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On simply connected, 4-dimensional polyhedra

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1. Introduction. Our main purpose is to show that the homotopy type of a simply connected, 4-dimensional polyhedron is completely determined by its inter-related co-homology rings, mod. m $(m = 0, 1, 2, \dots)$, together with one additional element of structure. The latter is defined in terms of a product, which was introduced by L. Pontrjagin[1]), and which has recently been studied in greater generality by N. E. Steenrod[2]). What we want here is Pontrjagin's method of associating a $2n$-dimensional co-homology class, $\mathfrak{p}x$, mod. $4r$, with every n-dimensional co-homology class, x, mod. $2r$. We shall call $\mathfrak{p}x$ the *Pontrjagin square* of x. If f is a co-cycle, mod. $2r$, in the co-homology class x, then $\mathfrak{p}x$ is represented by the co-chain which, in Steenrod's notation, is written as

$$f \cup f + f \cup_1 \delta f \ .$$

The co-homology rings of a polyhedron, P, with integers reduced mod. m $(m = 0, 1, 2, \dots)$ as coefficients, may be combined into a single ring by a method due to M. Bockstein[3]). We give this ring additional algebraic structure by introducing a certain operator Δ and also the Pontrjagin squares. We describe the result as the co-homology ring of P and prove that:

(1) any such ring, which satisfies the general algebraic conditions appropriate to a finite, simply connected polyhedron of at most four dimensions, can be realized geometrically. That is to say, it is possible to construct a polyhedron of this nature, whose co-homology ring is "properly" isomorphic to the given ring. Also

(2) two such polyhedra are of the same homotopy type if, and only if, their co-homology rings are properly isomorphic.

An example is given at the end of § 12 of two simply connected, 4-dimensional polyhedra, which are not of the same homotopy type though

1) See reference [1].

2) See [2].

3) See [4].

their co-homology rings are isomorphic if the Pontrjagin squares are ignored. They can be distinguished from each other by their third homotopy groups or by their Pontrjagin squares.

A note on the steps in the proof of the second of the two results stated above may be helpful. Let K and L be two simply connected, 4-dimensional polyhedra and let $R(K)$ and $R(L)$ be their co-homology rings, as we are using the term. Let $f^*: R(L) \to R(K)$ be what we call a "proper homomorphism" of $R(L)$ into $R(K)$. Then we prove that f^* can be "realized geometrically" by a map $f: K \to L$. That is to say, there is a map, $f: K \to L$, which induces the homomorphism f^*. The result (2) above then follows from the following theorem. Let K and L be two simply connected complexes, of any dimensionality, and let $f: K \to L$ be a map, which induces an isomorphism of each co-homology group, $H^n(L)$, with integral coefficients, onto the corresponding group, $H^n(K)$. Then K and L are of the same homotopy type and f is a homotopy equivalence[4]). This theorem shows the importance of the "realizability" of a given homomorphism $f^*: R(L) \to R(K)$. The conditions which we impose on the co-homology ring are designed to ensure that every proper homomorphism, $f^*: R(L) \to R(K)$, can be realized geometrically.

We consider this matter of realizability in two stages. We first confine ourselves to the additive group of the co-homology ring, or rather to the "spectrum" of co-homology groups, of which the additive group of the ring is the "finitely generated" direct sum. By the spectrum we mean the aggregate of all the absolute and modular co-homology groups, related by the homomorphisms, Δ, μ, defined in § 2 below[5]). We first consider the realizability of "proper" homomorphisms of the co-homology spectra by co-chain maps[6]), not by geometrical maps. It is obvious from the definition of Δ and μ that any homomorphism of the co-homology groups of L into those of K commutes with Δ and μ if it is the homomorphism induced by a co-chain map. Lemma 4, at the end of § 2, states the converse of this. This is valid for arbitrary (finite) complexes K, L, but the resulting co-chain map cannot, in general, be realized by a geometrical map $K \to L$. The step from the cochain map to the geo-

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[4]) i. e. there is a map $g: L \to K$ such that $gf \simeq 1$, $fg \simeq 1$, where $\simeq$ denotes the relation of homotopy and 1 denotes the identical map, both in K and in L.

[5]) Our μ includes and can be defined in terms of Bockstein's π and ω but our Δ is an additional element of algebraic structure.

[6]) i. e. homomorphisms of the groups of co-chains which commute with the co-boundary operator (cf. [5] and pp. 145 et seq. in [6]).

metrical map depends on the multiplicative structure of the rings, including the Pontrjagin squares, and the special nature of the complexes.

2. The co-homology groups. This section is concerned exclusively with additive properties. It is purely algebraic and depends only on a sequence, $C = \{C^n\}$ $(n = 0, 1, \ldots)$ of additive Abelian groups, related by a "co-boundary" homomorphism, $\delta : C^n \to C^{n+1}$, for each n, such that $\delta\delta = 0$. We assume that each C^n is a free Abelian group of finite rank, no two having an element in common, and that $C^n = 0$ for all sufficiently large values of n. We use the language of co-homology, but this is purely conventional. To translate this section into the language of homology we have only to delete the prefix "co" and re-write C^n as C_{N-n}, where N is such that $C^n = 0$ if $n > N$.

Let $H^n(m)$ be the n-dimensional co-homology group, defined in terms of C and δ, with integers reduced mod. m as coefficients ($m = 0, 1, 2, \ldots$; $H^n(1) = 0$). We shall write $H^n(0) = H^n$, the co-homology group with integers as coefficients. For convenience at a later stage we agree that all the groups $H^n(m)$ have the same zero element. Given $p > 0$, $q \geqslant 0$, we define operators,

$$\Delta_q : H^n(q) \to H^{n+1} \quad (q > 0) \;, \qquad \mu_{p,q} : H^n(q) \to H^n(p) \;,$$

as follows. Let $x \in H^n(q)$ and $x' \in x$. That is to say x' is a co-cycle, mod. q, in the co-homology class x. Then $\delta x' = qy'$, where y' is an (absolute) co-cycle. We define $\Delta_q x$ as the (absolute) co-homology class of y'. Thus we could write $\Delta_q = (1/q)\delta$. Let $c = (p, q)$. Then

$$\delta\left(\frac{p}{c}x'\right) = \frac{pq}{c}y' = p\left(\frac{q}{c}y'\right) . \tag{2.1}$$

Therefore $(p/c)x'$ is a co-cycle, mod. p, and we define $\mu_{p,q}x$ as its co-homology class mod. p. It is easily verified that $\Delta_q x$ and $\mu_{p,q}x$ depend only on $x \in H^n(q)$ and not on the particular choice of $x' \in x$. They are obviously homomorphisms. If $p \mid q$, in which case $c = p$, and in particular if $q = 0$, then $\mu_{p,q}x$ is the co-homology class of the same co-cycle x', but calculated mod. p instead of mod. q. Thus $\mu_{m,0}$ is the natural homomorphism of H^n onto $H^n_m = H^n - mH^n$. We shall sometimes write Δ_q, $\mu_{p,q}$ simply as Δ, μ.

As an immediate consequence of (2.1) we have

$$\Delta_p \mu_{p,q} = \frac{q}{(p, q)} \Delta_q . \tag{2.2}$$

Also I say that, if $p, q > 0$, $r \geqslant 0$, then

$$\mu_{p,q}\,\mu_{q,r} = \frac{q\,(p\,,r)}{(p\,,q)\,(q\,,r)}\,\mu_{p,r}\;. \tag{2.3}$$

For let $x \varepsilon H^n(r)$, $x' \varepsilon x$, and let $a = (q\,,r)$, $b = (r\,,p)$, $c = (p\,,q)$. Then $\mu_{q\,,r} x \varepsilon H^n(q)$ is represented by $y' = (q/a)\,x'$ and $\mu_{p\,,q}\mu_{q\,,r} x \varepsilon H^n(p)$ by $(p/c)\,y' = (pq/ca)\,x'$. On the other hand $\mu_{p\,,r} x \varepsilon H^n(p)$ is represented by $(p/b)\,x'$. Let $d = (p\,,q\,,r) = (c\,,a)$. Then $(ca/d) \mid q$, since $c \mid q$, $a \mid q$, $d = (c\,,a)$. Also $d \mid b$. Therefore $ca \mid qb$. That is to say qb/ca is an integer. But

$$\frac{p\,q}{c\,a}\,x' = \frac{q\,b}{c\,a}\cdot\frac{p}{b}\,x'\;,$$

whence both sides of (2.3), operating on x, are represented by the same co-cycle, mod. p.

The union of all the groups, $H^n(m)$, related by the homomorphisms, Δ, μ, will be called the *co-homology spectrum* of the set of groups C. We shall denote it by H. Notice that H is not the direct sum of the groups $H^n(m)$. An element of H is an element of an arbitrary one of the groups $H^n(m)$.

Let $\nu : H^n \to H^n$ be the endomorphism defined by $\nu\,x = m\,x$, for a given value of m. Let $\mu = \mu_{m,\,o}$, $\Delta = \Delta_m$. Then the sequence of homomorphisms

$$\cdots \xrightarrow[\Delta]{} H^n \xrightarrow[\nu]{} H^n \xrightarrow[\mu]{} H^n(m) \xrightarrow[\Delta]{} H^{n+1} \xrightarrow[\nu]{} \cdots \tag{2.4}$$

is exact, meaning that the kernel of each is the image group of the one preceding [7]) it. For it follows at once from the definition of ν, μ, Δ that $\nu\Delta = 0$, $\mu\nu = 0$, $\Delta\mu = 0$. Conversely, let $\nu x = m x = 0$, where $x \,\varepsilon\, H^n$, and let $x' \,\varepsilon\, x$. Then $m x' \sim 0$. That is to say $m x' = \delta u'$, for some $u' \,\varepsilon\, C^{n-1}$. Then u' is a co-cycle, mod. m, and $x = \Delta u$, where $u \,\varepsilon\, H^{n-1}(m)$ is the co-homology class of u'. Therefore $\nu^{-1}(0) = \Delta H^{n-1}(m)$.

Let $\mu x = 0$, where $x \,\varepsilon\, H^n$, and let $x' \,\varepsilon\, x$. Then x' is an absolute co-cycle, and μx is its co-homology class mod. m. Since $\mu x = 0$ there are co-chains $u' \,\varepsilon\, C^n$, $v' \,\varepsilon\, C^{n-1}$ such that $x' = m u' + \delta v'$. Then $m\,\delta u' = 0$. Since C^{n+1} is a free Abelian group it follows that $\delta u' = 0$. Therefore u' is an absolute co-cycle and $m u' \sim x'$. Therefore $x = m u$, where $u \,\varepsilon\, H^n$ is the co-homology class of u'. Therefore $\mu^{-1}(0) = \nu H^n$.

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[7]) H^0 is the (free Abelian) group of co-cycles in C^0, whence $\nu : H^0 \to H^0$ is an isomorphism of H^0 into itself.

Let $\Delta x = 0$, where $x \varepsilon H^n(m)$, and let $x' \varepsilon x$. Let $\delta x' = mu'$. Then $u' \varepsilon \Delta x$ and since $\Delta x = 0$ we have $u' = \delta v'$, for some $v' \varepsilon C^n$. Therefore $\delta x' = m\, \delta v'$, or $\delta(x' - mv') = 0$. That is to say $x' - mv'$ is an absolute co-cycle. Let y be its co-homology class. Then $x = \mu y$, since $x' = x' - mv'$, mod. m. Therefore $\Delta^{-1}(0) = \mu H^n$, which completes the proof that (2.4) is exact [8]).

It follows from the exactness of the sequence (2.4) that $\Delta_m H^n(m) = {}_mH^{n+1}$, the sub-group of H^{n+1} consisting of all the elements, $x \varepsilon H^{n+1}$, such that $mx = 0$. The group $H^n(m)$ $(m > 0)$ is a trivial extension of [9]) $\mu_{m,o} H^n = H^n_m$ by ${}_mH^{n+1}$. Hence there is an isomorphism Δ^*_m: ${}_mH^{n+1} \to H^n(m)$ (into, not necessarily onto) such that $\Delta_m \Delta^*_m = 1$. We shall describe Δ^*_m in detail, but first observe that, in consequence of this, $H^n(m)$ is the direct sum

$$H^n(m) = H^n_m + \Delta^*({}_mH^{n+1}) \;, \tag{2.5}$$

whence

$$H^n(m) = \theta_m(H^n + {}_mH^{n+1}) \;, \tag{2.6}$$

where $\theta_m \mid H^n = \mu_{m,o}$, $\theta_m \mid {}_mH^{n+1} = \Delta^*_m$. It will be convenient to allow $m = 0$ in (2.6), with $\theta_0 \mid H^n = 1$, $\theta_0 H^{n+1} = 0$ (N. B. ${}_0H^{n+1} = H^{n+1}$), but we do not define Δ^*_0. We shall often write Δ^*_m and θ_m as Δ^* and θ.

We may subject the operator Δ^*, which is not determined uniquely, to the condition

$$\mu_{p,q} \Delta^*_q\, y = \Delta^*_p \left\{ \frac{q}{(p\,,q)}\, y \right\} \qquad (y \,\varepsilon\, {}_qH^{n+1}) \;. \tag{2.7}$$

To prove this we recall that C^{n+1} and hence H^{n+1} has a finite number of generators. Therefore H^{n+1} is a direct sum of a free Abelian group and finite cyclic groups $U_1, \ldots, U_t$. Obviously

$${}_mH^{n+1} = {}_mU_1 + \cdots + {}_mU_t$$

for any $m > 0$. Therefore it is enough to prove (2.7) with $y \,\varepsilon\, {}_qU_i$ for an arbitrary value of $i = 1, \ldots, t$. Let us write $U_i = U$, let u be a generator of U and σ its order. Let $u' \varepsilon u$. Since $\sigma u = 0$ there is an n-dimensional co-chain, v', such that $\delta v' = \sigma u$. Let $v \,\varepsilon\, H^n(\sigma)$ be the

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[8]) It can be proved in much the same way that the sequence $\ldots, \mu_{lm,l}, \mu_{m,lm}, \mu_{l,o}\Delta_m, \mu_{lm,l}, \ldots$ is exact for any $l, m > 0$. If $l = 0$ this is the same as (2.4), with the convention that $\mu_{lm,l} = \nu$, $\mu_{l,o} = 1$ if $l = 0$.

[9]) Cf. [7], pp. 218 et seq. A proof of this, which is in any case elementary, is included in the proof of (2.7) below.

co-homology class of v'. Then $\Delta_\sigma v = u$. Let $ku \in {}_mU$. Since $mku = 0$ we have $mk \equiv 0$, mod. σ, whence $\sigma \mid k(m, \sigma)$. We define

$$\Delta_m^* \, k \, u = \frac{k\,(m\,,\sigma)}{\sigma} \, \mu_{m,\sigma} \, v \quad . \tag{2·8}$$

Then it follows from (2.2) that

$$\begin{aligned} \Delta_m \Delta_m^* \, k \, u &= \frac{k\,(m\,,\sigma)}{\sigma} \, \Delta_m \, \mu_{m,\sigma} \, v \\ &= \frac{k\,(m\,,\sigma)}{\sigma} \cdot \frac{\sigma}{(m\,,\sigma)} \, \Delta_\sigma \, v \\ &= k \, u \ . \end{aligned}$$

Therefore $\Delta_m \, \Delta_m^* = 1$. Let $p, q > 0$ be given and let $a = (q, \sigma)$, $b = (\sigma, p)$, $c = (p, q)$. Let $ku \in {}_qU$ and let $k' = kq/c$. Then $pk'u = (p/c)qku = 0$, whence $k'u \in {}_pU$. It follows from (2.8) and (2.3) that

$$\begin{aligned} \mu_{p,q} \Delta_q^* \, k \, u &= \frac{k\,(q\,,\sigma)}{\sigma} \, \mu_{p,q} \, \mu_{q,\sigma} \, v \\ &= \frac{k\,a}{\sigma} \cdot \frac{q\,b}{c\,a} \, \mu_{p,\sigma} \, v \\ &= \frac{k\,q}{c} \cdot \frac{b}{\sigma} \, \mu_{p,\sigma} \, v \\ &= \frac{k'\,(p\,,\sigma)}{\sigma} \, \mu_{p,\sigma} \, v \\ &= \Delta_p^* \, k' \, u \\ &= \Delta_p^* \left\{ \frac{q}{(p\,,q)} \, k \, u \right\} , \end{aligned}$$

which establishes (2.7).

