is defined and *generalize the idea that* $\sin(A,B) = 1$ *when A is perpendicular to B.*

Let A, B be flats of dimensions a and b which intersect at a d-flat D, $d \geq 0$ such that (7) holds. If Q is the q-flat of lowest dimension containing A and B then $a + b = q + d$ by (8). *Then A is called normal to B and B transversal to A in Q at D if for any D-flat A^* through D in Q*

$$sm(A^*,B) \leq sm(A,B) \ . \qquad (17)$$

[8]) In that case we can say: G is normal to V at f if and only if for any orientation G^+ of G and any oriented line L^+ through f in V the relation $cm(G^+, L^+) = 0$ holds.

We then put
$$sm(A, B) = \alpha(B, D, Q) \ . \tag{18}$$

Obviously $\alpha(B', D', Q) = \alpha(B, D, Q)$ if $B' \parallel B$ and $D' \parallel D$. If D is a point then trivially $D' \parallel D$ and we simply write $\alpha(B, Q)$. If in addition $q = n$ so that Q is unique, namely the whole space, and B is either a line or a hyperplane, we write $\alpha(B)$ instead of $\alpha(B, D, Q)$.

That an a-flat A normal to B in Q at D exists follows from trivial continuity considerations. The existence of a B transversal to A in Q at D lies deeper, because our discussion will show that it is equivalent to the following non-trivial theorem proved in [14]:

(19) In the s-dimensional euclidean space let Λ be a convex body with interior points and Φ an $(s-2)$-flat which contains an interior point of Λ. Let Π be a 2-flat perpendicular to Φ at a point p. A half-hyperplane ψ bounded by Φ intersects Λ in a set $\psi \cap \Lambda$ with $|\psi \cap \Lambda|_{s-1}^{L} > 0$. If $|\psi \cap \Lambda|_{s-1}^{L}$ is laid off on the ray $\psi \cap \Pi$ from p, then the resulting curve Γ in Π is convex. Γ is strictly convex if Λ is strictly convex.

Γ is differentiable if Λ is differentiable.

Since this is not proved in [14] a proof will be given here. We introduce rectangular coordinates $x, y, z_1, \ldots, z_{s-2}$ with p as origin and such that Π is the (x, y)-plane. We also introduce polar coordinates ϱ, φ $(\varrho \geq 0)$ in Π, then $\varrho, \varphi, z_1, \ldots, z_{s-2}$ may be regarded as a sort of cylindrical coordinates in E^s. The $(s-2)$-dimensional cylinder parallel to Φ circumscribed to Λ has then an equation of the form $\varrho = R(\varphi)$, were $R(\varphi)$ is continuous (even continuously differentiable) and positive. Let $R_0 < \min\limits_{0 \leq \varphi < 2\pi} R(\varphi)$. If $A(\varphi_0)$ denotes the $(s-1)$-dimensional volume of the intersection of the half-hyperplane $\varphi = \varphi_0$ with Λ we have to show that $A(\varphi)$ is a differentiable function of φ. Let $A_1(\varphi_0)$ denote the volume which is cut out of Λ by the hyperplane $\varphi = \varphi_0$ inside the cylinder $\varrho \leq R_0$ and $A_2(\varphi_0)$ the remainder, that is the part for which $R_0 \leq \varrho \leq R(\varphi)$. It is quite easy to see that $A_2(\varphi)$ is differentiable, so that we may restrict the discussion to $A_1(\varphi)$.

For any point (x_0, y_0) in the circle D: $\varrho \leq R_0$ in Π denote by $f(x_0, y_0)$ the $(s-2)$-dimensional volume of the intersection of the $(s-2)$-flat $x = x_0$, $y = y_0$ with Λ. For any chord of D the function $f(x, y)$ is a differentiable function of the length, as a simple differentiation under the integral sign shows (the cylinder $\varrho \leq R_0$ was introduced to avoid difficulties at the endpoints of the chord).

Moreover, by the Brunn-Minkowski Theorem, $f^{1/n-2}(x, y)$ is on this chord a concave function[9]). It follows that $f^{1/n-2}(x, y)$ is a concave function of x, y in D. Since $f(x, y)$ is differentiable and positive, $f^{1/n-2}(x, y)$ is differentiable; as a concave function it is continuously differentiable (see for instance [15, p. 9]), hence $f(x, y)$ has continuous first derivatives.

It is to be shown that $A_1(\varphi) = \int_0^{R_0} f(\varrho \cos \varphi, \varrho \sin \varphi)\, d\varrho$ is differentiable. Since for $\varrho \neq 0$

$$\frac{\partial f(\varrho \cos \varphi, \varrho \sin \varphi)}{\partial \varphi} = - f_x \varrho \sin \varphi + f_y \varrho \cos \varphi$$

the function $f(\varrho \cos \varphi, \varrho \sin \varphi)$ has for $0 < \varrho \leq R_0$ a continuous and bounded derivative with respect to φ, therefore differentiation under the integral sign is permitted, so that $A_1'(\varphi)$ exists.

With the previous notations let $\overline{R}$ be a euclidean space associated to the given Minkowski space R. „Perpendicular'' will be used for the ordinary normality in $\overline{R}$; "rectangular" will also refer to $\overline{R}$. We are going to construct a flat A normal to B, and a flat B transversal to A at D in Q. Chose a proper rectangular box $\overline{P}_d$ in D. Since all definitions are invariant under translations we may assume that the origin z is a vertex of D. We distinguish the two cases $a = d + 1$ and $b = d + 1$.

1) $a = d + 1$ or $b = q - 1$.

Let $\overline{P}_b$ a proper rectangular b-box in B with $\overline{P}_d$ as face, and $B^* \neq B$ a b-flat in Q parallel to B. The q-boxes $\overline{P}_q^x$ spanned by a variable point x in B^* and $\overline{P}_b$ have all the same Minkowski or euclidean volume. If $\overline{P}_a^x$ is the a-box spanned by x and $\overline{P}_d$ and A^x the a-flat through D and x then by definition

$$sm(A^x, B) = P_d P_q^x / P_a^x P_b \, .$$

Since P_q^x is constant $sm(A^*, B)$ will be maximal when P_a^x is minimal.

In the case $d = 0$ or $a = 1$ we have $P_a^x = zx$ so that z must be a foot of x on B. This shows that *the definition of normality by* (17) *agrees with the accepted concept,* as recalled in the beginning of this section, *where the latter is defined.*

As an application consider a box $\overline{P}_n$ whose sides have Minkowski length 1 and which has maximal volume P_n. Such a box obviously exists. If z is

[9]) See [7, pp. 88].

one of its vertices and $A_1, \ldots, A_n$ are the lines which carry the sides of P_n through z, call H_i the hyperplane spanned by A_j with $j \neq i$, and $\overline{P}{}^i_{n-1}$ the face of $\overline{P}_n$ in H_i. Because $P_n = P^i_{n-1}\, sm(A_i, H_i)$ is maximal, $sm(A_i, H_i)$ must be maximal so that A_i is normal to H_i. Hence

(20) *There are lines* $A_1, \ldots, A_n$ *through* z *such that each* A_i *is normal to the hyperplane spanned by the remaining* A_j.

Theorem (20) and its proof are due to Taylor [16]. If P_n is replaced by the box $\overline{P}{}^*_n$ with center z homothetic to P_n and sides of length 2, then A_i intersects the faces of $\overline{P}{}^*_n$ at the centers of the faces parallel to H_i, and these faces are supporting planes of C, at their intersections with A_i. Therefore

(20a) For a given convex surface C with center z there is a box $\overline{P}{}^*_n$ circumscribed to C such that the midpoint of each $(n-1)$-dimensional face of $\overline{P}{}^*_n$ lies on C.

Returning to the general case we observe that P^x_a is constant as long as x moves in the same a-flat through D. Therefore we may restrict the attention to x which lie in the $(q-d)$-flat V in Q prependicular to D at z. If z^* is the euclidean foot of z on B^* then

$$e(z, x) = e(z, z^*) \sec (xzz^*)$$

hence

$$\begin{aligned} P^x_a = P^{xL}_a\, \sigma(A^x) &= P^L_d\, e(z, x)\, \sigma(A^x) \\ &= e(z, z^*)\, P^L_d\, \sigma(A^x) \sec (xzz^*) \end{aligned}$$

so that P^x_a is minimal if $\sigma(A^x)\sec(xzz^*)$ is minimal or $|\, U(A^x)\,|^L_a \cos(xzz^*)$ is maximal.

Any ray T with origin z in V determines with D an a-flat A^T (the A^x are particular A^T). If $|\, U(A^T)\,|^L_a$ is laid off on T from z, then the end-points traverse a closed hypersurface W in V with center z. Theorem (19) implies that every section of W by a 2-flat through z in V is convex, therefore W is convex.

Let the ray from z through x intersect W in $y(x)$. Then

$$|\, U(A^x)\,|^L_a \cos(xzz^*)$$

will be maximal if the projection of $y(x)$ on $g(z, z^*)$ has maximal (euclidean) distance from z. This is the case if and only if $y(x)$ lies in a supporting $(q-d-1 = b-a+1)$-flat of W (in V) perpendicular to

$g(z, z^*)$ or parallel to B, such a supporting flat H exists (and would also exist if W were not convex) and any point $y(\bar{x})$ in H yields an $A^{\bar{x}}$ normal to B at D in Q. The point $\bar{x}$ is unique if H contains only one point of W, which is by (19) the case when C is strictly convex.

This construction shows also that a b-flat through D in Q transversal to a given a-flat A through D in Q exists. For if V denotes again the $(q-d)$-flat perpendicular to D at z in Q, then the surface W can be constructed as before. The given flat A intersects W in a point y. *Because W is convex* it has a supporting $(b-a+1)$-flat H at y. The parallel to H through z determines with D an $(b-a+1+d=b)$-flat B which is by the preceding construction transversal to A. Moreover H, and therefore B, is unique if W is differentiable. This will be the case when C is differentiable.

2) $b = d+1$ or $a = q-1$.

Let A_1 be the a-flat through D in Q perpendicular to B (which is given). In A_1 chose a proper rectangular box $\bar{P}_a$ with $\bar{P}_d$ as face and form the union Z of all lines in Q perpendicular to A_1 at points of $\bar{P}_a$. Then Z is a cylindrical set of dimension q. On the line G in B perpendicular to D at z chose an arbitrary point z^*. If $\bar{P}^*_a$ denotes the intersection of a variable a-flat A^* through D in Q with Z and $\bar{P}^*_q$ the box spanned by z^* and $\bar{P}^*_a$ then $\bar{P}^*_q$ and P^{*L}_q are constant. Each $\bar{P}^*_q$ has in B the same face, namely the box $\bar{P}_b$ spanned by z^* and $\bar{P}_d$. Hence

$$sm(A^*, B) = P_d P^*_q / P^*_a P_b$$

so that $sm(A^*, B)$ is maximal when P^*_a is minimal. If G^* is the (properly oriented) line in Q perpendicular to A^* at z, then $P^{*L}_a = P^L_a \sec(G, G^*)$, hence

$$P^*_a = \omega^{(a)} P^L_a / \mid U(A^*) \mid^L_a \cos(G, G^*)$$

so that A^* is normal to B when $\mid U(A^*) \mid^L \cos(G, G^*)$ is maximal.

The perpendiculars to G^* to the various a-flats A^x through D in Q form a $(q-d)$-flat V in Q. If $\mid U(A^x) \mid^L_a$ is laid off in both directions from z on G^*, then the end points traverse a convex hypersurface W in V (see [14, Theorem II]) and $\mid U(A^*) \mid^L_a \cos(G, G^*)$ equals the length of the projection on G of either of the two points $W \frown G^*$. This projection is maximal if the points $W \frown G^*$ lie in supporting $(q-d-1)$-flats of W in V which are perpendicular to G. The line G^* and therefore A^* is uniquely determined if these supporting flats touch W in only one point each. This is the case when W is strictly convex, which is in turn true when C is strictly convex.

Again, this construction leads to the solution of the problem to find a B through D in Q transversal to a given A^*. The surface W is determined by D and Q as above. Let G^* be the perpendicular to A^* in Q. Because W is convex it has (parallel) supporting $(q-d-1)$-flats at the points $G^* \cap W$. If $\overline{G}$ is the perpendicular to these supporting flats, then the b-flat B perpendicular to $\overline{G}$ and through D will be transversal to A^* by the preceding discussion. Moreover, B is unique, if the supporting flats to W at $G^* \cap W$ are unique. This is true when W is differentiable, hence when C is differentiable. Thus the following main result on normality has been established:

(21) *Theorem. Let* a, b, d, q *be non-negative integers with* $\min(a, b) = d + 1$ *and* $q = a + b - d$. *In* R *let a* q*-flat* Q *and a* d*-flat* D *in* Q *begiven.*

For a given b*-flat* B *through* D *in* Q *an* a*-flat* A *in* Q *normal to* B *at* D *exists.* A *is unique if* C *is strictly convex.*

For a given a*-flat* A *through* D *in* Q *a* b*-flat* B *in* Q *transversal to* A *at* D *exists.* B *is unique when* C *is differentiable.*

4. The isoperimetrix

Normality of a hyperplane to a line is of particular importance and is closely connected with the isoperimetric problem. We discuss this connection first.

Let $x_1, \ldots, x_n$ be rectangular coordinates in a euclidean space $\overline{R}$ associated to the given Minkowski space R. Then $\sum x_i u_i = c$, $|u| = 1$, represents a hyperplane H in normal form and $\sum x_i u_i = 0$ intersects U in $U(H)$. We put $\sigma(H) = \sigma(u)$ and extend the definition of $\sigma(u)$ to arbitrary vectors by

$$\begin{aligned} \sigma(u) &= |u| \cdot \sigma(u/|u|) \quad \text{for} \quad u \neq 0 \\ &= 0 \qquad\qquad\qquad\;\; \text{if} \quad u = 0 \end{aligned} \tag{22}$$

Then $\sigma(u) = 1$ is the locus Z obtained by laying off $\sigma^{-1}(u/|u|) = U(H)/\omega^{(n-1)}$ in the direction u. Since Z is by (19) convex, $\sigma(u)$ is a convex function (see [7, section 14]). By (22) it is positive homogeneous of degree 1, hence $\sigma(u)$ is supporting function of another convex body ([7, section 17]) whose boundary we denote by T^*. *The surfaces homothetic to* T^* *are the solutions of the Minkowskian isoperimetric problem,* to find among all surfaces with a given Minkowski area those which band the largest volume (see [2, section 6]). Among these solutions we call one, T, the *isoperimetrix*. T is determined by the requirement that it has the origin z as center and that its Minkowski area equals n-times its volume. The supporting function of T in $\overline{R}$ is (compare [2, section 6] or (28) of the present paper)

$$T: \ \sigma(u)\,\sigma^{-1} = \sigma(u) \mid U \mid_n^L / \omega^{(n)} \ . \tag{23}$$

The supporting planes of the unitsphere C of R at a point p are the planes transversal to the radius zp. The following counterpart to this fact holds:

(24) *The supporting planes of T at a point p are the planes normal to the radius zp at p.*

Proof. With the notations of the preceding section we have $q = n$, $a = n - 1$, $b = 1$, $d = 0$ and are in case 2. The discussion of this case shows that the hyperplane H with perpendicular u will be normal to the ray G if and only if $\mid U(H) \mid_{n-1}^l \cos(u, G)$ is maximal or $\sigma(u)\sigma^{-1}\sec(u, G)$ is minimal. But $\sigma(u)\sigma^{-1}$ is the (euclidean) distance from z of the supporting plane H of T with perpendicular u, and $\sigma(u)\sigma^{-1}\sec(u, G)$ is the length of the segment which H intercepts on G. The endpoint of this segment will always be outside of T unless H passes through the point $p = G \frown T$. But this means that H is supporting plane of T at p.

Since this supporting plane is unique everywhere if and only if T is differentiable, it follows from (21) that T is differentiable when C is strictly convex. This can also be seen from the fact that Z is strictly convex with C and that T^* is the polar reciprocal of Z with respect to the unitsphere $\mid x \mid = 1$, see [17, § 8].