Since $\theta_p \mid H^n = \mu_{p,o}$, $\theta_p \mid {}_pH^{n+1} = \Delta_p^*$, $\Delta_p \mu_{p,o} = 0$ and $\Delta_p \Delta_p^* = 1$ $(p > 0)$ we have

$$\begin{aligned} \Delta_p \, \theta_p (x + y) &= \Delta_p \, \mu_{p,o} \, x + \Delta_p \Delta_p^* \, y \\ &= y \ , \end{aligned} \tag{2.9}$$

where $x \in H^n$, $y \in {}_pH^{n+1}$. Also, if $x \in H^n$, $y \in {}_qH^{n+1}$ $(q > 0)$ it follows from (2.3), with $r = 0$, and (2.7) that

$$\begin{aligned}\mu_{p,q}\,\theta_q(x+y) &= \mu_{p,q}\,\mu_{q,o}\,x + \mu_{p,q}\,\Delta_q^*\,y \\ &= \frac{p}{(p\,,q)}\,\mu_{p,0}\,x + \Delta_p^*\left\{\frac{q}{(p\,,q)}\,y\right\} \\ &= \theta_p\left\{\frac{p}{(p\,,q)}\,x + \frac{q}{(p\,,q)}\,y\right\}\,. \qquad (2.10)\end{aligned}$$

Thus Δ and[10]) μ are expressible in terms of Δ^* and $\mu_{m,o}$. The conditions $\Delta\Delta^* = 1$ and (2.7) are the same as (2.9) and (2.10) with $x = 0$. They may therefore be interpreted as necessary and sufficient conditions, which Δ^* must satisfy in order that the operators Δ, μ, defined in terms of Δ^* and $\mu_{m,o}$ by (2.9), (2.10), shall be the same as those with which we started. Any operator $\Delta^* : {}_mH^{n+1} \to H^n(m)$, which is defined and is a homomorphism for every $m > 0$, $n \geqslant 0$ and satisfies the conditions $\Delta\Delta^* = 1$ and (2.7), will be called an *admissible right inverse* of Δ.

Now let a system of additive Abelian groups, $H^n(m)$ $(m, n{=}0, 1, \ldots;$ $H^n(1) = 0)$ be given abstractly. We assume that each of them has a finite number of generators and also that $H^n(m) = 0$ for all sufficiently large values of n. All the groups $H^n(m)$ shall have the same zero element but no two shall have a non-zero element in common. As before let $H^n = H^n(0)$ and let $H^n(m)$ $(m > 0)$ be related to H^n_m and ${}_mH^{n+1}$ by the equations (2.5), where $\Delta^*_m : {}_mH^{n+1} \to H^n(m)$ is a given isomorphism (into). We admit two possibilities. The first is that Δ, μ are given, satisfying (2.2), (2.3). In this case we require Δ^* to satisfy $\Delta\Delta^* = 1$ and (2.7), and hence (2.9) and (2.10). The second possibility is that Δ, μ are not given, in which case we define them by (2.9) and (2.10). In either case an admissible right inverse of Δ will mean the same as before and any two admissible right inverses of Δ will be regarded as equivalent. Thus it is Δ and μ which are fundamental, not a particular one of the operators Δ^*.

It is easily verified that (2.2) and (2.3) are formal consequences of (2.9) and (2.10). It also follows formally from (2.9) and (2.10) that the sequences (2.4) and $\ldots, \mu_{lm,l}, \mu_{m,lm}, \mu_{l,o}\Delta_m, \mu_{lm,l}, \ldots$ $(l, m > 0)$ are exact. For example, if $x \,\varepsilon\, H^n$, $y \,\varepsilon\, {}_lH^{n+1}$ it follows from (2.10) that

$$\begin{aligned}\mu_{m,lm}\,\mu_{lm,l}\,\theta_l(x+y) &= \mu_{m,lm}\,\theta_{lm}(m\,x + y) \\ &= \theta_m(mx + ly) \\ &= 0\,,\end{aligned}$$

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[10]) i. e. Δ_q and $\mu_{p,q}$ with $q > 0$. It is always to be understood that $\mu_{m,o}$ is the natural homomorphism $\mu_{m,o} = \theta_m \mid H^n : H^n \to H^n_m$ $(n = 0, 1, \ldots)$.

since $\theta_m m H^n = 0$, $y \varepsilon {}_l H^{n+1}$. Conversely, let

$$0 = \mu_{m,lm} \theta_{lm}(x + y) = \theta_m(x + ly) \ ,$$

where $x \varepsilon H^n$, $y \varepsilon {}_{lm}H^{n+1}$. Then $x = mu$ $(u \varepsilon H^n)$ and $ly = 0$, since Δ_m^* is an isomorphism. Therefore $y \varepsilon {}_l H^{n+1}$ and it follows from (2.10) that

$$\mu_{lm,l} \theta_l(u + y) = \theta_{lm}(mu + y) = \theta_{lm}(x + y) \ .$$

Therefore $\mu_{m,lm}^{-1}(0) = \mu_{lm,l} H^n(l)$. The other relations follow from similar arguments.

The system of groups $H^n(m)$, related in this way by Δ and μ, will be called a *spectrum of co-homology groups* or simply a *spectrum*. As before an element in such a spectrum will be an element in an arbitrary one of the groups $H^n(m)$. Notice that the whole spectrum may be defined in terms of the groups H^n, in case we are given these groups alone. For $H^n(m)$ may be defined by (2.5), where Δ_m^* is an isomorphism of ${}_mH^{n+1}$ onto a newly defined group $\Delta_m^*({}_mH^{n+1})$ $(m > 0)$.

By a proper homomorphism $f : H \to \bar{H}$ of a spectrum, H, into a spectrum, $\bar{H}$, we mean a transformation such that $f \mid H^n(m)$ is a homomorphism of $H^n(m)$ into $\bar{H}^n(m)$, for all values of m, n, and $f\Delta = \Delta f$, $f\mu = \mu f$. If also $f \mid H^n(m)$ is an isomorphism onto $\bar{H}^n(m)$ for all values of m, n, then f will be called a *proper isomorphism* of H onto $\bar{H}$ and H will be described as *properly isomorphic* to $\bar{H}$. If $f : H \to \bar{H}$ is a proper isomorphism of H onto $\bar{H}$, then its inverse, f^{-1}, given by $f^{-1} \mid \bar{H}^n(m) = \{f \mid H^n(m)\}^{-1}$, is a proper isomorphism of $\bar{H}$ onto H. For $f\Delta = \Delta f$, $f\mu = \mu f$ obviously imply $f^{-1}\Delta = \Delta f^{-1}$, $f^{-1}\mu = \mu f^{-1}$. A proper homomorphism $f : H \to \bar{H}$ is a (proper) isomorphism onto if, regarded simply as a transformation of the set of elements in H into those of $\bar{H}$, it is onto and $(1 - 1)$. It is therefore a proper isomorphism if there is a transformation, $f' : \bar{H} \to H$, such that $f'f = 1$, $ff' = 1$, in which case $f' = f^{-1}$.

Since an admissible right inverse, Δ^*, of Δ is not, in general, uniquely defined, a proper homomorphism, $f : H \to \bar{H}$, does not necessarily commute with Δ^*. But let Δ^* be given in H and let $f_0 : H^n \to \bar{H}^n$ be the homomorphism $f \mid H^n$ for each value of n. Then we have the lemma

Lemma 1. *If each* $f_0 : H^n \to \bar{H}^n$ *is an isomorphism onto* $\bar{H}^n$, *then there is an admissible right inverse,* $\bar{\Delta}^*$, *of* Δ *in* $\bar{H}$, *such that* $\bar{\Delta}^* f_0 = f\Delta^*$.

Let $\bar{\Delta}^* : {}_m\bar{H}^{n+1} \to \bar{H}^n(m)$ be defined by $\bar{\Delta}^* \bar{y} = f \Delta^* f_0^{-1} \bar{y}$, for each $\bar{y} \varepsilon {}_m\bar{H}^{n+1}$. Since $\bar{\Delta} f = f \Delta = f_0 \Delta$, where $\bar{\Delta}$ denotes the operator Δ in $\bar{H}$, and since $\Delta \Delta^* = 1$ in H, we have

$$\bar{\Delta}\bar{\Delta}^* = \bar{\Delta} f \Delta^* f_0^{-1} = f_0 \Delta\Delta^* f_0^{-1} = f_0 f_0^{-1} = 1 \ .$$

Similarly (2.7), in $\bar{H}$, follows from $\mu f = f \mu$ and (2.7) in H. Therefore $\bar{\Delta}^*$ is an admissible right inverse of $\bar{\Delta}$. Obviously $\bar{\Delta}^* f_0 = f \Delta^*$.

Lemma 2. *Let $f : H \to \bar{H}$ be a proper homomorphism such that $f \mid H^n$ is an isomorphism onto $\bar{H}^n$ for every value of n. Then f is a proper isomorphism of H onto $\bar{H}$.*

It follows from Lemma 1 that $f \Delta^* = \bar{\Delta}^* f_0$ with a suitable choice of $\bar{\Delta}^*$ in $\bar{H}$. Let θ mean the same as in (2.6) and let $\bar{\theta}$ be the corresponding homomorphism in $\bar{H}$. Since $\theta_m \mid H^m = \mu_{m,o}$, $\theta_m \mid {}_mH^{n+1} = \Delta_m^*$, with similar relations in $\bar{H}$, and since $f \mu_{m,o} = \mu_{m,o} f_0$, $f \Delta^* = \bar{\Delta}^* f_0$ we have

$$f\, \theta_m(x + y) = \bar{\theta}_m(f_0 x + f_0 y) \qquad (x \varepsilon H^n, \ \ y \varepsilon {}_mH^{n+1}) \ . \tag{2.11}$$

Let $f' : \bar{H} \to H$ be defined by

$$f'\, \bar{\theta}_m(\bar{x} + \bar{y}) = \theta_m(f_0^{-1} \bar{x} + f_0^{-1} \bar{y}) \qquad (\bar{x} \varepsilon \bar{H}^n, \ \ \bar{y} \varepsilon {}_m\bar{H}^{n+1}) \ .$$

Then

$$\begin{aligned} f' f\, \theta_m(x + y) &= f'\, \bar{\theta}_m(f_0\, x + f_0\, y) \\ &= \theta_m(x + y) \ , \end{aligned}$$

whence $f' f = 1$. Similarly $f f' = 1$. Therefore f is a proper isomorphism of H onto $\bar{H}$, and the lemma is established.

Lemma 3. *Any set of homomorphisms $f_0 : H^n \to \bar{H}^n$ $(n = 0, 1, \dots)$ can be extended to a proper homomorphism $f : H \to \bar{H}$.*

For $f \mid H^n(m)$ $(m > 0)$ may be defined by (2.11). Then it follows from (2.9) and (2.10) that $\Delta f = f_0 \Delta$, $\mu f = f \mu$. This establishes the lemma.

Let $H^n(m)$ be defined in terms of co-chain groups C^n $(n = 0, 1, \dots)$ and an operator δ, as at the beginning of the section. Let $\bar{H}^n(m)$ be similarly defined in terms of co-chain groups $\bar{C}^n$. We shall use j_m for the natural homomorphism of the group of co-cycles, mod. m, in C^n onto $H^n(m)$. Thus if $a \varepsilon C^n$ is a co-cycle, mod. m, then $j_m a \varepsilon H^n(m)$ is its co-homology class. We shall also use j_m to denote the natural homomorphism of co-cycles, mod. m, in $\bar{C}^n$ onto $\bar{H}^n(m)$. By a *co-chain map, or mapping*, of C^n into $\bar{C}^n$ we shall mean a set of homomorphisms $g: C^n \to \bar{C}^n$ $(n = 0, 1, \ldots)$ such that $g\delta = \delta g$, where δ also denotes the co-boundary operator in $\bar{C}^n$. A co-chain map, g, obviously induces a proper homomorphism, $f: H \to \bar{H}$, which is defined by $f j_m a = j_m g a$, where a is any co-cycle mod. m. Subject to this condition, which we write simply as $fj = jg$, we say that f is *realized* by the co-chain map g.

Lemma 4. *Any proper homomorphism $f: H \to \bar{H}$ can be realized by a co-chain map $g: C^n \to \bar{C}^n$.*

As in the theory of finite complexes, there is a basis $a_1, \ldots, a_q$, for each group C^n, such that $\delta a_i = \sigma_i b_i$ $(i = 1, \ldots, t \leqslant q)$, $\delta a_i = 0$ $(i = t+1, \ldots, q)$, where $\sigma_i \geqslant 1$ and $b_1, \ldots, b_t$ are basis elements in the basis for C^{n+1}. We write $\delta a_i = \sigma_i b_i$ $(i = 1, \ldots, q)$, with $\sigma_i = 0$, $b_i = 0$ if $i > t$. A co-chain $\lambda_1 a_1 + \cdots + \lambda_q a_q$ is a co-cycle mod. m $(m > 0)$ if, and only if, $\lambda_i \sigma_i \equiv 0$, mod. m, for each $i = 1, \ldots, q$. This is equivalent to $\lambda_i \equiv 0$, mod. ϱ_i, where[11] $\varrho_i = m/(m, \sigma_i)$. Therefore the group of co-cycles mod. m is generated by $\varrho_1 a_1, \ldots, \varrho_q a_q$ and $j_m \varrho_1 a_1, \ldots, j_m \varrho_q a_q$ generate $H^n(m)$. Notice that

$$j_m \varrho_i a_i = \mu_{m,\sigma_i} j_{\sigma_i} a_i \ , \tag{2.12}$$

according to the original definition of μ. I say that a cochain map, $g: C^n \to \bar{C}^n$, realizes f provided only that

$$f j_{\sigma_i} a_i = j_{\sigma_i} g a_i \qquad (i = 1, \ldots, q) \ , \tag{2.13}$$

the analogous conditions being satisfied for all values of n. For, writing $\varrho_i = \varrho$, $\sigma_i = \sigma$, $a_i = a$, it follows from the relations (2.12) and from $f\mu = \mu f$ and (2.13) that

$$\begin{aligned} f j_m \varrho a &= f \mu_{m,\sigma} j_\sigma a = \mu_{m,\sigma} f j_\sigma a \\ &= \mu_{m,\sigma} j_\sigma g a = j_m \varrho g a \\ &= j_m g \varrho a \ , \end{aligned}$$

or that $fj = jg$.

Let N be such that $C^r = 0$ if $r > N$, let $n \leqslant N$ and, starting with $g C^r = 0$ if $r > N$, assume that $g: C^k \to \bar{C}^k$ has been defined for each

[11]) $(m, o) = m$, whence $\varrho_i = 1$ if $i > t$.

$k > n$ in such a way that $fj = jg$, $g\delta = \delta g$. We proceed to define g: $C^n \to \overline{C}^n$. As before we write $a_i = a$, $\sigma_i = \sigma$, and also $\delta a = \sigma b$ for an arbitrary value of $i = 1, \ldots, q$ $(\sigma = 0, b = 0$ if $i > t)$. If $i > t$, then a is an absolute co-cycle and $j_0 a \varepsilon H^n$. In this case we write $g a = \overline{a}$, when $\overline{a}$ is any (absolute) co-cycle in the co-homology class $f j_0 a$. Then $g \delta a = \delta g a = 0$ and $j_0 g a = j_0 \overline{a} = f j_0 a$. Let $i \leqslant t$, so that $\varrho \neq 0$, $b \neq 0$. Then b is an absolute co-cycle and $j_0 b \varepsilon {}_\sigma H^{n+1}$, since $\sigma b = \delta a$. Also $g b$ is defined, since $b \varepsilon C^{n+1}$, and $j_0 g b = f j_0 b \varepsilon {}_\sigma \overline{H}^{n+1}$. Therefore $\sigma g b = \delta \overline{a}$, for some $\overline{a} \varepsilon \overline{C}^n$. According to the original definition of Δ we have $\Delta_\sigma j_\sigma a = j_0 b$, $\Delta_\sigma j_\sigma \overline{a} = j_0 g b$. Since $j_0 g b = f j_0 b$ and since $f \Delta = \Delta f$ it follows that

$$\begin{aligned} \Delta_\sigma j_\sigma \overline{a} &= j_0 g b = f j_0 b = f \Delta_\sigma j_\sigma a \\ &= \Delta_\sigma f j_\sigma a \ . \end{aligned}$$

Therefore $\Delta_\sigma (f j_\sigma a - j_\sigma \overline{a}) = 0$. It follows from the exactness of the sequence (2.4), with $m = \sigma$, that

$$f j_\sigma a - j_\sigma \overline{a} \, \varepsilon \, \mu_{\sigma,0} \overline{H}^n \ .$$

Therefore $f j_\sigma a = j_\sigma \overline{a} + j_\sigma \overline{u} = j_\sigma(\overline{a} + \overline{u})$, where $\overline{u} \varepsilon \overline{C}^n$ is an absolute co-cycle. Let $g a = \overline{a} + \overline{u}$ and define $g a_i$ in this way for each value of $i = 1, \ldots, q$, thus defining $g : C^n \to \overline{C}^n$. Since $j_\sigma g a = j_\sigma(\overline{a} + \overline{u}) = f j_\sigma a$ it follows from our preliminary result that $fj = jg$ in C^n. Also $\delta g a = \delta \overline{a} = \sigma g b = g \sigma b = g \delta a$. Therefore $g \delta = \delta g$. Repeating this construction we define g throughout all the groups C^n $(n = N, N - 1, \ldots, 0)$ and the lemma is established.