A consequence of (24) and (20a) applied to T is

(25) *There are lines $B_1, \ldots, B_n$ through a given point z such that each B_i is transversal to the hyperplane spanned by the remaining B_j.*

The representation (5) for area shows that the Minkowski area $A(K)$ of a convex surface K bounding the convex body $\overline{K}$ is

$$A(K) = n\,V^L(\overline{K}, \ldots, \overline{K}, \overline{T}^*) = n\,V_1^L(\overline{K}, \overline{T}^*)$$

where $\overline{T}^*$ is the body bounded by T^* and $V^L(\overline{K}, \ldots, \overline{K}, \overline{T}^*)$ is the mixed volume of $(n - 1)$-times $\overline{K}$ and $\overline{T}^*$ (compare [7, sections 29 and 37]). Therefore:

If $\overline{K}_1, \overline{K}_2$ are convex bodies with boundaries K_1, K_2 and $\overline{K}_1 \subset \overline{K}_2$, then $A(K_1) \leq A(K_2)$.

If T possesses no point at which n-linearly independent supporting planes exist, then $A(K_1) < A(K_2)$ when $\overline{K}_1$ is properly contained in $\overline{K}_2$.

For the first part see [7, section 29], for the second [17, § 27].

The mixed Minkowskian volumes $V_i(\overline{K}, \overline{L})$ of two convex bodies can be defined in the same way as in the euclidean case. If we put

$$V_i(\overline{K}, \overline{L}) = V_i^L(\overline{K}, \overline{L})\sigma \quad \text{then} \quad V_0(\overline{K}, \overline{L}) = |\overline{K}|_n, \ V_n(\overline{K}, \overline{L}) = |\overline{L}|_n$$

and

$$|\overline{K} + h\overline{L}|_n = |\overline{K} + h\overline{L}|_n^L \cdot \sigma = \sum_{i=0}^{n} \binom{n}{i} h^i V_i^L(\overline{K}, \overline{L})\sigma = \sum_{i=0}^{n} \binom{n}{i} h^i V_i(\overline{K}, \overline{L}) . \tag{26}$$

In particular, if $\overline{T}$ is the body bounded by T, then

$$|\overline{K} + h\overline{T}|_n - |\overline{K}|_n = nhV_1(\overline{K}, \overline{T}) + \cdots = nhV_1^L(\overline{K}, \overline{T})\sigma + \cdots$$
$$= nhV_1(\overline{K}, \overline{T}^*) + \cdots = nA(K) + \cdots ,$$

so that

$$A(K) = nV_1(\overline{K}, \overline{T}) = \lim_{h \to 0+} (|\overline{K} + h\overline{T}|_n - |\overline{K}|_n)/h . \tag{27}$$

That the first and third terms in (27) are equal was already proved in [2] for a wider class of surfaces.

The importance of the isoperimetrix makes it desirable to obtain an intrinsic Minkowskian equation for it. As a convex surface T has almost everywhere a tangentplane (see for instance [15, p. 24]). If $d\overline{S}$ and dS denote the euclidean and Minkowskian area elements of T at a point p where the tangent hyperplane H exists, and if u is the euclidean unit vector in the direction of the exterior perpendicular to T at p, then $dS = \sigma(u) d\overline{S}$. If $d\overline{V}$ and dV are the euclidean and Minkowskian volumes of the cones with vertex z and base dS then by (3a) and (23)

$$dV = d\overline{V} \cdot \sigma = n^{-1} \sigma(u) \sigma^{-1} d\overline{S} \cdot \sigma = n^{-1} dS . \tag{28}$$

If r is the Minkowski distance of z and p and W is the ray from z through p, then

$$dV = n^{-1} dS \cdot r \cdot sm(W, H) = n^{-1} dS \cdot r \cdot \alpha(W) \tag{29}$$

because H is by (24) normal to W ($\alpha(W)$ is defined in (18)). Comparison of (28) and (29) yields $r\,\alpha(W) = 1$. This relation was proved for those

W for which T is differentiable at $p = T \cap W$. By continuity it extends to all W, hence

$$r = \alpha^{-1}(W) \tag{30}$$

is what may be called the *Minkowskian polar equation of* T.

A corollary of (30) is

(31) *T coincides with C if and only if* $\alpha(W) \equiv 1$.

Statements (24) and (31) suggest the following, probably difficult problem: When are the solutions of the isoperimetris problem of the Minkowskian geometry with T as unitsphere homothetic to C? This is always the case for $n = 2$, but not for $n < 2$.

In Minkowskian geometry line segments are shortest connections of their endpoints and the only ones if C is strictly convex. Our next aim is to show how the existence of a line transversal to a given hyperplane implies that pieces of this hyperplane minimize the area.

Let the hyperplane H be perpendicular to the line G. In H take any set M with positive finite $| M |_{n-1}$ and denote by Z the cylindrical set formed by the lines parallel to G through points of M. If the hyperplane H^* is not parallel to G then it intersects M in a set M^* and if G^* denotes the perpendicular to H^*, then the discussion of case 2 in the preceding section shows that

$$| M^* |_{n-1} = | M |_{n-1}^{L} \omega^{(n-1)} / | U(H^*) |_{n-1}^{L} \cos (G, G^*)$$

so that $| M^* |_{n-1}$ is minimal when H^* is normal to G.

Therefore

(32) *Let Z be a cylindrical set with generators parallel to G such that a hyperplane H not parallel to G intersects Z in a set $M(H)$ with*

$$0 < | M(H) |_{n-1} < \infty .$$

Then the planes normal to G minimize $| M(H) |_{n-1}$ among all hyperplanes not parallel to G.

Let now the hyperplane H be given, in H consider an $(n-2)$-dimensional polyhedron B homeomorphic to an $(n-2)$-sphere, and let F be any $(n-1)$-dimensional polyhedron of the type of the solid $(n-1)$-dimensional sphere with B as boundary (for the notations compare [18, p. 124]). Let F_i, $i = 1, \ldots, m$ be the faces of F. If G is transversal to H and $\overline{F}_i$ the projection of F_i parallel to G on H, then

$$|\overline{F}_i|_{n-1} \leq |F_i|_{n-1}$$

by (32). On the other hand $\sum \overline{F}_i$ contains the closed domain E vounded by B in H, so that

$$|E|_{n-1} \leq \sum |\overline{F}_i|_{n-1} \leq \sum |F_i|_{n-1} = A(F) .$$

By methods of approximation by polyhedra known from the theory of surface area the last relation leads to the following more general theorem.

(33) *In a given hyperplane H let B be a surface of class D' homeomorphic to the $(n-2)$-sphere. Then the area $A(F)$ of any surface F of class D' and the type of the solid $(n-1)$-sphere in R with B as boundary is at least as large as the area (or measure) of the domain E bounded by B in H.*

If the transversal to H at a given point is unique then $A(F) > |E|_{n-1}$ unless F coincides with E.

5. The curvatures of curves

If $(x_1, \ldots, x_n) = x$ are rectangular coordinates in $\overline{R}$ and $x^0, \ldots, x^r$, $2 \leq r \leq n$ are the vertices of the simplex $\overline{T}_r$ (possibly degenerate) and $T_r^L = |\overline{T}|_r^L$, then

$$T_r^L = \frac{1}{r!}\left[\sum_{i_1 < \cdots < i_r} \begin{vmatrix} x_{i_1}^0 & \ldots & x_{i_r}^0 & 1 \\ \vdots & & \vdots & \vdots \\ \vdots & & \vdots & \vdots \\ x_{i_1}^r & \ldots & x_{i_r}^r & 1 \end{vmatrix}^2\right]^{\frac{1}{2}} . \tag{33}$$

For this and the following formulas see [19].

Let $x(\overline{S})$ be a curve of class C^r with the euclidean arc length $\overline{S}$ as parameter. If $x^i = x(\overline{S}_i)$, $\overline{S}_i \neq \overline{S}_j$ for $i \neq j$ then for $\overline{S}_i \to \overline{S}$

$$\overline{D}_r(\overline{S}) \,/\, r\,!\, \prod_{i=1}^{r} i\,! = \lim T_r^L \cdot \Pi\, e^{-1}(x^i, x^j) =$$

$$= \left[\sum_{i_1 < \cdots < i_r} \begin{vmatrix} x'_{i_1}(\overline{S}) & \ldots & x'_{i_r}(\overline{S}) \\ \vdots & & \vdots \\ x_{i_1}^{(r)}(\overline{S}) & \ldots & x_{i_r}^{(r)}(\overline{S}) \end{vmatrix}^2\right]^{\frac{1}{2}} \Bigg/ \, r\,!\, \prod_{i=1}^{r} i\,! \,. \tag{34}$$

The $(r-1)$-st curvature $\overline{\varkappa}_{r-1}(\overline{S})$ of $x(\overline{S})$ at $\overline{S}$ in $\overline{R}$ is given analytically by

$$\overline{\varkappa}_{r-1}(\overline{S}) = \overline{D}_r(\overline{S})\, \overline{D}_{r-2}(\overline{S})\, \overline{D}_{r-1}^{-2}(\overline{S}) \text{ if } \overline{D}_{r-1}(\overline{S}) \neq 0, \text{ where } \overline{D}_0(\overline{S}) = 1 \,. \tag{35}$$

If $\overline{T}^*_{r-1}$ is the simplex with vertices $x^1, \ldots, x^r$ and $T^{*L}_{r-1} = | \overline{T}^*_{r-1} |^L_{r-1}$ then (33) and (34) yield the following geometric interpretation of $\overline{\varkappa}_{r-1}(S)$:

$$\overline{\varkappa}_{r-1}(S) = \frac{r^2}{r-1} \lim \frac{1}{e(x^0, x^r)} \frac{T^L_r \, T^L_{r-2}}{T^L_{r-1} \, T^{*L}_{r-1}} \tag{36}$$

If V_{r-1} and V^*_{r-1} are the $(r-1)$-flats which carry $\overline{T}_{r-1}$ and $\overline{T}^*_{r-1}$ respectively and if it is kept in mind that an i-box spanned by $\overline{T}_i$ has i-dimensional measure $i! \; T^L_i$ then (36) may by (9) also be written as

$$\overline{\varkappa}_{r-1}(\overline{S}) = r \lim \sin (V_{r-1}, V^*_{r-1}) \, e^{-1}(x^0, x^r) \ . \tag{37}$$

Now consider $x(\overline{S})$ as a curve $x(S)$ in R with the Minkowskian arc length as parameter. S is a continuously differentiable function of $\overline{S}$ with a positive derivative, so that the relations $\overline{S}_i \to \overline{S}$ and $S_i \to S$ are equivalent for corresponding values $\overline{S}_i$, $\overline{S}$ and S_i, S. But S has in general not a second derivative with respect to $\overline{S}$. The preceding discussion suggests *to define the* $(r-1)$-*st Minkowskian curvature of the curve* $x(S)$, if $x(\overline{S})$ is of class C^r and $x^i = x(S_i) = x(\overline{S}_i)$, *by the formula*

$$\varkappa_{r-1}(S) = \frac{r^2}{r-1} \lim_{S_i \to S} \frac{1}{x(S_0)\, x(S_r)} \frac{T_r \, T_{r-2}}{T_{r-1} \, T^*_{r-1}} \tag{38}$$

where

$$T_i = | \overline{T}_i |_i \quad \text{and} \quad T^*_{r-1} = | \overline{T}^*_{r-1} |_{r-1} \ .$$

If

$$D_r(S) = r! \prod_{i=1}^{r} i! \lim T_r / \prod_{i<j} x(S_i)\, x(S_j)$$

then for $D_{r-2}(S) \neq 0$ by (10)

$$\varkappa_{r-1}(S) = D_r(S)\, D_{r-2}(S)\, D^2_{r-1}(S) = r \lim sm (V_{r-1}, V^*_{r-1}) / x^0 x^r \ . \tag{39}$$

The lines $g(x^i, x^j)$, $i \neq j$, tend for $S_i \to S$ to the tangent t_1 of $x(S)$ at S, hence by (1) and (2)

$$x^i x^i \cdot e^{-1}(x^i, x^j) \to 2/U(t_1) = \sigma(t_1) \ .$$

More generally V_r tends to the osculating r-flat t_r of $x(S)$ at S hence by (2) and (3)

$$T_r / T^L_r \to \sigma(t_r)$$

whence it follows that

$$\varkappa_{r-1}(S) = \frac{r^2}{r-1} \lim \frac{1}{e(x^0, x^r)} \frac{T_r^L T_{r-2}^L}{T_{r-1}^L T_{r-1}^{*L}} \frac{\sigma(t_r)\,\sigma(t_{r-2})}{\sigma^2(t_{r-1})\,\sigma(t_1)} = \tag{40}$$

$$= \overline{\varkappa}_{r-1}(\overline{S}) \frac{\sigma(t_r)\,\sigma(t_{r-2})}{\sigma^2(t_{r-1})\,\sigma(t_1)} \;;$$

for the first curvatures $\varkappa(S) = \varkappa_1(S)$ and $\overline{\varkappa}(\overline{S}) = \overline{\varkappa}_1(\overline{S})$ we find therefore

$$\varkappa(S) = \overline{\varkappa}(\overline{S})\,\sigma(t_2)\,\sigma^{-3}(t_1)\ . \tag{41}$$

If the definition of the $\sigma(t_i)$ is kept in mind then (40) shows clearly that $\varkappa_{r-1}$ is an integro-differential expression and that $\varkappa_{r-1}$ changes continuously when C various continuously.

To agree entirely with the usual definitions an agreement on the signs of the curvatures has to be added: $\varkappa_1, \ldots, \varkappa_{n-2}$ are always non-negative because the square root in (33) precludes a geometric interpretation of the signs. However, in the definition of $\varkappa_{n-1}$ the square root in (33) for T_n^L is accidental and we give $\varkappa_{n-1}(S)$ and $\overline{\varkappa}_{n-1}(S)$ the same sign as the determinant $|\, x^1(\overline{S}), \ldots, x^{(n)}(\overline{S})\,|$, or which amounts to the same, the sign of $|\, x_1^i, \ldots, x_n^i, 1\,|$ for $\overline{S}_0 < \overline{S}_1 < \cdots < \overline{S}_n$ and $\overline{S}_i$ close to $\overline{S}$.

One of the important properties of the curvatures $\overline{\varkappa}_i(\overline{S})$ is that they „determine" the curve in $\overline{R}$, at least if they are different from 0 [10]. The expression „determine" may mean either of the following three statements.

1) *If* $\overline{\varkappa}_1(\overline{S}), \ldots, \overline{\varkappa}_{n-1}(\overline{S})$ *and* $x(0)$ *and* $x'(0)$ *are given then the functions* $x_i(S)$ *and therefore the curve* $x(\overline{S})$ *are uniquely by determined.*

2) *If two curves* $x^1(\overline{S})$ *and* $x^2(\overline{S})$, *where* $\overline{S}$ *is the arc length on both curves have the same curvatures* $\overline{\varkappa}_i(\overline{S}) \neq 0$, $i = 1, \ldots, n-1$ *in some interval* $[a, b] : a \leq \overline{S} \leq b$, *then* $e[x^1(\overline{S}'), x^1(\overline{S}'')] = e[x^2(\overline{S}'), x^2(\overline{S}'')]$ *for any* $\overline{S}', \overline{S}''$ *in* $[a, b]$.

3) *Under the assumptions of* 2) *a motion of* $\overline{R}$ *exists which carries* $x^1(\overline{S})$ *into* $x^2(\overline{S})$ *for all* $\overline{S}$ *in* $[a, b]$.

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[10]) Precautions are necessary if $\overline{K}_i(\overline{s}) = 0$ for some i, which are however usually not discussed in books on differential geometry. For a completely satisfactory treatment see [20]. A reader not familiar with these difficulties will easily discover them when he considers the following two curves B_1, B_2 in 3-space: for $x \leqslant 1$ both B_1 and B_2 are defined by the equations $y = e^{-x^{-2}}$, $z = 0$ for $-\infty < x < 0$, $y = z = 0$ for $0 \leqslant x \leqslant 1$. In the interval $1 < x < \infty$ the curve B_1 is defined by $y = e^{-(x-1)^{-2}}$, $z = 0$ and B_2 by $y = 0$, $z = e^{-(x-1)^{-2}}$.

In a euclidean space the three statements are equivalent. Of the corresponding statements in a Minkowski space 3) is the strongest, it implies 1) and 2), but neither 1) or 2) imply 3). A priori it is to be expected that more invariants will be necessary to determine a geometric object in a Minkowski space because the group of motions is narrower.