3. The co-homology ring. We now assume that the group C^n, in the system C, with which we started in § 2, is the group of n-dimensional co-chains[12]) in a finite simplicial complex K. The additive group of the co-homology ring, R, of K shall be the "finitely generated" direct sum of the co-homology groups $H^n(m)$, for all values of m, n. That is to say, the additive group of R shall consist of all finite sums $x_1 + x_2 + \cdots$, where each x_i is in an arbitrary one of the groups[13]) $H^n(m)$. The product $f \cup g$, which we write simply as fg, of co-chains $f \varepsilon C^p$, $g \varepsilon C^q$ shall be

[12]) We distinguish between a chain and a co-chain, which is to be an integral valued function of chains. All our chains will have integral coefficients.

[13]) As in § 2, the groups $H^n(m)$ all have the same zero element and we regard them as imbedded in the additive group of R.

defined by the Čech-Whitney method[14]) in terms of a fixed ordering of the vertices of K. The product, xy, of elements $x \varepsilon H^p(m)$, $y \varepsilon H^q(m)$, shall be the co-homology class, mod. m, of fg, where $f \varepsilon x$, $g \varepsilon y$. Then

$$x y \varepsilon H^{p+q}(m) \ , \qquad x y = (-1)^{pq} y x \ . \tag{3.1}$$

If $x \varepsilon H^p(r)$, $y \varepsilon H^q(s)$ we define xy as

$$x y = \mu_{c,r}(x) \mu_{c,s}(y) \ , \tag{3.2}$$

where $c = (r, s)$. Thus xy is the co-homology class of fg mod. (r, s), where $f \varepsilon x$, $g \varepsilon y$. We then define the product, xy, of any two elements, $x, y \subset R$, by the condition that xy shall be bilinear. It is easily verified that $(xy)z = x(yz)$ for any $x, y, z \subset R$. This defines R, except for the Pontrjagin squares, which will he discussed in the next section. Clearly $R(m)$, the co-homology ring with integral coefficients reduced mod. m, is a sub-ring of R for every $m \geqslant 0$. Also R has a unit element, $e \varepsilon H^0$, which is the co-homology class of the co-cycle, which has the constant value, unity, at every vertex of K.

Let $A(m)$ denote the additive group of $R(m)$. Then the homomorphisms $\mu_{r,s}$, Δ_s, defined in the last section, may clearly be extended through $A(s)$ to give homomorphisms $\mu_{r,s}: A(s) \to A(r)$, $\Delta_s : A(s) \to A(0)$. I say that, if $x \varepsilon R(r)$, $y \varepsilon R(s)$, then

$$\mu_{m,r}(x) \mu_{m,s}(y) = \frac{m\,(m, r, s)}{(m, r)\,(m, s)} \mu_{m,c}(x y) \ , \tag{3.3}$$

for any $m > 0$, where $c = (r, s)$. Because of the bi-linearity of xy and the linearity of μ it is sufficient to prove this for $x \varepsilon H^p(r)$, $y \varepsilon H^q(s)$. Let this be so and let $f \varepsilon x$, $g \varepsilon y$. Let $a = (m, r)$, $b = (m, s)$, $d = (m, r, s) = (m, c)$. Then $(m/a) f \varepsilon \mu_{m,r} x$, $(m/b) g \varepsilon \mu_{m,s} y$, whence

$$\frac{m^2}{a b} f g \ \varepsilon \ \mu_{m,r}(x) \mu_{m,s}(y) \ .$$

Also

$$(m/d) f g \ \varepsilon \ \mu_{m,c}(xy)$$

and

$$\frac{m d}{a b} \cdot \frac{m}{d} f g = \frac{m^2}{a b} f g \ .$$

Therefore both sides of (3.3) are represented by the same co-cycle,

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[14]) See [8], [9], [10], [11] and [6], chap. V.

mod. m, and (3.3) is established. Notice that (3.3) reduces to (3.2) if $m = c = (r, s)$. For then $m = (m, r) = (m, s) = (m, r, s)$ and $\mu_{m,c} = 1$.

Out of respect for the operator Δ we display the formula

$$\mu_{c,0}\, \Delta_c(xy) = (\Delta_c\, x_c)\, y_c + (-1)^p\, x_c(\Delta_c\, y_c) \ , \tag{3.4}$$

where $x \varepsilon H^p(r)$, $y \varepsilon R(s)$, $c = (r, s)$ and $x_c = \mu_{c,r} x$, $y_c = \mu_{c,s} y$. This follows without difficulty from the relations $\delta(fg) = (\delta f)g + (-1)^p f(\delta g)$ and (2.2), where $f \varepsilon C^p$, $g \varepsilon C^q$. However we shall not need this because of the special nature of our complexes.

4. The Pontrjagin squares. Let K mean the same as in § 3. Let $f \varepsilon C^p$, $g \varepsilon C^q$. Following Steenrod[15]) we use $f \cup_1 g$ to denote the $(p+q-1)$-dimensional co-chain, which was introduced mod. 2 by Pontrjagin[15]), and which is given by

$$(f \cup_1 g)\, \sigma^{p+q-1} = \sum_{j=0}^{p-1} (-1)^{(p-j)(q+1)} (f\, \sigma_j^p)\, (g\, \sigma_j^q) \ ,$$

where $\sigma^{p+q-1} = a_0 \ldots a_{p+q-1}$ is a given $(p+q-1)$ simplex and $\sigma_j^p = a_0 \ldots a_j\, a_{j+q} \ldots a_{p+q-1}$, $\sigma_j^q = a_j \ldots a_{j+q}$. Steenrod[16]) proves that

$$\begin{aligned} \delta(f \cup_1 g) = {} & (-1)^{p+q-1}(fg - (-1)^{pq} gf) \\ & + \delta f \cup_1 g + (-1)^p f \cup_1 \delta g \ . \end{aligned} \tag{4.1}$$

Let

$$\mathfrak{p} f = f^2 + f \cup_1 \delta f \qquad (f^2 = ff = f \cup f) \ , \tag{4.2}$$

where f is any co-chain. If f is a co-cycle, mod. $2r$, then $\mathfrak{p} f$ is a co-cycle mod. $4r$. For let $\delta f = 2ru$. Then, calculating mod. $4r$, we have

$$\begin{aligned} \delta \mathfrak{p} f &= (\delta f) f \pm f(\delta f) + f(\delta f) - (\delta f) f + \delta f \cup_1 \delta f \\ &= 2r(\pm fu + fu) + 4r^2\, u \cup_1 u \\ &= 0 \ . \end{aligned}$$

Let $\delta f = 2ru$, $\delta g = 2rv$, where $f, g \subset C^n$. Then

$$\begin{aligned} \mathfrak{p}(f+g) - \mathfrak{p} f - \mathfrak{p} g &= fg + gf + f \cup_1 \delta g + g \cup_1 \delta f \\ &= fg + gf + 2r(f \cup_1 v + g \cup_1 u) \ . \end{aligned} \tag{4.3}$$

15) [1] and [2].

16) [2], Theorem (5.1), p. 296.

It follows from Steenrod's co-boundary formula[16]), with $i=2$, that $g \cup_1 u \sim u \cup_1 g$ (mod. 2), both g and u being co-cycles mod. 2. Therefore

$$2rg \cup_1 u \sim 2ru \cup_1 g \qquad \text{mod.}\, 4r\ . \tag{4.4}$$

It follows from (4.1), with $p=q=n$, that

$$gf \sim (-1)^n fg - 2r(f \cup_1 v \pm u \cup_1 g)\ . \tag{4.5}$$

Calculating mod. $4r$, it follows from (4.3), (4.4) and (4.5) that

$$\begin{aligned} \mathfrak{p}(f+g) - \mathfrak{p}f - \mathfrak{p}g &\sim fg + gf + 2r(f \cup_1 v + u \cup_1 g) \\ &\sim (1+(-1)^n) fg\ . \end{aligned} \tag{4.6}$$

Let $f_1, \ldots, f_s \subset C^n$ be co-cycles mod $2r$. Then it follows from (4.6) and induction on s that

$$\mathfrak{p}(f_1 + \cdots + f_s) \sim \sum_{i=1}^{s} \mathfrak{p} f_i + \nu \sum_{i<j} f_i f_j \tag{4.7}$$

where $\nu = 1 + (-1)^n$.

It also follows from (4.6) that $\mathfrak{p} f_1 \sim \mathfrak{p} f_2$, mod. $4r$, if $f_1 \sim f_2$, mod. $2r$, where $f_1, f_2 \subset C^n$ are co-cycles mod. $2r$. For let $g = 2rh$ in (4.6). Clearly $\mathfrak{p}(2rh) = 4r^2 \mathfrak{p} h$, whence, calculating mod. $4r$, we have

$$\mathfrak{p}(f + 2rh) \sim \mathfrak{p} f + \bigl(1 + (-1)^n\bigr) 2rfh = \mathfrak{p} f\ .$$

Secondly, let $g = \delta h$. Then $\mathfrak{p} g = (\delta h)^2 = \delta(h\, \delta h) \sim 0$. Therefore, if $\delta f = 2ru$, we have, mod. $4r$,

$$\begin{aligned} \mathfrak{p}(f + \delta h) &\sim \mathfrak{p} f + \bigl(1 + (-1)^n\bigr) f \delta h \\ &\sim \mathfrak{p} f \pm \bigl(1 + (-1)^n\bigr) (\delta f) h \\ &= \mathfrak{p} f \pm 2r\bigl(1 + (-1)^n\bigr) uh \\ &= \mathfrak{p} f\ . \end{aligned}$$

Therefore

$$\mathfrak{p} f \sim \mathfrak{p}(f + 2rh) \sim \mathfrak{p}(f + 2rh + \delta h') \qquad \text{mod.}\, 4r\ .$$

The co-homology class, mod. $4r$, of $\mathfrak{p} f$, where f is a co-cycle, mod. $2r$, is independent of the ordering of the vertices of K. This follows from (8.2) $(i=1)$ and (8.3) $(i=0)$ on p. 302 of [2] and the (easily verified) fact that, if u, v are co-cycles, mod. 2, then $u \vee_0 v \sim v \vee_0 u$, mod. 2.

We now derive a rather cumbersome set of relations, (4.8), as a basis for the relations (4.11) between $\mathfrak{p}$ and μ. Let f be a co-cycle mod. m, which may be even or odd, and let $\delta f = mu$. Let $a = (2r, m)$, $\varrho = 2r/a$. Then ϱf is a co-cycle mod. $2r$ and we have

$$\begin{aligned}\mathfrak{p}(\varrho f) &= \varrho^2(f^2 + f \cup_1 \delta f)\\ &= \varrho^2 f^2 + \varrho^2 m f \cup_1 u \ .\end{aligned}$$

Since $2r \mid \varrho m$ it follows that $\varrho^2 m \equiv 0$, mod. $4r$, if ϱ is even, which it certainly is if m, and hence a, is odd. Also $(4r, 2m) = 2(2r, m) = 2a$, whence $\varrho = 2r/a = 4r/(4r, 2m)$. Also if m is odd we have $(4r, m) = (2r, m) = (r, m)$, whence

$$\varrho^2 = \frac{4r^2}{(r, m)(4r, m)} = \frac{r}{(r, m)} \cdot \frac{4r}{(4r, m)} \ .$$

Therefore, calculating mod. $4r$, we have

$$\begin{aligned}
\mathfrak{p}(\varrho f) &= \varrho^2 \mathfrak{p} f = \varrho \frac{4r}{(4r, 2m)} \mathfrak{p} f && \text{if } m \text{ is even}\\
\mathfrak{p}(\varrho f) &= \varrho^2 f^2 && \text{if } m \text{ is odd}\\
&= \frac{r}{(r, m)} \cdot \frac{4r}{(4r, m)} f^2 \ . && \qquad (4.8)
\end{aligned}$$

We now introduce Pontrjagin squares into the ring R, of § 3. These are a family of maps $\mathfrak{p}_{2r}: H^n(2r) \to H^{2n}(4r)$, which are defined for all values of n, $r \geqslant 0$. If $x \,\varepsilon\, H^n(2r)$, $f \,\varepsilon\, x$ then $\mathfrak{p}_{2r} x$, or simply $\mathfrak{p} x$, is the co-homology class, mod. $4r$, of $\mathfrak{p} f$. From (4.2) we have

$$\left.\begin{aligned}
&\text{(a)} \quad \mu_{2r,4r}\, \mathfrak{p}_{2r} x = x^2 \qquad (r > 0)\\
&\text{(b)} \qquad\quad\ \ \mathfrak{p}_0 x = x^2 \qquad (x \,\varepsilon\, H^n) \ .
\end{aligned}\right\} \qquad (4.9)$$

If $x_1, \ldots, x_s \subset H^n(2r)$ it follows from (4.7) that

$$\mathfrak{p}_{2r}(x_1 + \cdots + x_s) = \sum_{i=1}^{s} \mathfrak{p}_{2r}\, x_i + \sum_{i<j} \nu\, x_i\, x_j \ , \qquad (4.10)$$

where $\nu\, x_i\, x_j = 0$ or $\mu_{4r,2r}(x_i x_j)$ according as n is odd or even. Let $x \,\varepsilon\, H^n(m)$. Then it follows from (4.8) that

$$\begin{aligned}
\mathfrak{p}_{2r}\, \mu_{2r,m}\, x &= \frac{2r}{(2r, m)} \mu_{4r,2m}\, \mathfrak{p}_m\, x && \text{if } m \text{ is even}\\
&= \frac{r}{(r, m)} \mu_{4r,m}\, x^2 && \text{if } m \text{ is odd.} \qquad (4.11)
\end{aligned}$$

It is obvious that

$$\mathfrak{p}_0 e = e , \tag{4.12}$$

where e is the unit element in R.

Notice that the relation $2\mathfrak{p}_{2r}x = \mu_{4r,2r}x^2$ $(r>0)$, which is an obvious consequence of (4.2), may be proved formally by operating on both sides of (4.9a) with $\mu_{4r,2r}$. The result then follows from (2.3). Since $s + 2s(s-1)/2 = s^2$ it follows from this and (4.10), with $x_1 = \ldots = x_s = x \,\varepsilon\, H^n(2r)$, that

$$\begin{aligned} \mathfrak{p}_{2r}(sx) &= s\,\mathfrak{p}_{2r}x && \text{if } n \text{ is odd} \\ &= s^2\,\mathfrak{p}_{2r}x && \text{if } n \text{ is even .} \end{aligned} \tag{4.13}$$

The ring R, complete with Pontrjagin squares, will be called the *co-homology ring* of the complex K. Let $R(K)$ and $R(L)$ be the co-homology rings of complexes K and L. By a *proper homomorphism* $R(K) \to R(L)$ we shall mean one, other than the trivial homomorphism $R(K) \to 0$, which induces a proper homomorphism, $H(K) \to H(L)$, and which commutes with the operator $\mathfrak{p}$, where $H(K)$ and $H(L)$ are the co-homology spectra imbedded in $R(K)$ and $R(L)$.

5. Simple 4-dimensional co-homology rings. We now lay down the conditions on a ring, R, which is given abstractly, in order that it may be the co-homology ring of a finite simply-connected polyhedron of at most four dimensions. We shall describe such a polyhedron as a *simple, 4-dimensional polyhedron* and the corresponding ring as a *simple, 4-dimensional co-homology ring*. First of all the additive group of R shall be the finitely generated direct sum of the groups, $H^n(m)$, in a spectrum of the kind discussed in § 2, such that H^0 is cyclic infinite and $H^n(m) = 0$ if $n = 1$ or if $n > 4$. Since $H^1(m) = 0$ it follows from (2.5) that H^2 contains no element of finite order. It is therefore a free Abelian group.

As regards multiplication, R shall have a unit element, which is to be a generator of H^0. Also (3.1) and (3.3), and hence (3.2), shall be satisfied.

A map $\mathfrak{p}_{2r}\colon H^n(2r) \to H^{2n}(4r)$ is defined, for all values of n, $r \geqslant 0$, which satisfies (4.9), (4.10), (4.11) and (4.12). We shall call $\mathfrak{p}_{2r}x$ the *Pontrjagin square* of $x \,\varepsilon\, H^n(2r)$. Sometimes we shall write $\mathfrak{p}_{2r}$ simply as $\mathfrak{p}$.