A second set of invariant curvatures is immediately available: Let $E(R)$ be the euclidean metric intrinsically determined by R and defined in section 1, and call $\varkappa_1^E(\overline{S}),\ldots,\varkappa_{n-1}^E(\overline{S})$ the curvatures $\overline{\varkappa}_i(\overline{S})$ for $\overline{R} = E(R)$. The $\varkappa_i^E(\overline{S})$ are invariant under all motions of R. It is clear that they determine the curve in the sense of 1). The problem to find invariants which determine a curve in R in the sense of 2) oder 3) seems to be quite difficult for $n > 2$. For $n = 2$ it will be shown below that $\varkappa^E(\overline{S})$ and $\varkappa(S)$ determine the curve in a sense which lies between 2) and 3). Because of the novelty of the situation an *example* will be discussed first:

In the plane with rectangular coordinates x, y and corresponding polar coordinates r, φ, let C be the curve defined in the first quadrant by:

$$\begin{aligned} r &= \sec\varphi \cos\pi/8 && \text{for} \quad 0 \le \varphi \le \pi/8 \\ &= 1 && \text{for} \quad \pi/8 \le \varphi \le \pi/3 \\ &= \csc\varphi \sin\pi/3 && \text{for} \quad \pi/3 \le \varphi \le \pi/2 \end{aligned}$$

and extended to the other quadrants by reflection in the x- and y-axis.

The Loewner ellipse for C is the circle $r = 1$. This ellipse must have the coordinate axes as axes because otherwise at least two such ellipses would exist contradicting the uniqueness. If we write the ellipse in the standard form $x^2/a^2 + y^2/b^2 = 1$ then we know that its (euclidean) area $\pi a b$ is at most π since $r = 1$ is an ellipse which contains C. For the point $x_0 = y_0 > 0$ of the ellipse we must have $x_0 \ge 2^{-\frac{1}{2}}$ because $\left(2^{-\frac{1}{2}}, 2^{-\frac{1}{2}}\right)$ lies on C. Therefore $(a^{-2} + b^{-2})/2 \le 1$. This relation yields together with $ab \le 1$ readily that $a = b = 1$ [11]). Now let $p_1(\overline{S}) = (x_1(\overline{S}), y_1(\overline{S}))$ be any oriented arc of class C^2 whose oriented tangent has direction $\varphi_1(\overline{S})$ with $\pi/8 \le \varphi_1(\overline{S}) \le \pi/3$ and let $p_2(\overline{S})$ originate from $p_1(\overline{S})$ by a rotation in $E(R)$ about the origin through $\pi(7/8 - 1/3) = 13\pi/24$. Then $p_1(\overline{S})$ and $p_2(\overline{S})$ are in the relation of statement 2) and have the same curvature functions $\varkappa^E(\overline{S})$ and $\varkappa(\overline{S})$. Moreover, there

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[11]) This argument yields the following fact which often allows to determine the Loewner ellipse: If the ellipse E' contains C and has four points with C in common which are endpoints of conjugate diameters of E', then E' is the Loewner ellipse for C.

exists a Minkowski metric $m(x, y)$ (in the present case $E(x, y)$) which coincides with xy for points of $p_1(S)$ and $p_2(S)$ and for which $p_1(S)$ and $p_2(S)$ have the relation 3).

Next let $p_3(\overline{S})$ be any oriented arc of class C^2 whose oriented tangent has direction $\varphi_2(\overline{S})$ with $-\pi/8 \leq \varphi_2(\overline{S}) \leq \pi/8$ and let $p_4(\overline{S}_1)$ originate from $p_3(\overline{S})$ by a rotation about the origin through $\pi/2$ followed by a contraction in the ratio $\lambda = \sin(\pi/3) \sec \pi/8$. Then $\overline{S}_1/\overline{S} = \lambda$ and $\overline{S}_1$ is the euclidean arc length on p_4. If $t(\overline{S})$ and $t(\overline{S}_1)$ are the tangents of p_3 and p_4 at corresponding points, then $\sigma[t(\overline{S})]/\sigma[t(\overline{S}_1)] = \lambda^{-1}$, hence the Minkowski lengths of p_3 and p_4 at corresponding points $\overline{S}$ and $\overline{S}_1$ are equal, moreover $p_3(S')p_3(S'') = p_4(S')p_4(S'')$. The curves p_3 and p_4 are therefore in the relation 2). Since for any three values S_0, S_1, S_2 the areas of the triangles with vertices $p_3(S_i)$, $i = 1, 2, 3$, and $p_4(S_i)$ have the ratio λ^2, it follows that the curvatures $\varkappa$ of p_3 and p_4 have the constant ratio λ^2 and are therefore not equal. Hence the curvature $\varkappa(S)$ may be different for curves which are in the relation 2).

We consider now two curves $p_1(\overline{S})$ and $p_2(\overline{S})$ in an arbitrary Minkowski plane $E(R)$ which have the same curvatures $\varkappa^E(\overline{S})$ and $\varkappa(S)$. If $t^i(\overline{S})$ denotes the tangent of $p_i(\overline{S})$ at $\overline{S}$, then (41) implies

$$\sigma[t^1(\overline{S})] = \sigma[t^2(\overline{S})] \ . \tag{42}$$

Therefore the Minkowski lengths corresponding to equal values of $\overline{S}$ are equal. Moreover, for any $\overline{S}'$, $\overline{S}''$ there is a value $\overline{S}_0$ between $\overline{S}'$ and $\overline{S}''$ such that $t'(\overline{S}_0)$ is parallel to the chord $g(\overline{p}_1(\overline{S}'), p(\overline{S}''))$. The motion of $E(R)$ which carries $p_1(\overline{S})$ into $p_2(\overline{S})$ carries $t'(\overline{S}_0)$ into the tangent $t^2(\overline{S}_0)$ of $p_2(\overline{S})$ at $\overline{S}_0$ and $g(\overline{p}_1(\overline{S}'), p(\overline{S}''))$ into $g(\overline{p}_2(\overline{S}'), \overline{p}_2(\overline{S}''))$, so that this line is parallel to $t^2(\overline{S}_0)$. Therefore

$$\begin{aligned} p_1(\overline{S}')\, p_2(\overline{S}'') &= E\,[p_1(\overline{S}')\,,\, p_1(\overline{S}'')\,]\,\sigma\,[t'(\overline{S}_0)\,] = \\ &= E\,[p_2(\overline{S}')\,,\, p_2(\overline{S}'')\,]\,\sigma\,[t^2(\overline{S}_0)\,] = p_2(\overline{S}')\, p_2(\overline{S}'') \ . \end{aligned}$$

By (42) there is a rotation of $E(R)$ which carries the arc C_1 of C corresponding to the radii which are parallel to the tangent of p_1 into the arc C_2 of C correspondingly derived from p_2. If C_1 contains at least a semicircle of C, then this rotation of $E(R)$ carries all of C into itself. Therefore

(43) *An arc $p_1(S)$ of class C^2 whose tangent varies through at least π can be carried into the arc $p_2(S)$ by a motion of the Minkowski plane if and only if $p_1(S)$ and $p_2(S)$ have the same curvature functions $\varkappa(S)$ and $\overline{\varkappa}^E(\overline{S})$.*

The tangent of $p_1(S)$ will, of course, always vary through at least π if $p_1(S)$ is closed. The discussion of the general case leads to the following result which is stated without complete proof:

(44) *The curvature functions $\varkappa(S)$ and $\varkappa^E(\bar{S})$ are equal for two curves $p_1(S)$ and $p_2(S)$ of class C^2 in a Minkowski plane R if and only if a Minkowski metric R' exists such that* 1) $E(R) = E(R')$, 2) *distances of pairs of points on $p_i(S)$ are the same for R and R'*, 3) *there is a motion of R' which carries $p_1(S)$ into $p_2(S)$.*

6. The Theorems of Meusnier and Euler for the Minkowski curvatures. Comparison with other curvatures.

Finsler showed in [13] that his curvatures (see the end of this section) satisfy with his cosine function (14) the theorem of Meusnier. *The present curvatures also satisfy Meusnier's Theorem, but with the present sine function.* The formulation of the underlying facts for the euclidean case given below, from which we derive the Minkowskian case, follows Finsler [13].

Let M be an m-dimensional manifold in $\bar{R}$, $m<n$, of class C^m. All curves A through a point p of M which have non-vanishing curvatures up to order $r-1$ at p and have a common r-dimensional osculating r-flat t_r intersecting the tangent m-flat H of M at p in a line L (hence $r \leq n-m+1$) have at p the same osculating flats $t_1 = L, t_2, \ldots, t_r$ and the same curvatures $\bar{\varkappa}_1, \ldots, \bar{\varkappa}_{r-1}$.

All those curves A whose osculating $(r+1)$-flats contain some perpendicular to H (hence $r<n-m+1$) contain the same perpendicular P and all the osculating $(r+1)$-flats of the various curves A form a pencil through t_r. If t'_{r+1} is the element of the pencil through P and $\bar{\varkappa}'_r$ is the r-th curvature of $t'_{r+1} \cap M$ at p, then Meusnier's theorem states that curvature $\bar{\varkappa}_r$ of any curve A with osculating $(r+1)$-flat t_{r+1} satisfies the relation

$$\bar{\varkappa}_r \cos(t_{r+1}, t'_{r+1}) = \bar{\varkappa}'_r . \tag{45}$$

If t^*_{r+1} is the element of the pencil perpendicular to t'_{r+1}, that is, the $(r+1)$-flat in the pencil intersecting H in a two-flat, then $\cos(t_{r+1}, t'_{r+1}) = \sin(t_{r+1}, t^*_{r+1})$ so that (45) may be replaced by the statement that $\bar{\varkappa}_r \sin(t_{r+1}, t^*_{r+1})$ is constant.

It follows from (40) that the curvatures $\varkappa_1, \ldots, \varkappa_{r-1}$ are the same for all curves A. If Q is the space of the pencil, then (45) may be written as

$$\sin(t_{r+1}, t^*_{r+1}) \frac{\sigma(Q)\,\sigma(t_r)}{\sigma(t_{r+1})\,\sigma(t^*_{r+1})} \bar{\varkappa}_r \frac{\sigma(t_{r+1})\,\sigma(t_{r-1})}{\sigma^2(t_r)\,\sigma(t_1)} =$$

$$= \sin(t'_{r+1}, t^*_{r+1}) \frac{\sigma(Q)\,\sigma(t_r)}{\sigma(t'_{r+1})\,\sigma(t^*_{r+1})} \bar{\varkappa}'_r \frac{\sigma(t'_{r+1})\,\sigma(t_{r-1})}{\sigma^2(t_r)\,\sigma(t_1)}$$

which means by (40) and (11) that

$$\varkappa_r\, sm(t_{r+1}, t^*_{r+1}) = \varkappa'_r\, sm(t'_{r+1}, t^*_{r+1}) \ .$$

Hence $\varkappa_r\, sm(t_{r+1}, t^*_{r+1})$ *is constant in the pencil, which is Meusnier's Theorem.* A relation similar to (45) is obtained if a t''_{r+1} in the pencil normal to t^*_{r+1} at t_r in Q is selected. If $\varkappa''_r$ denotes the r-th curvature of $t''_{r+1} \frown M$, at p, then

$$\varkappa_r\, sm(t_{r+1}, t^*_{r+1}) = \varkappa''_r\, \alpha(t^*_{r+1}, t_r, Q) \ . \tag{46}$$

A remarkable consequence of this relation is that $\varkappa''_r$ is the same for all t''_{r+1} normal to t^*_{r+1} at t_r in Q.

In a (two-dimensional) Minkowski plane let Minkowskian parallel coordinates be introduced, that is ordinary parallel coordinates but such that the units on the x- and y-axis have Minkowski length 1. If β is the Minkowski sine of the x- and y-axis then a trivial calculation shows that a curve of class C^2 with the x-axis as tangent at the origin has at $(0, 0)$ the curvature

$$\varkappa = \beta f''(0) = \beta \lim_{x \to 0} 2f(x)\, x^{-2} \ . \tag{47}$$

We use this remark to establish by a well known method a close analogue to Euler's theorem for the normal sections of a hypersurface.

In an n-dimensional Minkowski-space R let M be a hypersurface of class C^2, so that it may locally be represented in the form $z = f(x_1, \ldots, x_n)$, where $x_1, \ldots, x_{n-1}, z$ are Minkowskian parallel coordinates and $z = 0$ is the tangent plane of M at the origin. We form the intersection of $z = f(x)$ with the planes $z = \pm h/2$, $h > 0$, and project this intersection parallel to the z-axis on the plane $z = 0$. The locus thus obtained has the equation

$$f(x) = \pm h/2, \qquad z = 0 \ . \tag{48}$$

For a ray R in $z = 0$ issuing from the origin 0 let $r(h, R)$ be the Minkowski distance from 0 of the point $x(h, R)$ in which R intersects the locus (48). Since $2f[x(h, R)] = \pm h$ it follows from (47) that

$$\lim_{h \to 0} h/r^2(h, R) = 1/[\varrho(R)\, sm(z, R)] \ , \tag{49}$$

where $1/\varrho(R)$ is the curvature $\varkappa$ at 0 of the intersection of M with the two-flat determined by the z-axis and R. Dilation of the locus (48) in the ratio $1 : h^{\frac{1}{2}}$ from 0 yields the locus

$$I_h : f\left(h^{\frac{1}{2}} x_1, \ldots, h^{\frac{1}{2}} x_{n-1}\right) = \pm h/2 \; , \quad z = 0 \; , \tag{50}$$

or

$$\pm h/2 = f\left(h^{\frac{1}{2}} x\right) = 2^{-1} h \sum f_{x_i x_j} x_i x_j + h \varepsilon(x, h)$$

where ε tends to 0 when h tends to 0. Therefore I_h tends to

$$I : \sum f_{x_i x_j}(0) \, x_i x_j = \pm 1 \; , \tag{51}$$

so that I (unless degenerate) is a pair of conjugate quadrics one of which may be imaginary. The point $I_h \frown R$ has distance $r(h, R) h^{-\frac{1}{2}}$ from 0, so that I intersects R in a point whose distance from 0 has by (49) the value $[\varrho(R) sm(z, R)]^{\frac{1}{2}}$. Thus we have *Euler's Theorem*:

(52) *Let M be a hypersurface of class C^2 in a Minkowski space, p a point of M, and L a line not in the tangent hyperplane H of M at p. If on each ray R in H with origin p a segment of Minkowski length $[\varrho(R) sm(L, R)]^{\frac{1}{2}}$ is laid off from p, where $\varrho^{-1}(R)$ is the curvature $\varkappa$ at p of the intersection of the (two)-plane through L and R with M, then the endpoints of these segments traverse a pair of conjugate quadrics in H.*

If L is normal to H, and $P(R)$ is the plane through L and R, we have $sm(z, R) = \alpha(R, \pi(R))$.

Curvatures for curves in Finsler spaces were introduced for $n = 2$ by Underhill [3] and Landsberg [4]. For general n they were introduced by Finsler [5, 13]. The definitions coincide for $n = 2$. The underlying idea of these definitions is this: If $x(S)$ is a curve with tangent t at a given point q, then the parallel to t through z intersects C in a point q' (or rather in a pair of points, but it will not matter which point is chosen). There is exactly one ellipsoid with z as center through q' which has at q' the same second differential as C. This ellipsoid determines a euclidean metric $E(q)$. *Finsler defines the curvatures of $x(S)$ at q as the curvatures at q of $x(S)$ as a curve in $E(q)$.* Obviously $E(q)$ exists only if C has a second differential at q' and the indicatrix is a non-degenerate ellipse. Actually the idea is significant only if C is of class C^2 and has positive Gauss curvature. Thus $x(S)$ may not even have a curvature when it is analytic[12]. Moreover, small changes of C may induce large changes of the curvatures of $x(S)$.

There exists another definition of curvature for curves in general spaces which is due to Menger [6] (for modifications of this concept see [21]). Haantjes' curvature, see [21], coincides with Finsler's (see [5, p. 59]). Hence Haantjes' main result in [21] means that *Menger's definition of* $\varkappa_1$[13] *coincides in Minkowski (and even in general Finsler) spaces with Finsler's definition*. Although this may seem surprising at first sight, the reason becomes obvious as soon as Menger's procedure is analysed: Menger departs also from (36), but expresses the volumes T_r^L in terms of the distances $e(x^i, x^j)$ of the vertices. In these new expressions he replaces $e(x^i, x^j)$ by $x^i x^j$ and then passes to the limit. Since the points x^i approach the same point $q = x(S)$ the distances $x^i x^j$ become better and better approximations of the distances in $E(q)$.