Let R and $\bar{R}$ be two such rings. A homomorphism $f : R \to \bar{R}$ will be called a *proper homomorphism* if, and only if,

(1) f is not the trivial homomorphism $R \to 0$,

(2) f induces a proper homomorphism, as defined in § 2, of the spectrum of groups $H^n(m) \subset R$ into the corresponding spectrum in $\bar{R}$.

(3) $f\mathfrak{p} = \mathfrak{p}f$, where $\mathfrak{p}$ denotes the Pontrjagin square operator, both in R and $\bar{R}$.

In particular a proper homomorphism may be a *proper isomorphism*, and R and $\bar{R}$ will be described as *properly isomorphic* if, and only if, there is a proper isomorphism of one onto the other. The inverse of a proper isomorphism is obviously a proper isomorphism. If $f : R \to \bar{R}$ is a proper homomorphism, then $f \mid H^0$ is an isomorphism of H^0 onto the corresponding group $\bar{H}^0 \subset \bar{R}$. For let $e, \bar{e}$ be the unit elements of R, $\bar{R}$. Since e, $\bar{e}$ generate H^0, $\bar{H}^0$ and since $fH^0 \subset \bar{H}^0$ we have $fe = k\bar{e}$ for some value of k. Since e, $\bar{e}$ are idempotents and f is a homomorphism we have $k^2\bar{e} = k^2\bar{e}^2 = f e^2 = f e = k\bar{e}$. Since $(k^2 - k)\bar{e} = 0$ implies $k^2 = k$ it follows that $k = 0$ or 1. If $f(e) = 0$, then $f(x) = f(ex) = 0$ for every $x \in R$, which is excluded. Therefore $f(e) = \bar{e}$ and $f \mid H^0$ is an isomorphism of H^0 onto $\bar{H}^0$.

Our main theorems are:

Theorem 1. *Any simple, 4-dimensional co-homology ring can be realized geometrically by a simple, 4-dimensional polyhedron. That is to say, it is possible to construct a simple, 4-dimensional polyhedron, whose co-homology ring is properly isomorphic to a given ring of this type.*

Theorem 2. *Two simple, 4-dimensional polyhedra are of the same homotopy type if, and only if, their co-homology rings are properly isomorphic.*

The part of Theorem 2, which is contained in the clause "only if" asserts that the co-homology ring is an invariant of the homotopy type. This is true of any simplicial complex since the Pontrjagin squares of co-homology classes, like the products, are independent of the ordering of the vertices. More generally, let K and K' be two simple, 4-dimensional polyhedra and let R and R' be their co-homology rings. Then a given homotopy class of maps, $f : K \to K'$, determines a unique proper homomorphism $f^* : R' \to R$. Theorem 2 will be proved with the help of a partial converse of this, namely:

Theorem 3. *Let R and R' be the co-homology rings of simple 4-dimensional polyhedra, K and K'. Any proper homomorphism, $f^* : R' \to R$, is the one determined by at least one homotopy class of maps $f : K \to K'$.*

The fact that the homotopy class of maps $f : K \to K'$ is, in general, not determined uniquely by $f^* : R' \to R$ may be seen by taking K to be a 3-sphere, S^3, and K' a 2-sphere, S^2. Then $H^n = 0$ unless $n = 0$ or 3 and H^3 is cyclic infinite. Similarly $H'^n = 0$ unless $n = 0$ or 2 and H'^2 is cyclic infinite. There is only one proper homomorphism $f^* : R' \to R$, which is determined by $f^*(e') = e$, $f^* H'^2 = 0$, where e, e' are the unit elements of R, R'. This is induced by any map $f : S^3 \to S^2$. In order to establish a $(1-1)$ dual correspondence between homotopy classes of maps $f : K \to K'$ and proper homomorphisms $f^* : R' \to R$ one must, presumably, enrich the concept of a proper homomorphism. For example[17]), we may demand that, if $u \varepsilon H'^p$, $v \varepsilon H'^q$ are such that $f^*(u) = 0$, $uv = 0$ then an element

$$[f^*, u, v] \varepsilon H^{p+q-1} - (f^* H'^{p+q-1} + H^{p-1} f^* v)$$

is uniquely determined by f^*, u, v. However we shall leave these possible refinements aside and shall proceed to prove our theorems.

6. Theorem 3 implies Theorem 2. Let K and L be finite simply connected complexes[18]) of arbitrary dimensionality and let $f : K \to L$ be a map which induces an isomorphism of each co-homology group $H^n(L)$, with integral coefficients, onto the corresponding group $H^n(K)$. Then Theorem 2 obviously follows from Theorem 3 and:

Theorem 4. *Under these conditions K and L are of the same homotopy type and f is a homotopy equivalence.*

In proving this we may assume, after a suitable deformation of the map f, if necessary, that $f K^n \subset L^n$ for each value of n, where K^n, L^n are the n-sections of K, L. Then f determines a co-chain mapping, $f^* : C^n(L) \to C^n(K)$, where $C^n(K)$ and $C^n(L)$ are the groups of n-dimensional co-chains in K and L. By hypothesis the homomorphisms $H^n(L) \to H^n(K)$, which are induced by f, and hence by f^*, are all isomorphisms onto. It follows from a theorem due to Lefschetz[19]) that the homomorphisms, $H_n(K) \to H_n(L)$, of homology groups, which are in-

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[17]) See [3].

[18]) These shall be simplicial or, more generally, cell complexes of the kind discussed in § 7 below.

[19]) [6], p. 148. There is a flaw in the paragraph following (10.11). For let there be just one 2-dimensional chain, b, and just one 3-dimensional chain, d, such that $Fd = \partial d = 5b$. Then the chains $2b$, $2d$ satisfy the conditions (a) and (b), with $p = 2$, but are not basis elements in any canonical basis for the chains. This may be repaired by an elaboration of Lefschetz' argument. See also the following paragraph.

duced by the chain mapping dual to f^*, are also isomorphisms onto. Since these are the homomorphisms induced by f the theorem follows from a theorem which is proved in [12].

We give an alternative proof of the theorem of *Lefschetz*[19]. Let $H(K)$ and $H(L)$ be the cohomology spectra of K and L and let $h: H(L) \to H(K)$ be the proper homomorphism induced by the co-chain mapping f^*. Since $h \mid H^n(L)$ is an isomorphism onto, for each value of n, it follows from Lemma 2 that h is a proper isomorphism of $H(L)$ onto $H(K)$. By Lemma 4 its inverse, $h^{-1}: H(K) \to H(L)$, can be realized by a co-chain mapping, $g^*: C^n(K) \to C^n(L)$. Then $f^* g^*$ induces the identical automorphism of $H(K)$ and $g^* f^*$ induces the identical automorphism of $H(L)$. It follows from another Theorem of Lefschetz[20], with homology replaced by co-homology, that $f^* g^*$ and $g^* f^*$ are each co-chain homotopic to the identity. That is to say, there are families of homomorphisms, $a^*: C^{n+1}(K) \to C^n(K)$, $b^*: C^{n+1}(L) \to C^n(L)$, such that

$$f^* g^* - 1 = \delta a^* + a^* \delta \;, \qquad g^* f^* - 1 = \delta b^* + b^* \delta \;.$$

Therefore

$$g_* f_* - 1 = a\,\partial + \partial a \;, \qquad f_* g_* - 1 = b\,\partial + \partial b \;,$$

where $f_*: C_n(K) \to C_n(L)$, $g_*: C_n(L) \to C_n(K)$ are the chain mappings dual to f^*, g^* and $a: C_n(K) \to C_{n+1}(K)$, $b: C_n(L) \to C_{n+1}(L)$ are the duals of a^*, b^*. Hence f_* is a chain equivalence, which establishes the theorem in question.

7. Cell complexes. Simplicial complexes are unsuitable for what follows and we shall use instead a type of complex, which we shall describe as a (finite) *cell complex*, or simply as a complex. Let σ^n be a fixed n-simplex and let $\dot{\sigma}^n$ be its boundary. A (finite) cell complex, K, consists of a finite number of open cells, no two of which have a point in common and each of which is homeomorphic to the interior of σ^n for some value of n. Moreover the closure, $\bar{e}^n$, of each n-cell, $e^n \in K$, is the image of σ^n in a map, $f: \sigma^n \to \bar{e}^n$, such that

(1) $f\,\dot{\sigma}^n \subset K^{n-1}$, *where* K^{n-1} *is the* $(n-1)$*-section*[21] *of* K, *and*

(2) $f \mid (\sigma^n - \dot{\sigma}^n)$ *is a homeomorphism onto* e^n.

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[20] [6], Theorem (17.3), p. 155. This theorem, like the results in § 2, can equally well be stated in terms of homology or co-homology.

[21] i. e. the aggregate of cells in K, whose dimensionalities do not exceed $n-1$. This is a sub-complex, whose dimensionality does not exceed $n-1$, but need not be as great. For example, an n-sphere is covered by a complex, K, consisting of one 0-cell and one n-cell, in which $K^r = K^0$ if $0 \leqslant r < n$.

Such a map will be called a *characteristic map* for e^n. We do not, at this stage, impose any other condition on a characteristic map. For example, $f\,\dot{\sigma}^n$ need not coincide, as a point set, with a sub-complex of K.

Examples of cell complexes are:

(1) the complex projective plane, regarded as a complex, K, consisting of a 0-cell, a 2-cell and a 4-cell, e^4. The 2-section, K^2, is a 2-sphere and the "homotopy boundary" βe^4, of e^4, as defined below, is a generating element[22]) of $\pi_3(K^2)$.

(2) the topological product, $S^2 \times S^2$, of a 2-sphere with itself, regarded as a complex $K = e^0 + e_1^2 + e_2^2 + e^4$, where $e^0 = p_0 \times p_0$ $(p_0 \varepsilon S^2)$, $\overline{e}_1^2 = S^2 \times p_0$, $\overline{e}_2^2 = p_0 \times S^2$. In this case[23]) $\beta e^4 = a_1 a_2$, where a_i is a generating element of $\pi_2(\overline{e}_i^2)$ $(i = 1, 2)$.

A cell complex may be built up cell by cell, starting with a finite set of points K^0. Assume that a complex, K, has been constructed and let E^n be an n-element (i. e. a homeomorph of σ^n), which has no point in common with K. Let $f: \dot{E}^n \to K^{n-1}$ be any map of $\dot{E}^n$, the boundary of E^n, into K^{n-1}. Let $e^n = E^n - \dot{E}^n$ and let $\varphi: K \cup E^n \to K \cup e^n$ be the map which is given by $\varphi \mid K \cup e^n = 1$, $\varphi p = f p$ if $p \,\varepsilon\, \dot{E}^n$. Let $K + e^n$ be the space which consists of the points in $K \cup e^n$ with the identification topology determined by φ. That is to say a set $X \subset K + e^n$ is closed if, and only if, $\varphi^{-1} X \subset K \cup E^n$ is closed. It follows from this definition that $\Phi \mid K$, $\Phi \mid e^n$ are homeomorphisms, i. e. that K and e^n retain their topologies in $K + e^n$. Therefore $K + e^n$ is a complex, whose cells are the cells in K, together with e^n. If $g: \sigma^n \to E^n$ is a homeomorphism (onto), then $\varphi g: \sigma^n \to K + e^n$ is a characteristic map for e^n. We shall say that $K + e^n$ is formed by *attaching* e^n *to* K *by means of the map* $f: \dot{E}^n \to K^{n-1}$.

Let K be a given complex and let $K_0 = K + e_0^n$, $K_1 = K + e_1^n$, be complexes, each of which consists of K and one other cell, e_i^n $(i = 0, 1)$. Let $f_i: \sigma^n \to \overline{e}_i^n$ be a characteristic map for e_i^n in K_i.

Lemma 5. *If* $f_0 \mid \dot{\sigma}^n \simeq f_1 \mid \dot{\sigma}^n$ *in* K, *then* K_0 *and* K_1 *are of the same homotopy type.*

Let p_0 be the centroid of σ^n and let p be a variable point in $\dot{\sigma}^n$. We refer σ^n to "polar" coordinates r, p, such that (r, p) is the point which

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22) Cf. [2], § 19, pp. 310, 311.

23) Cf. [13], § 3, where ab was denoted by $a.b$ (it is now often denoted by $[a, b]$). See also a passage in the proof of Theorem 5, in § 12 below.

divides the segment $p_0 p$ in the ratio $r:(1-r)$. Then $(0, p) = p_0$, $(1, p) = p$. Let $g_t : \dot{\sigma}^n \to K$ be a homotopy of $g_0 = f_0 \mid \dot{\sigma}^n$ into $g_1 = f_1 \mid \dot{\sigma}^n$ and let $h_0 : K_0 \to K_1$ be given by $h_0 \mid K = 1$ and

$$\begin{aligned} h_0 f_0(r, p) &= f_1(2r, p) && \text{if } 0 \leqslant 2r \leqslant 1 \\ &= g_{2-2r}\, p && \text{if } 1 \leqslant 2r \leqslant 2\ . \end{aligned}$$

Let $h_1 : K_1 \to K_0$ be given by $h_1 \mid K = 1$ and

$$\begin{aligned} h_1 f_1(r, p) &= f_0(2r, p) && \text{if } 0 \leqslant 2r \leqslant 1 \\ &= g_{2r-1}\, p && \text{if } 1 \leqslant 2r \leqslant 2\ . \end{aligned}$$

It is easily verified that h_0, h_1 are single-valued and hence continuous [24]. We have

$$\begin{aligned} h_1 h_0 f_0(r, p) &= h_1 f_1(2r, p) && \text{if } 0 \leqslant 2r \leqslant 1 \\ &= h_1 g_{2-2r}\, p && \text{if } 1 \leqslant 2r \leqslant 2\ . \end{aligned}$$

Since $h_1 \mid K = 1$ it follows that

$$\begin{aligned} h_1 h_0 f_0(r, p) &= f_0(4r, p) && \text{if } 0 \leqslant 4r \leqslant 1 \\ &= g_{4r-1}\, p && \text{if } 1 \leqslant 4r \leqslant 2 \\ &= g_{2-2r}\, p && \text{if } 1 \leqslant 2r \leqslant 2\ . \end{aligned}$$

Let $\xi_t : K_0 \to K_0$ be the homotopy, which is defined by $\xi_t \mid K = 1$ and

$$\begin{aligned} \xi_t f_0(r, p) &= f_0\,\{(4-3t)\, r, p\} && \text{if } 0 \leqslant r \leqslant 1/(4-3t) \\ &= g_{(4-3t)r-1}\, p && \text{if } \frac{1}{4-3t} \leqslant r \leqslant \frac{2-t}{4-3t} \\ &= g_{\frac{1}{2}(4-3t)(1-r)}\, p && \text{if } \frac{2-t}{4-3t} \leqslant r \leqslant 1\ . \end{aligned} \tag{7.1}$$

It is easily verified that ξ_t is single-valued, and hence continuous [24], and that $\xi_0 = h_1 h_0$, $\xi_1 = 1$. A homotopy $\eta_t : K_1 \to K_1$, such that $\eta_0 = h_0 h_1$, $\eta_1 = 1$, is defined by (7.1) with f_0, g_λ replaced by $f_1, g_{1-\lambda}$, where $\lambda = (4-3t)\, r - 1$ or $(4-3t)(1-r)/2$, and the lemma is therefore established.

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[24]) See [15], § 5.

If K and K' are two complexes covering the same set of points and if every cell of K' is contained in a cell of K, then K' will be called a *sub-division* of K and a *simplicial sub-division* if K' is a simplicial complex. The following lemma is useful when considering complexes which are constructed by the method of attaching cells. Let K_0 be a complex which has a simplicial sub-division K'_0. Let $K = K_0 + e^n$, where the cell e^n is attached to K_0 by a map $\dot{E}^n \to K_0^{n-1}$, which is simplicial with respect to K'_0 and some triangulation of $\dot{E}^n$.

Lemma 6. *Under these conditions K has a simplicial sub-division, of which K'_0 is a sub-complex.*

This will be proved elsewhere.

8. Homology and co-homology in a cell complex. It follows from [15] or [16] that a (finite) cell complex, K, is a locally contractible compactum. Therefore all the standard homology theories (Čech, singular, etc.) are equivalent and all our remarks on homology are to be interpreted in terms of the singular theory. We orient each cell, $e^n \varepsilon K$, and also the fixed simplex, σ^n, and restrict the characteristic maps, $f: \sigma^n \to \bar{e}^n$, to those which are of degree $+1$ in e^n. Let $e_1^n, \ldots, e_q^n$ be the n-cells in K. Let e_i^n also denote the element of the relative homology group, $H_n(K^n, K^{n-1})$, which is represented by a characteristic map for e_i^n, the vertices of σ^n being positively ordered[25]). Then $H_n(K^n, K^{n-1})$ is a free Abelian group[26]) which is freely generated by $e_1^n, \ldots, e_q^n$. The natural homomorphism $H_n(K^n) \to H_n(K)$ is onto and its kernal is

$$\partial H_{n+1}(K^{n+1}, K^n) \subset H_n(K^n) ,$$

where ∂ is the homology boundary operator. Also the natural homomorphism $H_n(K^n) \to H_n(K^n, K^{n-1})$ is an isomorphism onto the subgroup $\partial^{-1}(0) \subset H_n(K^n, K^{n-1})$. Therefore, identifying each element of $H_n(K^n)$ with the corresponding element of $H_n(K^n, K^{n-1})$, it follows that the homology groups of K may be defined in terms of chain groups $C_n(K) = H_n(K^n, K^{n-1})$, and the boundary operator ∂.

We define co-chains as integral valued functions of chains and $C^n(K)$ will denote the group of n-dimensional co-chains in K. We shall use

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[25]) Cf. [5]. $H_n(K^n, K^{n-1}) = H_0(K^0)$ if $n = 0$. Every 0-dimensional chain is a cycle.

[26]) This and other assertions in this paragraph will be proved in a forthcoming book by S. Eilenberg and N. E. Steenrod. I have been greatly helped by a set of notes, prepared by Eilenberg and Steenrod, for a course of lectures, which Eilenberg gave at Princeton in 1945—46.

$\varphi_1^n, \ldots, \varphi_q^n$ to denote the basis for $C^n(K)$, which is defined by $\varphi_i^n e_i^n = 1$, $\varphi_i^n e_j^n = 0$ if $i \neq j$. We shall describe $\varphi_1^n, \ldots, \varphi_q^n$ as the co-chains *dual* to $e_1^n, \ldots, e_q^n$. Following Eilenberg and Steenrod we shall describe a map, $f : K \to L$, of K into a cell complex L, as cellular if, and only if, $f K^r \subset L^r$ for each value of $r = 0, 1, \ldots$. A cellular map $f : K \to L$ determines a chain map, $g : C_n(K) \to C_n(L)$, as follows. Let h_i be a characteristic map for the cell $e_i^n \varepsilon K$. Then $g\, e_i^n \varepsilon C_n(L) = H_n(L^n, L^{n-1})$ is the element which is represented by the map $f h_i : (\sigma^n, \dot{\sigma}^n) \to (L^n, L^{n-1})$. The dual co-chain map $g^* : C^n(L) \to C^n(K)$ is defined, as usual, by $(g^* \psi)\, c = \psi(gc)$, where $\psi \varepsilon C^n(L)$, $c \varepsilon C_n(K)$. We shall say that g, and likewise g^*, is *realized* by $f : K \to L$. Any map $f_0 : K \to L$ is homotopic to a cellular map[27] $f_1 : K \to L$ and if two cellular maps of K into L are homotopic to each other then the corresponding chain maps are chain homotopic.

We now assume that all our complexes have simplicial sub-divisions. Let K' be a simplicial sub-division of K. Then the identical map $i : K \to K'$ is cellular. Let $i^* : C^n(K') \to C^n(K)$ be the induced co-chain mapping. The identical map $i' : K' \to K$ is not cellular, unless $K' = K$. However it is homotopic to cellular map $j : K' \to K$, with the property[27] that $j K_0' \subset K_0$, where K_0 is any sub-complex of K and K_0' is the sub-complex of K' which covers K_0. Let $j^* : C^n(K) \to C^n(K')$ be the co-chain mapping which is induced by j. Since the maps $j\, i : K \to K$ and $i\, j : K' \to K'$ are each homotopic to the identity it follows that each of $i^* j^*$ and $j^* i^*$ is co-chain homotopic to the identity. Therefore i^* induces a proper isomorphism, as defined in § 2, of the co-homology spectrum $H(K')$ onto the spectrum $H(K)$ and j^* induces its inverse.

Let the co-homology ring, $R(K')$, be defined as in §§ 3, 4 and let $A(K')$ be its additive group. Let $A(K)$ be the finitely generated direct sum of all the (absolute and modular) co-homology groups $H^n(K, m)$ $(m, n = 0, 1, \ldots)$. Then i^* determines a proper isomorphism[28] $h : A(K') \to A(K)$. We define $R(K)$ by making h a (proper) isomorphism of $R(K')$ onto $R(K)$. That is to say, if $x, y \subset A(K)$, $z \varepsilon H^n(K, 2r)$ we define xy and $\mathfrak{p}\, z$ by

$$xy = h\{(h^{-1}x)(h^{-1}y)\}\ , \qquad \mathfrak{p}\, z = h\, \mathfrak{p}\, h^{-1} z\ . \tag{8.1}$$

If K is a simple, 4-dimensional complex the ring $R(K')$, and hence $R(K)$, satisfies the conditions in § 5 for a simple, 4-dimensional co-

[27]) See § 16 below.

[28]) i. e. an isomorphism which induces a proper isomorphism $H(K') \to H(K)$.

homology ring. Moreover the unit element has the same geometric interpretation in $R(K)$ as in $R(K')$. For let $u \varepsilon C^0(K)$, $u' \varepsilon C^0(K')$ be the co-chains with constant value 1. Then the unit element, $e' \varepsilon R(K')$, is the co-homology class of u'. Obviously $i^*u' = u$, whence he', the unit element in $R(K)$, is the co-homology class of u.

Let $f: K \to L$ be any map of K in a complex L. Let $R(L)$ be defined in the same way as $R(K)$, by means of a simplicial sub-division, L', of L. The map f determines a unique proper homomorphism $R(L') \to R(K')$ and hence, in the obvious way, a unique proper homomorphism $R(L) \to R(K)$.

For purposes of calculation, and especially for the sake of (8.4) below, we need to define products and Pontrjagin squares of co-chains in K. If φ, ψ are given co-chains in K we define $\varphi\psi$ and $\mathfrak{p}\varphi$ by

$$\varphi \psi = i^* \{(j^* \varphi)(j^* \psi)\} , \qquad \mathfrak{p} \varphi = i^* \mathfrak{p} j^* \varphi . \tag{8.2}$$

When we pass from co-cycles to co-homology classes this obviously leads to (8.1).

Let K_0 be any sub-complex of K and let K_0' be the sub-complex of K' which covers K_0. If $\varphi \varepsilon C^n(K)$ we shall denote the function φ, restricted to chains in K_0, by $\varphi \mid K_0$. Also $\varphi' \mid K_0'$ will have a similar meaning if $\varphi' \varepsilon C^n(K')$. Let products of co-chains and Pontrjagin squares in K_0' be defined by the same rule as in K', the vertices in K_0' taking their order from the ordering in K'. Then[29])

$$\left.\begin{aligned} \varphi' \psi' \mid K_0' &= (\varphi' \mid K_0')(\psi' \mid K_0') \\ (\mathfrak{p}\, \varphi') \mid K_0' &= \mathfrak{p}(\varphi' \mid K_0') , \end{aligned}\right\} \tag{8.3}$$

for any co-chains φ', ψ' in K'.

Clearly $iK_0 = K_0'$ and we recall that $jK_0' \subset K_0$. Let $i_0 = i \mid K_0$, $j_0 = j \mid K_0'$ and let $i_0^* : C^n(K_0') \to C^n(K_0)$, $j_0^* : C^n(K_0) \to C^n(K_0')$ be the co-chain maps induced by i_0, j_0. Let products and Pontrjagin squares of co-chains in K_0 be defined by (8.2), with i^*, j^* replaced by i_0^*, j_0^*. Then I say that

$$\left.\begin{aligned} \varphi \psi \mid K_0 &= (\varphi \mid K_0)(\psi \mid K_0) \\ (\mathfrak{p}\, \varphi) \mid K_0 &= \mathfrak{p}(\varphi \mid K_0) , \end{aligned}\right\} \tag{8.4}$$

[29]) Cf. [9], condition P_1, p. 403.

for any co-chains φ, ψ in K. To prove this we first show that

$$(i^* \varphi') \mid K_0 = i_0^* (\varphi' \mid K_0') , \qquad (j^* \psi) \mid K_0' = j_0^* (\psi \mid K_0) , \tag{8.5}$$

for given co-chains $\varphi' \varepsilon C^n(K')$, $\psi \varepsilon C^n(K)$. Let $u_0' = u' \mid K_0'$, $v_0 = v \mid K_0$, where u', v are any co-chains in K', K. Then (8.5) may be written in the form

$$(i^* \varphi')_0 = i_0^* \varphi_0' , \qquad (j^* \psi)_0 = j_0^* \psi_0 .$$

Let $c_0 \varepsilon C_n(K_0)$ and let i, i_0 also denote the chain mappings induced by $i : K \to K'$, $i_0 : K_0 \to K_0'$. Then $i\, c_0 = i_0\, c_0 \varepsilon C_n(K_0)$, taking $C_n(K_0) \subset C_n(K)$. Therefore $\varphi' i\, c_0 = \varphi_0' i_0\, c_0$ and

$$(i^* \varphi')\, c_0 = \varphi' i\, c_0 = \varphi_0' i_0\, c_0 = (i_0^* \varphi_0')\, c_0 .$$

Therefore $(i^* \varphi')_0 = i_0^* \varphi_0'$. Similarly $(j^* \psi)_0 = j_0^* \psi_0$. It follows from (8.5) and (8.3) that

$$\begin{aligned} \varphi \psi \mid K_0 &= \left(i^* \{(j^* \varphi)(j^* \psi)\}\right) \mid K_0 & \\ &= i_0^* \{(j^* \varphi)(j^* \psi) \mid K_0'\} & \text{by } (8.5) \\ &= i_0^* \{(j^* \varphi)_0 (j^* \psi)_0\} & \text{by } (8.3) \\ &= i_0^* \{(j_0^* \varphi_0)(j_0^* \psi_0)\} & \text{by } (8.5) \\ &= \varphi_0 \psi_0 . & \end{aligned}$$

Similarly $(\mathfrak{p}\, \varphi)_0 = \mathfrak{p}\, \varphi_0$, which establishes (8.4).

9. A lemma on extensions of maps. Let K be a simply connected complex and let $n \geqslant 3$. Then it is an easy extension of a theorem due to W. Hurewicz[30]) that the natural homomorphism of the relative homotopy group, $\pi_n(K^n, K^{n-1})$, into $C_n(K) = H_n(K^n, K^{n-1})$ is an isomorphism onto. We identify corresponding elements in this isomorphism and, as before, we use e^n to stand both for an n-cell in K and for the corresponding element in $C_n(K) = \pi_n(K^n, K^{n-1})$. Let h be the natural homomorphism $h : \pi_n(K^n) \to \pi_n(K^n, K^{n-1})$. If a given element, $a \varepsilon \pi_n(K^n)$ is represented by a map, $f : \dot{\sigma}^{n+1} \to K^n$, then ha is represented by the same map. It follows that $\partial c = h\, \beta\, c$, for any $c \varepsilon C_{n+1}(K)$, where $\beta : \pi_{n+1}(K^{n+1}, K^n) = C_{n+1}(K) \to \pi_n(K^n)$ is the boundary homomorphism in the sense of homotopy. Since $\pi_1(K^n) = 1$ an element in $\pi_{n+1}(K^{n+1}, K^n)$ or $\pi_n(K^n)$ is uniquely, determined by a map of the form $(\sigma^{n+1}, \dot{\sigma}^{n+1}) \to (K^{n+1}, K^n)$, or $\dot{\sigma}^{n+1} \to K^n$, without reference to a base point.

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30) [17], p. 522.

Let K and L be simply connected complexes, let e^n be a principal cell (i. e. one which is an open sub-set) of K and let $K_0 = K - e^n$ $(n \geqslant 3)$. Let $g: C_r(K) \to C_r(L)$ be a chain mapping such that the map $g \mid C_r(K_0)$ $(r = 0, 1, \ldots)$ can be realized by a cellular map $f_0: K_0 \to L$. Let $f_0 \beta e^n$ denote the element of $\pi_{n-1}(L^{n-1})$ into which βe^n is carried by the map $f_0: K_0 \to L$.

Lemma 7. *If $f_0 \beta e^n = \beta g e^n$ then f_0 can be extended to a map $f: K \to L$, which realizes the chain mapping g.*

Let $f_0 \beta e^n = \beta g e^n$. Let $h: \sigma^n \to \bar{e}^n$ be a characteristic map for e^n and let $k: (\sigma^n, \dot{\sigma}^n) \to (L^n, L^{n-1})$ be a map which represents the element $g e^n \varepsilon C_n(L) = \pi_n(L^n, L^{n-1})$. The element $\beta e^n \varepsilon \pi_{n-1}(K^{n-1})$ is represented by the map $h \mid \dot{\sigma}^n$ and $f_0 \beta e^n$ is represented by $f_0 h \mid \dot{\sigma}^n$. Also $\beta g e^n$ is represented by $k \mid \dot{\sigma}^n$. Since $f_0 \beta e^n = \beta g e^n$ it follows that $k \mid \dot{\sigma}^n \simeq f_0 h \mid \dot{\sigma}^n$ in L^{n-1}. A given homotopy of $k \mid \dot{\sigma}^n$ into $f_0 h \mid \dot{\sigma}^n$ can be extended throughout σ^n and we may therefore assume that $k \mid \dot{\sigma}^n = f_0 h \mid \dot{\sigma}^n$. This being so, I say that the map $k h^{-1}: \bar{e}^n \to L^n$ is single-valued. For it is single-valued in e^n, since $h \mid (\sigma^n - \dot{\sigma}^n)$ is a homeomorphism onto e^n and $h \dot{\sigma}^n = \bar{e}^n - e^n$. If $p \varepsilon \bar{e}^n - e^n$ we have $h^{-1} p \subset \dot{\sigma}^n$, whence $k h^{-1} p = f_0 h h^{-1} p = f_0 p$. Therefore $k h^{-1}$ is single valued, and hence continuous[24]). Moreover $k h^{-1} \mid \bar{e}^n \cap K_0 = f_0 \mid \bar{e}^n \cap K_0$, since $\bar{e}^n \cap K_0 = \bar{e}^n - e^n$ and $k h^{-1} p = f_0 p$ if $p \varepsilon \bar{e}^n - e^n$. Also $(k h^{-1}) h = k$. For $k p \varepsilon k h^{-1} h p$ $(p \varepsilon \bar{e}^n)$ and since $k h^{-1}$ is single-valued it follows that $k p = k h^{-1} h p$. Therefore the required map $f: K \to L$ is defined by $f \mid K_0 = f_0$, $f \mid \bar{e}^n = k h^{-1}$.

10. Reduced complexes. A simple, 4-dimensional complex, K, will be called a *reduced complex* if, and only if, it satisfies the conditions

(1) $K^1 = K^0$, a single point.

(2) The closures of the 2-cells, $e_1^2, \ldots, e_n^2$, are 2-spheres, $S_1^2, \ldots, S_n^2$, attached to K^0.

(3) $K^3 = K^2 + e_1^3 + \cdots + e_t^3 + e_{t+1}^3 + \cdots + e_{t+l}^3$, where e_i^3 $(i = 1, \ldots, t \leqslant n)$ is attached to K^2 by a map $\dot{E}_i^3 \to S_i^2$ of degree $\sigma_i > 0$ and e_{t+j}^3 by a map of the form $\dot{E}_{t+j}^3 \to K^0$ $(j = 1, \ldots, l)$. Thus $\bar{e}_{t+j}^3$ is a 3-sphere, S_j^3, and K^3 consists of clusters of 2-spheres and of 3-spheres attached to K^0, together with the bounded 3-cells $e_1^3, \ldots, e_t^3$.

(4) Each 4-cell is attached to K^3 by a map of the form $\dot{E}^4 \to K^2 + S_1^3 + \cdots + S_l^3$.

Notice that each of the bounded 3-cells, $e_1^3, \dots, e_t^3$, is a principal cell of K.

In the above conditions we may have $n = 0$, in which case $K^2 = K^0$. Similarly we may have $1 = 0$ or $t = 0$ and there need be no 4-cells, in which case $K = K^3$.

A map $f_i : \dot{E}_i^3 \to S_i^2$, by which e_i^3 is attached to K^2 $(i = 1, \dots, t)$, is homotopic in K^2 to a map of the form $\dot{E}_i^3 \to S_i^2$, which is simplicial with respect to a given triangulation of K^2 and a suitable triangulation of $\dot{E}_i^3$. Since e_i^3 is a principal cell of K, for each $i = 1, \dots, t$, it follows from Lemma 5 that K is of the same homotopy type as a reduced complex, K_1, in which each of the 3-cells e_i^3 is attached to K^2 by a simplicial map $\dot{E}_i^3 \to S_i^2$. The 3-spheres $S_1^3, \dots, S_l^3$ may be triangulated and it follows from Lemma 6 that K_1^3 has a simplicial sub-division. On applying a similar argument to the 4-cells it follows that K_1 and hence K, is of the same homotopy type as a reduced complex, which has a simplicial sub-division. Therefore the condition that each of our complexes is to have a simplicial sub-division does not restrict the homotopy type of a reduced complex.