It is worth while to see the connection of these curvatures with the present ones at least in the simplest case. Let $F(x)$ be the distance function of C and put

$$\Phi(x, \xi) = \sum x_i x_k F_{x_i x_k}(\xi) \; .$$

To evaluate the Finsler curvature $\varkappa^f$ of a given curve $x(t)$ at a given point t_0 we may by [13, p. 158] procede as follows. If H is a hyperplane through $x(t_0)$ but not through the tangent t_1 of $x(t)$ at $x(t_0)$ we assume that the parameter t is chosen such that it is proportional to the distance from $x(t_0)$ of the point at which a hyperplane parallel to H through $x(t)$ intersects t_1. Then the Finsler curvature $\varkappa^f$ of $x(t)$ at t_0 is given by

$$(\varkappa^f)^2 = \Phi(x'', x') F^{-3}(x') \; .$$

In case of a plane we put $x_1 = x$, $x_2 = y$ and choose the coordinate system such that the $x(t_0)$ is the origin, and the x-axis the tangent t_1. We can then choose $t = x$ and find that because of $x'' = 0$

$$(\varkappa^f)^2 = y''^2 F_{yy}(1,0) F^{-3}(1,0) \; . \tag{53}$$

[12]) The contributors to the theory of Finsler spaces were never interested in reducing differentiability hypotheses, so that the idea of curves of high differentiability in spaces with metrics of low differentiability could not enter their discussions. It is less understandable that this distinction escaped Menger when he introduced his curvature in [6] because this concept was meant for general spaces.

[13]) Menger does not define higher curvatures. How this can be done along Menger's lines is shown in [19].

Since $F(1,0) = \sigma(t_1)$ and $y'' = \bar{\varkappa}$ it remains only to find a geometric interpretation for $F_{yy}(1,0)$.

The surface Z defined in section 5 is in the plane case obtained from C by rotation through $\pi/2$ about z, so that $H(x, y) = F(y, -x)$ is the distance function of Z. Then T^* has $H(x, y)$ as supporting function.

If $x_0^2 + y_0^2 = 1$, then $H_{xx}(x_0, y_0) + H_{yy}(x_0, y_0)$ is the euclidean radius of curvature at a point of T^* (see [7, p. 65]) where the normal has direction x_0, y_0; in particular, $H_{xx}(0, 1) + H_{yy}(0, 1)$ is the radius of curvature where the tangent of T^* is parallel to the x-axis t_1. The homogeneity of $H(x, y)$ implies $xH_{xy} + yH_{yy} = 0$, hence $H_{yy}(0, 1) = 0$. Since $H_{xx}(0, 1) = F_{yy}(1, 0)$ it follows that $F_{yy}(1, 0)$ is the radius of curvature of T^*, and therefore $\sigma^{-1}F_{yy}(1, 0)$ the radius of curvature $\bar{\varkappa}_T^{-1}$ of T at the points where the tangent is parallel to t_1. Therefore

$$(\varkappa^f)^2 = \sigma\,\sigma^{-3}(t_1)\,\bar{\varkappa}^2\,\bar{\varkappa}_T^{-1} = \frac{\sigma^2\bar{\varkappa}^2}{\sigma^6(t_1)}\cdot\frac{\sigma^3(t_1)}{\bar{\varkappa}_T\,\sigma}\,. \tag{54}$$

(41) and (54) yield:

The Finsler curvature $\varkappa^f$ and the curvature $\varkappa$ of a curve A (in a Minkowski plane) at a point p are related by

$$\big(\varkappa^f(p)\big)^2 = \varkappa^2(p)\,\varkappa_T^{-1}(\bar{p}) \tag{55}$$

where $\varkappa_T(\bar{p})$ is the curvature of the isoperimetrix T at a point $\bar{p}$ where the tangent of T is parallel to the tangent of A at p.

This relation and similar ones for the curvatures of curves in space show that Finsler's curvatures can be expressed in terms of the $\varkappa_i$. (55) also exhibits how the differentiability properties of C enter the definition of $\varkappa^f$. Due to the exclusive use of differential invariants the trend in Finsler spaces (compare [22]) was to consider these spaces not as point spaces, but as spaces of line elements, where a euclidean metric is associated with each line element, just as $E(q)$ was associated above with the tangent of $x(s)$ at q. *The use of integro-differential expressions like the present area element and the curvatures $\varkappa_i$ make a return to the geometrically much more natural idea of a point space possible.*

7. The elements of surface theory

Let R be a three dimensional Minkowski space and $p(u, v) = [x(u, v), y(u, v), z(u, v)]$ a surface of class C^2 where x, y, z are rectangular coordinates in $E(R)$[14]. We denote the first and second

[14]) The first part of this section extends without change to hypersurfaces in an n-dimensional space.

fundamental forms of $p(u, v)$ with respect to $E(R)$ by $P^E(du, dv)$ and $Q^E(du, dv)$. They determine $p(u, v)$ up to a motion in $E(R)$. They do not determine $p(u, v)$ up to a motion in R because in general not every motion of $E(R)$ is a motion of R.

If $F(x, y, z)$ is the distance function of C, then $P(du, dv) = F^2(x_u\, du + x_v\, dv, y_u\, du + y_v\, dv, z_u\, du + z_v\, dv)$ is, up to terms of higher order, the square of the Minkowski distance of the points $p(u, v)$ and $p(u + du, v + dv)$. Of course $P^E(du, dv) \equiv P(du, dv)$ if $E(R) = R$. The three forms $P^E(du, dv)$, $Q^E(du, dv)$ and $P(du, dv)$ determine the surface $p(u, v)$ in the same sense as $\varkappa$ and $\varkappa^E$ determine a plane curve.

For let p be such that for any two points $p_i = p(u_i, v_i)$, $i = 1, 2$, a point $p_3 = p(u_3, v_3)$ exists, such that a suitable tangent t of p at p_3 is parallel to $g(p_1, p_2)$. Assume that $q(u, v)$ is a surface in R for which the forms P^E, Q^E, P are identical to those of $p(u, v)$. Then a motion φ of $E(e)$ exists which carries $p(u, v)$ into $q(u, v)$.

If $p_i = p(u_i, v_i)$, $i = 1, 2$, are arbitrary and p_3 is chosen as above, and $p_i\, \varphi = q_i = q(u_i, v_i)$ then $t\, \varphi$ will be parallel to $g(q_1, q_2)$ because φ is a motion of $E(R)$. If t corresponds to the ratio $\overline{du} : \overline{dv}$ of du and dv, then $t\varphi$ will correspond to the same ratio. But

$$P(\overline{du}, \overline{dv}) \,/\, P^E(\overline{du}, \overline{dv}) = \sigma^2(t)$$

at $p(u_3, v_3)$ and has by hypothesis the same value at $q(u_3, v_3)$ so that $\sigma^2(t) = \sigma^2(t\, \varphi)$. It follows from $E(p_1, p_2) = E(q_1, q_2)$ and (1) that $p_1\, p_2 = q_1\, q_2$.

If $p(u, v)$ represents a closed manifold of cass C^2 (which does not exclude selfintersections of $p(u, v)$ in the large), then tangent planes with any (non-oriented) normal, hence tangents with a given direction exist. The tangents t for which $\sigma(t) = \sigma(t\, \varphi)$ fill at least a hemisphere of C so that φ carries C into itself and is therefore a motion of R. Therefore we proved:

(56) *Theorem: The closed manifolds* $p(u, v)$ *and* $q(u, v)$ *have identical fundamental forms* P, P^E, Q^E *if and only if a motion of* R *exists which carries* $p(u, v)$ *into* $q(u, v)$.

The form Q^E is up to terms of higher order twice the euclidean distance of $p(u + du, v + dv)$ from the tangent plane of p at (u, v). The Minkowskian distance of $p(u + du, v + dv)$ from the tangent plane is proportional to the euclidean distance, so that the Minkowskian second fundamental form $Q(du, dv)$ is obtained from Q^E by multiplication with an easily evaluated factor $\beta(u, v)$.

Therefore the asymptotic directions are given either by $Q^E = 0$ or by $Q = 0$; also, conjugate directions can be defined either by means of Q^E or by Q and have therefore a Minkowskian meaning. This is, of course, due to the fact that these concepts are affine invariants.