Lemma 8. *Any simple, 4-dimensional complex is of the same homotopy type as some reduced complex.*

This will be proved in § 15 below.

If $t = 0$, in which case the bounded 3-cells $e_1^3, \dots, e_t^3$ are absent from K, then every 2-dimensional co-chain, $\varphi \, \varepsilon \, C^2(K)$, is an absolute co-cycle. Therefore it follows from (4.2) that $\mathfrak{p}\,\varphi = \varphi\,\varphi$. Let

$$K_0 = K - (e_1^3 + \dots + e_t^3) = K^2 + S_1^3 + \dots + S_l^3 + e_1^4 + \dots + e_r^4 \, .$$

Then $\mathfrak{p}'\,\varphi = \varphi\,\varphi$, where $\varphi \, \varepsilon \, C^2(K_0) = C^2(K)$ and $\mathfrak{p}'$ denotes the operator $\mathfrak{p}$ in K_0. Also $\psi \mid K_0 = \psi$ if $\psi \, \varepsilon \, C^n(K)$ for $n = 2$ or 4 since K_0 contains all the 2-cells and 4-cells in K. Therefore it follows from (8.4) that

$$\begin{aligned} \mathfrak{p}\,\varphi &= (\mathfrak{p}\,\varphi) \mid K_0 = \mathfrak{p}'(\varphi \mid K_0) = \mathfrak{p}'\,\varphi \\ &= \varphi\,\varphi \, . \end{aligned} \tag{10.1}$$

Therefore if $x \, \varepsilon \, H^2(K, 2r)$ and if $\varphi \, \varepsilon \, x$, then $\mathfrak{p}\,x$ is the co-homology class[31]) of $\varphi\,\varphi$, mod. $4r$. Thus if $\varphi \, \varepsilon \, C^2(K)$ is a co-cycle, mod. $2r$, then

[31]) N. B. $\varphi\varphi \, \varepsilon \, C^4(K)$ and every 4-dimensional co-chain is an absolute co-cycle.

$\varphi\varphi$ has invariant significance as a co-cycle mod. $4r$, as apart from mod. $2r$. Notice also that, if $\varphi, \psi \subset C^2(K)$, then it follows from similar arguments that $\varphi\psi \varepsilon C^4(K)$ is the same whether it is calculated in K or in K_0.

11. $\pi_3(K^3)$, where K^3 is reduced. Let K^3 satisfy the conditions (1), (2), (3) in § 10 and let

$$K_1^3 = K^2 + e_1^3 + \cdots + e_t^3 ,$$

so that

$$K^3 = K_1^3 + S_1^3 + \cdots + S_l^3 .$$

Let b_i be a generating element of $\pi_3(S_i^3)$ $(i = 1, \ldots, l)$. Then $\pi_3(K^3)$ is the direct sum[32])

$$\pi_3(K^3) = \pi_3(K_1^3) + (b_1, \ldots, b_l) ,$$

where $(b_1, \ldots, b_l)$ is a free Abelian group, which is freely generated by $b_1, \ldots, b_l$. We study $\pi_3(K_1^3)$. Let $a_i \,\varepsilon\, \pi_2(K^2)$ be the element which corresponds to a map $\dot{\sigma}^3 \to S_i^2$ of degree $+1$. Then $\pi_2(K^2)$ is a free Abelian group, which is freely generated by $a_1, \ldots, a_n$, and $\beta e_i^3 = \sigma_i a_i$ $(i = 1, \ldots, t)$. Since $\beta e_1^3, \ldots, \beta e_t^3$ are linearly independent the natural homomorphism $i : \pi_3(K^2) \to \pi_3(K_1^3)$ is onto[33]). That is to say, each element in $\pi_3(K_1^3)$ has a representative map in K^2. The group $\pi_3(K^2)$ is a free Abelian group, which is freely generated[34]) by the $n(n+1)/2$ elements $e_{ij} = e_{ji}$ $(i, j = 1, \ldots, n)$, where $e_{ij} = a_i a_j$ if $i \neq j$ and e_{ii} is the generator of $\pi_3(S_i^2)$, which is represented by a map $\dot{\sigma}^4 \to S_i^2$ with Hopf invariant $+1$. It follows easily from the definition of the product $a_i a_j$ that $a_i a_i$ has Hopf invariant 2. Therefore $a_i a_i = 2 e_{ii}$.

Let us also use a_i and e_{ij} to denote the images of $a_i \,\varepsilon\, \pi_2(K^2)$ and $e_{ij} \,\varepsilon\, \pi_3(K^2)$ in the injection homomorphisms $i : \pi_2(K^2) \to \pi_2(K_1^3)$, $i : \pi_3(K^2) \to \pi_3(K_1^3)$. These homomorphisms do not alter the fact that $e_{ij} = a_i a_j$ in $\pi_3(K_1^3)$, if $i \neq j$, and that $a_i a_i = 2 e_{ii}$. Let σ_j be defined for $j = 1, \ldots, n$ by writing $\sigma_{t+1} = \cdots = \sigma_n = 0$. Then $\pi_3(K_1^3)$ is

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32) [14], Theorem 19, p. 285. Strictly speaking there is a natural isomorphism of this direct sum onto $\pi_3(K^3)$. We have implicitly identified corresponding elements in this isomorphism.

33) [13], Lemma 3, p. 417.

34) This is an obvious generalization of the special case $m = n = 2$ of Theorem 2 (p. 413) in [13], where the product ab was written as $a.b$. See also [1].

generated by e_{ij}, since $i : \pi_3(K^2) \to \pi_3(K_1^3)$ is onto, subject to the relations[35])

$$\left.\begin{array}{lll} \text{(a)} & \sigma_i\, a_i\, a_j = 0 & (i, j = 1, \ldots, n)\ , \\ \text{(b)} & \sigma_i^2\, e_{ii} \ = 0\ , & \end{array}\right\} \qquad (11.1)$$

which are a complete set. Since $a_i\, a_j = a_j\, a_i = e_{ij}$ if $i \neq j$ and since $a_i\, a_i = 2 e_{ii}$ it follows that (11.1a) are equivalent to $(\sigma_i, \sigma_j)\, e_{ij} = 0$ if $i \neq j$, $2 \sigma_i\, e_{ii} = 0$, on the understanding that $(\sigma_i, \sigma_j) = 0$ if $\sigma_i = \sigma_j = 0$. Therefore (11.1) are equivalent to

$$\sigma_{ij}\, e_{ij} = 0\ , \qquad (11.2)$$

where $\sigma_{ij} = (\sigma_i, \sigma_j)$ if $i \neq j$, $\sigma_{ii} = (2\sigma_i, \sigma_i^2) = 2\sigma_i$ or σ_i according as σ_i is even or odd.

Let
$$\gamma = \sum_{i \leqslant j} \gamma^{ij}\, e_{ij} \qquad (11.3)$$

be an arbitrary element in $\pi_3(K^2)$ and let γ be represented by a map $f : S^3 \to K^2$, which is simplicial with respect to triangulations of S^3 and K^2. Then γ^{ij} is the linking coefficient $L\,\{f^{-1}(p_i), f^{-1}(q_j)\}$, where p_i, q_i $(p_i \neq q_i)$ are inner points of 2-simplexes in S_i^2. The matrix $||\, \gamma^{ij}\, ||$ is called the *(generalized) Hopf invariant* of any map $S^3 \to K^2$ in the class γ, or simply the Hopf invariant of γ. In exactly the same way we may define the Hopf invariant of a map of the form

$$S^3 \to K_0^3 = K^2 + S_1^3 + \cdots + S_l^3$$

or the Hopf invariant of the corresponding element in $\pi_3(K_0^3)$.

Let $g : (K^2, K^0) \to (L^2, L^0)$ be a map of K^2 into a reduced complex, L^2, which consists of a set of 2-spheres, $S_1'^2, \ldots, S_p'^2$, attached to a point L^0. Let us also use g to denote the induced homomorphism $g : \pi_n(K^2) \to \pi_n(L^2)$ $(n = 2, 3)$ and let $a'_\alpha \,\varepsilon\, \pi_2(L^2)$ be the element represented by a map $\dot\sigma^3 \to S_\alpha'^2$ of degree $+1$. Then

$$g\, a_i = \sum_{\alpha=1}^{p} g_i^\alpha\, a'_\alpha\ ,$$

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[35]) [13], Lemma 4, p. 418, and [1]. The relations (11.1a) express the fact that $(\sigma_i\, a_i)\, a_j = 0$ in $\pi_3(K_1^3)$ since $\sigma_i a_i = 0$ in $\pi_2(K_1^3)$. The relations (11.1b) express the fact that a map of the form $S^3 \to \dot E_i^3 \to S_i^2$ $(i = 1, \ldots, t)$ is inessential in K_1^3, where $\dot E_i^3 \to S_i^2$ is a map by which e_i^3 is attached to K^2. If $S^3 \to \dot E_i^3$ has Hopf invariant 1, then $S^3 \to S_i^2$ has Hopf invariant σ_i^2. It is proved in [13] that these relations are complete.

where g_i^α is the degree with which $g \mid S_i^2$ covers $S_\alpha'^2$. I say that, if γ is given by (11.3), then

$$g\,\gamma = \sum_{\alpha \leqslant \beta} \eta^{\alpha\beta}\, e'_{\alpha\beta}\ , \tag{11.4}$$

where $e'_{\alpha\beta} = a'_\alpha\, a'_\beta$ if $\alpha \neq \beta$, $2e'_{\alpha\alpha} = a'_\alpha\, a'_\alpha$ and

$$\eta^{\alpha\beta} = \sum_{i=1}^{n} \sum_{j=1}^{n} \gamma^{ij}\, g_i^\alpha\, g_j^\beta\ . \tag{11.5}$$

For since $\gamma^{ij} = \gamma^{ji}$, $a_i\, a_j = a_j\, a_i$, $a_i\, a_i = 2e_{ii}$ we have

$$2\gamma = \sum_{i=1}^{n} \sum_{j=1}^{n} \gamma^{ij}\, a_i\, a_j\ .$$

It is obvious from the definition of the product $a_i\, a_j$ that $g(a_i\, a_j) = (g\, a_i)(g\, a_j)$. Also $b\, b'$ is bi-linear in $b, b' \subset \pi_2(L^2)$. Therefore, using the summation convention for $i, j = 1, \ldots, n$ and $\alpha, \beta = 1, \ldots, p$, we have

$$\begin{aligned} 2g\,\gamma &= \gamma^{ij}\, g(a_i\, a_j) = \gamma^{ij}\, (g\, a_i)(g\, a_j) \\ &= \gamma^{ij}\, (g_i^\alpha\, a'_\alpha)(g_j^\beta\, a'_\beta) = \gamma^{ij}\, g_i^\alpha\, g_j^\beta\, a'_\alpha\, a'_\beta \\ &= \eta^{\alpha\beta}\, a'_\alpha\, a'_\beta \\ &= 2 \sum_{\alpha \leqslant \beta} \eta^{\alpha\beta}\, e'_{\alpha\beta}\ , \end{aligned}$$

where $\eta^{\alpha\beta}$ is given by (11.5). Therefore (11.4) follows from the fact that $\pi_3(L^2)$, being a free Abelian group, contains no element of order 2.

Now let $f : K^3 \to L^3$ be a cellular map of K^3 into a reduced, simply connected complex, L^3, and let f also denote the induced homomorphism $f : \pi_3(K^3) \to \pi_3(L^3)$. Then $f\,i = i\,g$, where $g : \pi_3(K^2) \to \pi_3(L^2)$ is the homomorphism induced by $g = f \mid K^2$ and i stands for both injection homomorphisms $i : \pi_3(K^2) \to \pi_3(K^3)$ and $i : \pi_3(L^2) \to \pi_3(L^3)$. It follows from the relation $f\,i = i\,g$ and (11.4) that

$$f\,\gamma = \sum_{\alpha \leqslant \beta} \eta^{\alpha\beta}\, e'_{\alpha\beta}\ , \tag{11.6}$$

where γ and $\eta^{\alpha\beta}$ are given by (11.3) and (11.5) and, as in (11.2), e_{jk}, $e'_{\alpha\beta}$ ($= i\, e_{jk}, i\, e'_{\alpha\beta}$) are interpreted as generators of $i\,\pi_3(K^2)$, $i\,\pi_3(L^2)$.

12. Homotopy and co-homology in a reduced complex. Let K be a reduced complex. Let $e_1^4, \ldots, e_r^4$ be the 4-cells in K, and, using the same notation as in § 11, let

$$\beta\, e_\lambda^4 = \sum_{i \leqslant j} \gamma_\lambda^{ij}\, e_{ij} + b_\lambda^*\ , \tag{12.1}$$

where $b_\lambda^* \varepsilon (b_1, \ldots, b_l)$ and $||\gamma_\lambda^{ij}||$ is the Hopf invariant in $K_0^3 = K^2 + S_1^3 + \cdots + S_l^3$ of a map, $\dot{E}_\lambda^4 \to K_0^3$, by which e_λ^4 is attached to K^3. Let $\{\varphi_i^n\}$ be the basis for $C^n(K)$, which is dual to the basis $\{e_i^n\}$ for $C_n(K)$. The incidence relations between $\{e_i^2\}$ and $\{e_i^3\}$ are $\partial e_i^3 = \sigma_i e_i^2$ $(i = 1, \ldots, t)$, $\partial e_j^3 = 0$ if $j > t$. Therefore the co-boundaries of φ_i^2 are $\delta\varphi_i^2 = \sigma_i \varphi_i^3$ $(i = 1, \ldots, t)$, $\delta\varphi_j^2 = 0$ $(j = t+1, \ldots, n)$, whence φ_i^2 is a co-cycle mod. σ_i, with $\sigma_{t+1} = \cdots = \sigma_n = 0$. Also φ_λ^4 are absolute co-cycles. Let x_i be the co-homology class, mod. σ_i, of φ_i^2 and y_λ the (absolute) co-homology class of φ_λ^4. Then $H^4(K, m)$ is generated by $y_1(m), \ldots, y_r(m)$, where $y_\lambda(m) = \mu_{m,o} y_\lambda$. According to (3.2), $x_i x_j = (\mu_{m,\sigma_i} x_i)(\mu_{m,\sigma_j} x_j)$, where $m = (\sigma_i, \sigma_j)$, on the understanding that $m = 0$ and $\mu_{m,\sigma_i} = \mu_{m,\sigma_j} = 1$ if $\sigma_i = \sigma_j = 0$.

Theorem 5[36]. *If γ_λ^{ij} mean the same as in* (12.1), *then*

$$\left.\begin{aligned} &\text{(a)}\quad x_i x_j = \sum_{\lambda=1}^{r} \gamma_\lambda^{ij}\, y_\lambda(m) \quad \textit{with } m = (\sigma_i, \sigma_j)\,, \\ &\text{(b)}\quad \mathfrak{p}\, x_i = \sum_{\lambda=1}^{r} \gamma_\lambda^{ii}\, y_\lambda(2\sigma_i) \quad \textit{if } \sigma_i \textit{ is even.} \end{aligned}\right\} \tag{12.2}$$

Assume that the theorem is true if $t = 0$, the bounded 3-cells $e_1^3, \ldots, e_t^3$ being absent from K. Then it is true if K is replaced by

$$K_0 = K - (e_1^3 + \cdots + e_t^3)\ .$$

Since γ_λ^{ij} in (12.1) are the elements of the Hopf invariants in K_0^3, of the maps $\dot{E}_\lambda^4 \to K_0^3$, they are the same when calculated for K_0 as for K. It follows from the theorem, with K replaced by K_0, that, in K_0,

$$\varphi_i^2 \varphi_j^2 - \sum_{\lambda=1}^{r} \gamma_\lambda^{ij} \varphi_\lambda^4 \sim 0\ . \tag{12.3}$$

If $\varphi = \delta\psi_0$ in K_0, where $\varphi \,\varepsilon\, C^4(K_0) = C^4(K)$, $\psi_0 \,\varepsilon\, C^3(K_0)$, then $\varphi = \delta\psi$, where $\psi \,\varepsilon\, C^3(K)$ is any extension of ψ_0 (i. e. $\psi e_1^3, \ldots, \psi e_t^3$ have arbitrary values)[37]. Therefore $\varphi \sim 0$ in K_0 implies $\varphi \sim 0$ in K, whence (12.3) holds in K. It follows from the concluding remarks in § 10 that we obtain (12.2a), in K, by taking (12.3) mod. (σ_i, σ_j) and (12.2b) by taking (12.3) mod. $2\sigma_i$, in case $i = j$ and σ_i is even. Therefore the theorem is true in K if it is true in K_0.

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[36]) Cf. the main theorem in [1].

[37]) N. B. the group of cycles in $C_3(K)$ is generated by $e_{t+1}^3, \ldots, e_{t+l}^3$. Therefore $(\delta\psi)c = \psi(\partial c) = \psi_0(\partial c) = (\delta\psi_0)\, c$, where $c \,\varepsilon\, C_4(K)$, $\psi \,\varepsilon\, C^3(K)$, $\psi_0 = \psi \mid K_0$.

It remains to prove the theorem when $t = 0$. In this case $x_i \varepsilon H^2(K)$ and $\mathfrak{p} x_i = x_i^2$ for each $i = 1, \ldots, n$. Therefore we only have to prove (12.2a). We do this by constructing a complex, L, which consists of $Q = S^2 \times S^2$ together with complex projective planes, P_1 and P_2, each of which has one of the 2-spheres $S^2 \times p_0$ and $p_0 \times S^2$ $(p_0 \varepsilon S^2)$ as a basic 2-cycle. We use the Poincaré duality and intersection theory to prove the theorem in L and then prove (12.2a), for given values of i, j, by means of the proper homomorphism, $f^* : R(L) \to R(K)$, which is induced by a certain map $f : K \to L$.

Let $Q = S^2 \times S^2$, where S^2 is an oriented 2-sphere. Let $e'^0 = p_0 \times p_0$ $(p_0 \varepsilon S^2)$, let $S_1'^2 = S^2 \times p_0$, $S_2'^2 = p_0 \times S^2$ and let $e_\lambda'^2 = S_\lambda'^2 - e'^0$ $(\lambda = 1, 2)$. Let $h' : (\sigma^2, \dot\sigma^2) \to (S^2, p_0)$ be a map such that $h' | (\sigma^2 - \dot\sigma^2)$ is a homeomorphism onto $S^2 - p_0$ of degree $+1$. Let $E^4 = \sigma^2 \times \sigma^2$ and let $h : E^4 \to Q$ be given by $h(x_1, x_2) = (h' x_1) \times (h' x_2)$ $(x_1, x_2 \subset \sigma^2)$. Then $h(\sigma^2 \times x_2) = S^2 \times (h' x_2)$, $h(x_1 \times \sigma^2) = (h' x_1) \times S^2$, whence $h \dot E^4 = Q^2 = S_1'^2 + S_2'^2$. Clearly $h | (E^4 - \dot E^4)$ is a homeomorphism onto $e_0'^4 = Q - Q^2$. Therefore Q is a complex, $Q = e'^0 + e_1'^2 + e_2'^2 + e_0'^4$. We orient E^4 so that $\sigma^2 \times x_2$ intersects $x_1 \times \sigma^2$ with coefficient $+1$, where $x_1, x_2 \subset \sigma^2 - \dot\sigma^2$, and we orient $e_0'^4$ so that $h | (E^4 - \dot E^4)$ is of degree $+1$. Then it follows from [13] that $\beta e_0'^4 = a_1' a_2' = e_{12}'$, where $a_\lambda' \varepsilon \pi_2(Q^2)$ is represented by a characteristic map for $e_\lambda'^2$ and $e_{\lambda\mu}' \varepsilon \pi_3(Q^2)$ is defined in the same way as $e_{ij} \varepsilon \pi_3(K^2)$ $(\lambda, \mu = 1, 2)$.

Let $u_1, u_2 \subset C^2(Q)$ be the co-cycles dual to $e_1'^2, e_2'^2 \subset C_2(Q)$ and let $v_0 \varepsilon C^4(Q)$ be defined by $v_0 e_0'^4 = 1$. Since $\sigma^2 \times x_2$ and $x_1 \times \sigma^2$ intersect with coefficient $+1$ in E^4 so do $S^2 \times h' x_2 = h(\sigma^2 \times x_2)$ and $h' x_1 \times S^2 = h(x_1 \times \sigma^2)$, and hence $S_1'^2$ and $S_2'^2$ in Q. Therefore it follows from Poincaré duality that $u_1 u_2 = v_0$. Similarly $u_\lambda^2 = u_\lambda u_\lambda = 0$ $(\lambda = 1, 2)$.

Let $L = Q + e_1'^4 + e_2'^4$, where $e_\lambda'^4$ is attached to $S_\lambda'^2$ by a map $\dot E_\lambda^4 \to S_\lambda'^2$ $(\lambda = 1, 2)$, with Hopf invariant $+1$, of such a kind that $\bar e_\lambda'^4$ is a complex projective plane[38] P_λ. Then $\beta e_\lambda'^4 = e_{\lambda\lambda}'$. Let $u_1, u_2 \subset C^2(L) = C^2(Q)$ mean the same as before and let $v_0, v_1, v_2 \subset C^4(L)$ be the basic co-cycles dual to $e_0'^4, e_1'^4, e_2'^4 \subset C_4(L)$, v_0 being the same as before. It follows from (8.4) that $u_\lambda^2 e_\lambda'^4 = (u_\lambda u_\lambda) e_\lambda'^4 = (u_\lambda^2 | P_\lambda) e_\lambda'^4 = (u_\lambda | P_\lambda)^2 e_\lambda'^4$ $(\lambda = 1, 2)$. That is to say, $u_\lambda^2 e_\lambda'^4$ may be calculated in P_λ, ignoring the rest of L. Since $e_\lambda'^4$ is so oriented that $\beta e_\lambda'^4 = + e_{\lambda\lambda}'$, it follows from Poincaré duality and intersection theory[38] that $u_\lambda^2 e_\lambda'^4 = 1$. Similarly $u_\lambda^2 e_\mu'^4 = (u_\lambda | P_\mu)^2 e_\mu'^4 = 0$ if $\lambda \neq \mu$ $(\lambda, \mu = 1, 2)$. Also $u_\lambda^2 e_0'^4 = (u_\lambda | Q)^2 e_0'^4 = 0$, as proved above. Therefore $u_\lambda^2 = v_\lambda$ $(\lambda = 1, 2)$. Similarly $u_1 u_2 = v_0$.

[38]) [2], p. 311.

Let $f_0 : K^3 \to L^3 = L^2 = L^0 + S_1'^2 + S_2'^2$ be a map such that $f_0 K^0 = L^0$ and :

(1) $f_0 | S_i^2$ is a homeomorphism onto $S_1'^2$, of degree $+1$, for a given value of i,

(2) if $n > 1$, then $f_0 | S_j^2$ is a homeomorphism onto $S_2'^2$, of degree $+1$, for a given value of $j \neq i$,

(3) $f_0\{K^3 - (S_i^2 + S_j^2)\} = L^0$, or $f_0(K^3 - S_i^2) = L^0$ if $n = 1$.

Let $g_0 : C k(K^3) \to C k(L^3)$ be the chain mapping which is induced by f_0 and let $g : Ck(K) \to Ck(L)$ be defined by[39] $g | C_k(K) = g_0 | C_k(K^3)$ if $k < 4$ and

$$g\, e_\lambda^4 = \gamma_\lambda^{ij}\, e_0'^4 + \gamma_\lambda^{ii}\, e_1'^4 + \gamma_\lambda^{jj}\, e_2'^4 \qquad (\lambda = 1, \ldots, r) \ . \qquad (12.4)$$

Then g is a chain mapping, since $f_0 S_\varrho^3 = L^0$, whence $g \partial e_\lambda^4 = \partial g\, e_\lambda^4 = 0$. Let $a_i \varepsilon \pi_2(K)$ be the element which is represented by a characteristic map for e_i^2. Then $f_0 a_i = a_1'$, $f_0 a_j = a_2'$, $f_0 a_k = 0$ if $k \neq i$ or j, where $f_0 : \pi_k(K^3) \to \pi_k(L^3)$ is the homomorphism induced by $f_0 : K^3 \to L^3$. It follows from (11.5) and (11.6) that $f_0 e_{ij} = e_{12}'$, $f_0 e_{ii} = e_{11}'$, $f_0 e_{jj} = e_{22}'$, $f_0 e_{pq} = 0$ for all other pairs p, q. Also $f_0(b_1, \ldots, b_l) = 0$ since $f_0 S_p^3 = L^0$. Therefore

$$\begin{aligned} f_0 \beta\, e_\lambda^4 &= f_0 \Big(\sum_{p \leqslant q} \gamma_\lambda^{pq}\, e_{pq} + b_\lambda^* \Big) \\ &= \gamma_\lambda^{ij}\, e_{12}' + \gamma_\lambda^{ii}\, e_{11}' + \gamma_\lambda^{jj}\, e_{22}' \\ &= \gamma_\lambda^{ij}\, \beta\, e_0'^4 + \gamma_\lambda^{ii}\, \beta\, e_1'^4 + \gamma_\lambda^{jj}\, \beta\, e_2'^4 \\ &= \beta\, g\, e_\lambda^4 \ . \end{aligned}$$

It follows from Lemma 7 that f_0 can be extended to a map $f : K \to L$, which realizes the chain map g. Let $f^* : R(L) \to R(K)$ be the homomorphism induced by f and let $u_\lambda \varepsilon H^2(L)$, $v_s \varepsilon H^4(L)$ also denote the co-homology classes, which consist of the single co-cycles u_λ, v_s. Then $f^* u_1 = x_i$, $f^* u_2 = x_j$ and it follows from (12.4) that

$$f^* v_0 = \sum_{\lambda=1}^{r} \gamma_\lambda^{ij}\, y_\lambda\,, \qquad f^* v_1 = \sum_{\lambda=1}^{r} \gamma_\lambda^{ii}\, y_\lambda \ .$$

Therefore

$$x_i\, x_i = (f^* u_1)^2 = f^* (u_1^2) = f^* v_1 = \sum_{\lambda=1}^{r} \gamma_\lambda^{ii}\, y_\lambda\,,$$

$$x_i\, x_j = (f^* u_1)\,(f^* u_2) = f^* (u_1 u_2) = f^* v_0 = \sum_{\lambda=1}^{r} \gamma_\lambda^{ij}\, y_\lambda\,,$$

which establishes the theorem.

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[39]) It is obvious what modifications must be made here and in the following arguments if $n = 1$. In this case we only need the sub-complex $P_1 \subset L$.

This theorem shows the part played by the Pontrjagin squares. We illustrate this by an example. Let $K_\gamma = e^0 + e^2 + e^3 + e^4$ be the complex in Theorem 5 with $n = t = 1$, $\sigma_1 = 2$, $l = 0$, $r = 1$ and $\gamma_1^{11} = \gamma \geqslant 0$ in (12.1). Then $K_\gamma^3 = e^0 + e^2 + e^3$, where $\partial e^3 = 2e^2$, and $\pi_3(K^3)$ is generated by e_{11} and is of order $(2\sigma_1, \sigma_1^2) = 4$. Since $\beta\, e^4 = \gamma\, e_{11}$ it follows that $\pi_3(K_\gamma)$ is generated by e_{11}, subject to the relation $4e_{11} = 0$ and the additional relation[40] $\gamma\, e_{11} = 0$. It is therefore a cyclic group of order $(\gamma, 4)$. Thus $\pi_3(K_0) \not\approx \pi_3(K_2)$. The only non-trivial products which occur in the co-homology ring are multiples of $x^2 = \gamma\, y \,\varepsilon\, H^4(K_\gamma, 2)$, where x and y generate $H^2(K_\gamma, 2)$ and $H^4(K_\gamma, 2)$. Therefore the co-homology rings of K_0 and K_2 would be properly isomorphic to each other if the Pontrjagin squares were ignored. On the other hand $\mathfrak{p}\, x = \gamma\, y$ and now γ may be taken mod. 4 and not merely mod. 2. Therefore K_0 and K_2 can be distinguished from each other by means of the Pontrjagin squares.

13. Proof of Theorem 1. Let R be a given simple, 4-dimensional co-homology ring, as defined in § 5. Let the group $H^q = H^q(0) \subset R$ be the direct sum of a free Abelian group of rank p_q and t_{q-1} finite cyclic groups of orders $\sigma_1^{(q-1)}, \ldots, \sigma_{t_{q-1}}^{(q-1)}$ $(t_1 = 0)$. Let K mean the same as in § 12 with $n = t_2 + p_2$, $t = t_2$, $\sigma_i = \sigma_i^{(2)}$ $(i = 1, \ldots, t_2)$, $l = p_3 + t_3$, $r = t_3 + p_4$ and

$$\partial e_\lambda^4 = \sigma_\lambda^{(3)} e_{t_2+p_3+\lambda}^3 \qquad (\lambda = 1, \ldots, t_3)$$

$$\partial e_\lambda^4 = 0 \qquad (\lambda = t_3 + 1, \ldots, t_3 + p_4) .$$

Then the co-bounding relations are

$$\delta\varphi_i^2 = \sigma_i^{(2)} \varphi_i^3 \quad (i = 1, \ldots, t_2),\ \delta\varphi_i^2 = 0 \quad (i = t_2 + 1, \ldots, t_2 + p_2) ,$$

$$\delta\varphi_\lambda^3 = 0 \quad (\lambda = 1, \ldots, t_2 + p_3),\ \delta\varphi_{t_2+p_3+\lambda}^3 = \sigma_\lambda^{(3)} \varphi_\lambda^4 \quad (\lambda = 1, \ldots, t_3) .$$

Therefore $H^n(K) \approx H^n$ for $n = 2, 3, 4$ and hence for all values of n. It follows from Lemmas 3 and 2, in § 2, that there is a proper isomorphism of the co-homology spectrum, H, in R, onto the co-homology spectrum $H(K)$. This determines an isomorphism, h, of the additive group, A, of R, onto the additive group, $A(K)$, of $R(K)$. To simplify the notation we identify each element $x \,\varepsilon\, A$ with $h\, x \,\varepsilon\, A(K)$. Then R becomes a ring whose additive group is the same as that of $R(K)$ and the operators Δ, μ are the same in both. Let us write $R(K) = \bar{R}$ and let

[40]) [14], Theorem 18, p. 281.

xy denote the product of $x, y \subset A$ in R and $\overline{x\,y}$ their product in $\bar{R}$. The coefficients γ_λ^{ij} in (12.1) are still at our disposal and we shall show how to choose them so that $R = \bar{R}$.

Since xy is bi-linear in x and y and since both rings have the same additive group, it is sufficient to consider products of the form xy, where $x \varepsilon H^r(a)$, $y \varepsilon H^s(b)$. Moreover, if $x \varepsilon H^r(a)$, $y \varepsilon H^s(b)$, then $xy = \mu_{m,a}(x)\, \mu_{m,b}(y)$, where $m = (a, b)$. Since μ is the same for both rings we have only to ensure that $xy = \overline{x\,y}$ if $x \varepsilon H^r(m)$, $y \varepsilon H^s(m)$ for all values of m, r, s. Since $xy \varepsilon H^{r+s}(m)$ if $x \varepsilon H^r(m)$, $y \varepsilon H^s(m)$ and since $H^n(m) = 0$ if $n = 1$ or $n > 4$, we have $xy = \overline{x\,y} = 0$ unless $r = 0$ or $s = 0$ or $r = s = 2$. In either case $xy = yx$, $\overline{x\,y} = \overline{y\,x}$.

We first dispose of the case $x \varepsilon H^0(m)$, $y \varepsilon H^s(m)$. The unit element of each ring, R and $\bar{R}$, is one of the two generators of H^0. Altering the original identification of H^0 with $H^0(K)$, if necessary, we assume that both rings have the same unit element, e. Then $\mu_{m,0}e$ is the unit element of both rings $R(m)$, $\bar{R}(m)$. It is also a generator of $H^0(m)$, whence any element $x \varepsilon H^0(m)$ is of the form $x = k\,\mu_{m,0}e$, for some value of k. Therefore $xy = \overline{x\,y} = ky$.

We now consider the products $xy, \overline{x\,y}$, where $x, y \subset H^2(m)$. I say that $xy = \overline{x\,y}$ provided $x_i x_j = \overline{x_i x_j}$ $(i, j = 1, \ldots, n)$, where $x_i \varepsilon H^2(\sigma_i)$ is the co-homology class of φ_i^2. For consider first the case $m = 0$. The group $H^2 = H^2(0)$ is generated by $x_{t+1}, \ldots, x_n$, whence $xy = \overline{x\,y}$ for all $x, y \subset H^2$, provided $x_i x_j = \overline{x_i x_j}$ for $i, j = t+1, \ldots, n$. Secondly let $m > 0$. Then $H^2(m)$ is generated by $\mu_{m,\sigma_1} x_1, \ldots, \mu_{m,\sigma_n} x_n$, as shown in the proof of Lemma 4, in § 2. Therefore, if $x_i x_j = \overline{x_i x_j}$ it follows from (3.3) in the form

$$(\mu_{m,\sigma_i} x_i)\,(\mu_{m,\sigma_j} x_j) = \frac{m\,(m, \sigma_i, \sigma_j)}{(m, \sigma_i)\,(m, \sigma_j)}\,\mu_{m,c}(x_i\,x_j)\ ,$$

where $c = (\sigma_i, \sigma_j)$, that $xy = \overline{x\,y}$ for any elements $x, y \subset H^2(m)$, since μ is the same for both rings and products are bi-linear.