It also follows from (47) and Euler's Theorem (52) that

$$\frac{Q(du, dv)}{P(du, dv)} = \frac{1}{\alpha(du, dv)\, \varrho(du, dv)} \tag{57}$$

where $\alpha(du, dv)$ is the α-function (18) for the line with direction du, dv through $p(u, v)$ and the plane through this line and the Minkowski normal to $p(u, v)$ at p, and $\varrho^{-1}(du, dv)$ is the curvature $\varkappa$ of the intersection of this plane with $p(u, v)$ at p. (57) generalizes a well known relation of ordinary differential geometry (compare [23, p. 59]).

Now let $\overline{R}$ be an arbitrary associated space to R. If $p(u, v)$ represents again a surface of class C^1, and $\overline{n}(u, v)$ is the euclidean unitvector perpendicular to p at (u, v) so oriented that $| p_u, p_v, \overline{n} | > 0$, then

$$d\overline{A} = | p_u, p_v, \overline{n} |\, du\, dv \tag{58}$$

is the euclidean area element of p.

If $\sigma(u)$ is defined as in section 5, then

$$dA = | p_u, p_v, \overline{n} |\, \sigma(\overline{n})\, du\, dv \tag{59}$$

is the Minkowski area element of p. The supporting plane H of T parallel to the tangent plane of p at (u, v) has euclidean distance $\sigma(\overline{n})\, \sigma^{-1}$ from the origin z (see (23)). If w is a vector which leads from z to a point of $H \frown T$, then

$$| p_u, p_v, \overline{n} |\, \sigma(\overline{n}) = | p_u, p_v, w |\, \sigma\ .$$

If $| p_u, p_v, w |_m$ denotes the Minkowski volume of the box spanned by p_u, p_v, w then we can write dA in the form

$$dA = | p_u, p_v, w |\, \sigma\, du\, dv = | p_u, p_v, w |_m\, du\, dv\ . \tag{60}$$

This relation suggests that the isoperimetrix will be the appropriate surface to use in order to obtain some of the results connected with the sphere as carrier of the spherical image. In carrying this idea out we use the results of Müller [24], but follow the representation of Duschek [25], where the steps omitted in the following calculations are found.

First let S be any convex surface with z as center of class C^2 with positive Gauss curvature and choose the vector representation $q(u, v) = (\bar{x}(u, v), \bar{y}(u, v), \bar{z}(u, v))$ of S such that the tangentplanes of q and p are parallel for the same values u, v. Then

$$\begin{aligned} p_u &= a\, q_u + b\, q_v \ , \\ p_v &= c\, q_u + d\, q_v \ . \end{aligned}$$

We investigate those directions at a given point (u, v) which are parallel to their images on q: that is for which

$$dp = R\, dq \ . \tag{61}$$

Denoting by r any vector with $|\, r\, q_u\, q_v\,| \neq 0$ we multiply (61) first by $r \times q_n$, then by $r \times q_v$. Elimination of R from the relations thus obtained yields

$$|\, r\, p_u\, q_u\,|\, du^2 + (|\, r\, p_u\, q_v\,| + |\, r\, p_v\, q_u\,|)\, du\, dv + |\, r\, p_v\, q_v\,|\, dv^2 = 0 \tag{62}$$

as equation for the directions du, dv which satisfy (61). The corresponding values R_1^q, R_2^q of R are given by the relation

$$|\, r\, p_u\, p_v\,| - R(|\, r\, p_u\, q_v\,| + |\, r\, q_u\, p_v\,|) + R^2\, |\, r\, q_u\, q_v\,| = 0 \ . \tag{63}$$

Then

$$K = (R_1^q)^{-1}\, (R_2^q)^{-1} = |\, r\, q_u\, q_v\,| \,/\, |\, r\, p_u\, p_v\,| = (a d - b c)^{-1} \tag{64}$$

$$\begin{aligned} H = \tfrac{1}{2}\, [(R_i^q)^{-1} + (R_2^q)^{-1}] = (\,|\, r\, p_u\, q_u\,| \,+ \\ + |\, r\, q_u\, p_v\,|\,) \,/\, 2\, |\, r\, p_u\, p_v\,| = (a + d)\, 2^{-1}\, (a d - b c)^{-1} \ . \end{aligned} \tag{65}$$

The R_1^q can also be determined as follows : Let I and J be the indicatrices of p and q at (u, v). If J is not homothetic to I then exactly two values $|\, R_i^q\, |^2$ of R^2 exist for which $R^2 J$ is tangent to I. The points of contact lie along conjugate directions of both I and J. The directions du, dv defined by (62) are the only directions which are simultaneously conjugate for I and J.

R_1^q and R_2^q are undefined if I is homothetic to J ; that is $I = R^2\, J$ for some R^2. In that case we put $R_i^q = R$.

Finally it follows in the usual way (see [23, p. 63]) that the lines on p defined by the differential equation (62) are characterized by the property that the lines $p(u, v) + \lambda\, q(u, v)$ parallel to $q(u, v)$ through $p(u, v)$ form developable surfaces.

As ratios of distances measured along parallel lines the R_i^q have a Minkowski meaning. Also, the normal sections of p and q at (u, v) in the directions (du_i, dv_i) defined by (62) lie in parallel planes, so that by (41) and (57), if $\varrho(du_i, dv_i; p)$ and $\varrho(du_i, dv_i; q)$ denote the Minkowskian radii of curvature of p and q,

$$R_i^q = \varrho(du_i, dv_i; p) \,/\, \varrho(du_i, dv_i; q) \ . \tag{66}$$

(60) and (64) show that K is the ratio of corresponding area elements of p and q. We would naturally like K to be the ratio of the area of either the unit sphere or the isoperimetrix and p. The relation (60) points to T, and (68) will corroborate this choice. Using T presupposes that T is of class C^2 and has positive Gauss curvature, which will be assumed here. The results can be carried over with some modifications to general T if the methods of Fenchel and Jessen in [26] are used. The author intends to take this up in another paper.

With T as S the vector $q = w$ is by (24) transversal to the tangent plane of p at (u, v), so that $p(u, v) \to w(u, v)$ is the mapping of p on T by parallel transversals. We call principal directions on p those which are parallel to their images on T. Summarizing our results we have

(67) *The principal directions are simultaneously conjugate on p and T. If the corresponding principal radii R_i are defined by*

$$p_u\, du_i + p_v\, dv_i = R_i(w_u\, du_i + w_v\, dv_i)$$

then

$$R_i = \varrho(du_i, dv_i; p) \,/\, \varrho(du_i, dv_i; w) \ .$$

If dA_T denotes the area element of T, then

$$K = R_1^{-1}\, R_2^{-1} = dA_T \,/\, dA \ . \tag{68}$$

The lines on $p(u, v)$ whose directions are everywhere principal are characterized by the property that the transversals of $p(u, v)$ along these lines form developable surfaces.

(68) is not the only reason for distinguishing T among all S. Duschek [25] shows that for any variation

$$\delta p = \varepsilon_1 \cdot p_u + \varepsilon_2 \cdot p_v + \varepsilon_3 \cdot q \ , \qquad \varepsilon_i = \varepsilon\, \lambda_i(u, v) \ ,$$

of p the corresponding variation of $\iint |\, q\, p_u\, p_v \,|\, du\, dv$ is

$$\delta \iint | q p_u p_v | \, du \, dv = \oint | q \, \delta p \, dp | + \iint \varepsilon_3 \big(| q p_u q_v | + | q q_u p_v | \big) \, du \, dv \quad (69)$$

If S is the isoperimetrix T or $q = w$ then (60), (65), and (69) yield

$$\delta A = \oint | w \, \delta p \, dp |_m + \iint (R_1^{-1} + R_2^{-1}) \, \varepsilon_3 \, dA_T \quad (70)$$

a formula which is surprisingly similar to the euclidean expression for the first variation of the area (compare [23, p. 173]).

(70) yields

(71) *The first variation of the area of a surface in a Minkowski-space vanishes if and only* if $R_1^{-1} + R_2^{-1} = 0$.

These Minkowskian minimal surfaces have many other properties similar to those of ordinary minimal surfaces: their area can be represented by an integral along the boundary (see [9, p. 205] and [25, p. 6]). The asymptotic directions of a minimal surface form with the (imaginary) asymptotic directions at the corresponding point of T a harmonic quadruple. The transversals of a minimal surface form a Ribaucour congruence etc. For the last two statements compare [24].

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