Assume that $xy = \overline{x\,y}$ for every pair $x, y \subset R$ and consider the Pontrjagin squares, $\mathfrak{p}x$, $\bar{\mathfrak{p}}x$, in R and $\bar{R}$. Since $H^0(m)$ is generated by $\mu_{m,0}e$ $(m \geqslant 0, \mu_{0,0} = 1)$ it follows from (4.12), (4.11) and (4.13) that, if $x = s\,\mu_{2r,0}e$, then $\mathfrak{p}x = \bar{\mathfrak{p}}x = s^2\,\mu_{4r,0}e$. Because of dimensionality the only other non-zero Pontrjagin squares in either ring, if any, are of the form $\mathfrak{p}x$, $\bar{\mathfrak{p}}x$, where $x \varepsilon H^2(2r)$. Let x_i mean the same as in the preceding paragraph. Then it follows from (4.11) and (4.10) that $\mathfrak{p}_{2r}x = \bar{\mathfrak{p}}_{2r}x$ for every $x \varepsilon H^2(2r)$ and every value of r, provided $\mathfrak{p}\,x_i = \bar{\mathfrak{p}}\,x_i$ for every i such that σ_i is even. Therefore Theorem 1 will have been established when we have chosen γ_λ^{ij} in (12.1) in such a way that $x_i\, x_j = \overline{x_i x_j}$ $(i, j = 1, \ldots, n)$ and $\mathfrak{p}\, x_i = \bar{\mathfrak{p}}\, x_i$ if σ_i is even.

Let $y_1, \ldots, y_r$ mean the same as in § 12 and let $y_\lambda(m) = \mu_{m,o} y_\lambda$. It follows from (2.3) that $y_\lambda(p) = \mu_{p,q} y_\lambda(q)$ if $p \mid q$. Let $\tau_\lambda = \sigma_\lambda^{(3)}$ for $\lambda = 1, \ldots, t_3$ and let $\tau_\lambda = 0$ for $\lambda = t_3 + 1, \ldots, t_3 + p_4$. Then y_λ is of order τ_λ and $y_\lambda(m)$ is of order (m, τ_λ). In the given ring, R, let

$$\left.\begin{aligned} &\text{(a)} \quad x_i x_j = \sum_{\lambda=1}^{r} \varrho_\lambda^{ij}\, y_\lambda(m) \qquad && \text{with } m = (\sigma_i, \sigma_j) \\ &\text{(b)} \quad \mathfrak{p}\, x_i = \sum_{\lambda=1}^{r} \varrho_\lambda^{i}\, y_\lambda(2\sigma_i) \qquad && \text{if } \sigma_i \text{ is even.} \end{aligned}\right\} \qquad (13.1)$$

Since $y_\lambda(m)$ is of order $(m, \tau_\lambda) = (\sigma_i, \sigma_j, \tau_\lambda)$ the coefficients ϱ_λ^{ij} are only determined mod. $(\sigma_i, \sigma_j, \tau_\lambda)$. If $i \neq j$, or if $i = j$ and σ_i is odd, we give them arbitrary numerical values in the appropriate residue classes. The coefficients ϱ_λ^i are determined mod. $(2\sigma_i, \tau_\lambda)$ and we give them arbitrary values in the appropriate residue classes. It follows from (4.9) that $\mu_{\sigma_i, 2\sigma_i} \mathfrak{p}\, x_i = x_i^2$, in case σ_i is even. Also $\mu_{\sigma_i, 2\sigma_i} y_\lambda(2\sigma_i) = y_\lambda(\sigma_i)$. Therefore, operating on both sides of (13.1b) with $\mu_{\sigma_i, 2\sigma_i}$, we see that $\varrho_\lambda^i \equiv \varrho_\lambda^{ii}$, mod. (σ_i, τ_λ). Having assigned numerical values to ϱ_λ^i, in case σ_i is even, we take $\varrho_\lambda^{ii} = \varrho_\lambda^i$.

Let $\gamma_\lambda^{ij} = \varrho_\lambda^{ij}$ in (12.1). Then Theorem 1 follows from Theorem 5, the products $x_i\, x_j$ and the Pontrjagin squares $\mathfrak{p}\, x_i$, in Theorem 5, being formed in the ring $\overline{R} = R(K)$. This completes the proof of Theorem 1.

14. Proof of Theorem 3. Let P, Q be given simple, 4-dimensional complexes and let $f^* : R(Q) \to R(P)$ be a proper homomorphism. We have to prove that f^* can be realized by a map $f : P \to Q$. We first show that P, Q may be replaced by reduced complexes. By Lemma 8, in § 10, there is a reduced complex, K, which is of the same homotopy type as P. Let $u : K \to P$, $v : P \to K$ be maps such that $vu \simeq 1$, $uv \simeq 1$. Let $u^* : R(P) \to R(K)$, $v^* : R(K) \to R(P)$ be the proper homomorphisms induced by u and v. Since $vu \simeq 1$, $uv \simeq 1$ it follows that $u^* v^* = 1$, $v^* u^* = 1$. Assume that the homomorphism $u^* f^* : R(Q) \to R(K)$ can be realized by a map $h : K \to Q$. Then the homomorphism induced by $hv : P \to Q$ is $v^*(u^* f^*) = v^* u^* f^* = f^*$. Therefore we may replace P by the reduced complex K. Similarly Q may be replaced by a reduced complex, L, and the theorem will be established when we have proved it for K, L.

By Lemma 4, in § 2, the proper homomorphism of the co-homology spectrum, $H(L)$, into the co-homology spectrum, $H(K)$, which is induced by $f^* : R(L) \to R(K)$, can be realized by a co-chain mapping $g^* : C^n(L) \to C^n(K)$. Let $g : C_n(K) \to C_n(L)$ be the chain mapping dual to g.

Assume that g can be realized by a cellular $f: K \to L$ and let $f': R(L) \to R(K)$ be the proper homomorphism induced by f. In each co-homology group, $H^n(L, m)$, f' is the homomorphism induced by g^*. Therefore $f' = f$ in $H^n(L, m)$, whence $f' = f^*$ by linearity. Hence the theorem will follow when we have shown that the chain map g can be realized by a map $f: K \to L$, using the fact that the dual co-chain map g^* induces a proper homomorphism $f^*: R(L) \to R(K)$.

Each of K^0 and L^0 consists of a single 0-cell, $e^0 \varepsilon K$, $e'^0 \varepsilon L$. Let $u \varepsilon C^0(K)$, $v \varepsilon C^0(L)$ be the co-cycles defined by $u\, e^0 = 1$, $v\, e'^0 = 1$ and let u, v also denote the co-homology classes $u \varepsilon H^0(K)$, $v \varepsilon H^0(L)$, of which the co-cycles u, v are the solitary members. Then u, v are the unit elements of $R(K)$, $R(L)$. Therefore $f^* v = u$, as proved in § 5. Therefore $g^* v = u$, $g\, e^0 = e'^0$ and $g \,|\, C_0(K)$ is realized by the map $K^1 = K^0 \to L^0 = L^1$.

Because of the natural homomorphisms $\pi_2(K^2) \approx H_2(K^2)$, $\pi_2(L^2) \approx H_2(L^2)$ it follows from an argument, which is similar to the one at the beginning of the proof of Lemma 8, in § 15 below, that g, restricted to the chains in K^3, can be realized by a cellular map $f_0: K^3 \to L^3$. Let $f_0: \pi_3(K^3) \to \pi_3(L^3)$ be the homomorphism induced by $f_0: K^3 \to L^3$. If $f_0\, \beta\, e^4 = \beta\, g\, e^4$, for every 4-cell in K, then it follows from repeated applications of Lemma 7, in § 9, that f_0 can be extended to the required map $f: K \to L$. However, this is not always possible. For example, let $K^3 = S^2 + S^3$, where S^2 is a 2-sphere and S^3 a 3-sphere, which meets S^2 in a single point, K^0, and let $K = K^3 + e^4$ where e^4 is attached to K^3 by an essential map $\dot{E}^4 \to S^3$. Let $L = K$ and let $f^* = 1$. Then the identical chain map $g: C_n(K^3) \to C_n(K^3)$ is realized by a map, $f_0: K^3 \to K^3$, such that $f_0 \,|\, S^2 = 1$ and $f_0 \,|\, S^3$ covers S^3 with degree unity and has an arbitrary Hopf invariant, γ, in S^2. Then $f_0\, \beta\, e^4 \neq \beta\, e^4 = \beta\, g\, e^4$ unless $\gamma = 0$. This shows that we may have to modify f_0 before extending it.

As in § 11 let $\quad K_1^3 = K^2 + e_1^3 + \cdots + e_t^3$,

where $\partial e_i^3 = \sigma_i\, e_i^2$ and let

$$K^3 = K_1^3 + S_1^3 + \cdots + S_l^3 .$$

Let b_i be a generator of $\pi_3(S_i^3)$ and, as before, let us make the identification

$$\pi_3(K^3) = \pi_3(K_1^3) + (b_1, \ldots, b_l) .$$

Let $h: \pi_3(K^3) \to H_3(K^3)$ be the natural homomorphism. Clearly $h \,|\, (b_1, \ldots, b_l)$ is an isomorphism onto $H_3(K^3) = H_3(S_1^3 + \cdots + S_l^3)$.

Also $\pi_3(K_1^3) = i\,\pi_3(K^2)$, whence $\pi_3(K_1^3) = h^{-1}\,0$. Let us identify each element $b \,\varepsilon\, (b_1, \ldots, b_l)$ with $h\,b \,\varepsilon\, H_3(K^3)$. Then

$$\pi_3(K^3) = i\,\pi_3(K^2) + H_3(K^3) \ .$$

Let us also identify each element in $H_3(K^3)$ with the cycle, which is its unique representative. Then b_i is an element of $H_3(K^3) \subset C_3(K)$.

If $c \,\varepsilon\, C_4(K) = \pi_4(K, K^3)$, then $\beta\,c = \gamma\,c + \partial c$, where $\gamma\,c \,\varepsilon\, i\,\pi_3(K^2)$, $\partial c = h\,\beta\,c \,\varepsilon\, H_3(K^3)$ and $\partial = h\,\beta$ is the homology boundary operator. We write $\beta = \gamma + \partial$. Similarly

$$\pi_3(L^3) = i\,\pi_3(L^2) + H_3(L^3)$$

and $\beta, \gamma, \partial, h$ will mean the same in L as in K. Since $g : C_3(K) \to C_3(L)$ is induced by $f_0 : K^3 \to L^3$, we have $h\,f_0 = g\,h : \pi_3(K^3) \to H_3(L^3) \subset C_3(L)$, where f_0 now denotes $f_0 : \pi_3(K^3) \to \pi_3(L^3)$. Therefore

$$h(f_0\,\beta - \beta\,g) = g\,\partial - \partial g = 0 \ .$$

Since $h^{-1}0 = i\,\pi_3(L^2)$ it follows that $(f_0\,\beta - \beta\,g)\,c \,\varepsilon\, i\,\pi_3(L^2)$ for any $c \,\varepsilon\, C_4(K)$. Therefore $f_0\,\beta - \beta\,g$ is a homomorphism of the form $(f_0\,\beta - \beta\,g) : C_4(K) \to i\,\pi_3(L^2)$. That is to say, $f_0\,\beta - \beta\,g$ is a 4-dimensional co-cycle in K, with coefficients in $i\,\pi_3(L^2)$. Assume that $f_0\,\beta - \beta\,g \sim 0$ and let $f_0\,\beta - \beta\,g = \delta\,\psi$, where $\psi \,\varepsilon\, C^3\{K, i\,\pi_3(L^2)\}$. That is to say, ψ is a homomorphism of the form $\psi : C_3(K) \to i\,\pi_3(L^2)$ and

$$f_0\,\beta - \beta\,g = \delta\psi = \psi\,\partial \ .$$

Let $b_j \,\varepsilon\, \pi_3(K^3)$ be defined by a homeomorphism $h_j : (S^3, p_0) \to (S_j^3, K^0)$ $(j = 1, \ldots, l)$, where S^3, with base point $p_0 \,\varepsilon\, S^3$, is the standard 3-sphere in terms of which $\pi_3(K^3)$ and $\pi_3(L^3)$ are defined. Let $k_j : (S^3, p_0) \to (L^3, L^0)$ be a representative map of $f_0\,b_j - \psi\,b_j \,\varepsilon\, \pi_3(L^3)$ and let $f_1 : K^3 \to L^3$ be defined by $f_1 = f_0$ in K_1^3, $f_1 | S_j^3 = k_j h_j^{-1}$. Then $f_1\,b_j \,\varepsilon\, \pi_3(L^3)$ is represented by the map $f_1 h_j = k_j : S^3 \to L^3$. Therefore $f_1 b_j = f_0 b_j - \psi\,b_j$. Since $f_1 = f_0$ in K_1^3 and $\psi\,b_j \,\varepsilon\, i\,\pi_3(L^2)$ it follows that f_1 also realizes the chain mapping $g : C_n(K^3) \to C_n(L^3)$. If $c \,\varepsilon\, C_4(K)$, then ∂c is of the form $\partial c = n_1 b_1 + \cdots + n_l b_l$, whence $f_1 \partial c = f_0 \partial c - \psi\,\partial c$ or $f_1 \partial = f_0 \partial - \psi \partial$. Also $f_1 \gamma = f_0 \gamma$, since $\gamma c \,\varepsilon\, i \pi_3(K^2)$ and $f_1 | K^2 = f_0 | K^2$. Therefore

$$\begin{aligned} f_1\,\beta - \beta\,g &= f_1(\gamma + \partial) - \beta\,g \\ &= f_0\,\gamma + f_0\,\partial - \psi\,\partial - \beta\,g \\ &= f_0\,\beta - \beta\,g - \psi\,\partial \\ &= 0 \ . \end{aligned}$$

Therefore f_1 can be extended throughout K to give the required map $f: K \to L$.

We now prove that $f_0 \beta - \beta g \sim 0$. Since the coefficient group, $i\,\pi_3(L^2)$, is a direct sum of cyclic groups it is sufficient to prove that $(f_0 \beta - \beta g)\, c = 0$, mod. m, where $c \,\varepsilon\, C_4(K)$ is any cycle, mod. m, for any $m = 0, 1, 2, \ldots$ Since $\beta = \gamma + \partial$ and $h\gamma = 0$ we have $\partial = h\beta = h\,\partial$. Therefore $h(f_0 \partial - \partial g) = g\,\partial - \partial g = 0$, whence $f_0 \partial - \partial g$ is a co-cycle with coefficients in $i\,\pi_3(L^2)$. Since $(f_0 \partial - \partial g)\, c = f_0\, \partial c - g\, \partial c = 0$, mod. m, if $\partial c = 0$ mod. m, it follows that $f_0 \partial - \partial g \sim 0$. Therefore it is sufficient to prove that

$$(f_0 \beta - \beta g) - (f_0 \partial - \partial g) = f_0 \gamma - \gamma g \sim 0 \ .$$

Let the notations, with respect to K, be the same as in the preceding sections and let $c = n_1 e_1^4 + \cdots + n_r e_r^4$ be a cycle mod. m. Then

$$\begin{aligned} \gamma c &= \sum_{\lambda=1}^{r} n_\lambda \gamma e_\lambda^4 \\ &= \sum_{\lambda=1}^{r} n_\lambda \sum_{i \leqslant j} \gamma_\lambda^{ij} e_{ij} \\ &= \sum_{i \leqslant j} \gamma^{ij} e_{ij} \ , \end{aligned}$$

where

$$\begin{aligned} \gamma^{ij} &= \sum_{\lambda=1}^{r} \gamma_\lambda^{ij} n_\lambda \\ &= \sum_{\lambda=1}^{r} \gamma_\lambda^{ij} \varphi_\lambda^4 c \\ &= (\varphi_i^2 \varphi_j^2)\, c \qquad (\text{mod. } m)\,, \end{aligned}$$

the last step following from (12.3) and the fact that c is a cycle mod. m. Therefore

$$\gamma\, c = \sum_{i \leqslant j} e_{ij}\, (\varphi_i^2 \varphi_j^2)\, c \qquad (\text{mod. } m)\,. \tag{14.1}$$

Let $e_1'^2, \ldots, e_p'^2$ be the 2-cells in L and let $\psi_1^2, \ldots, \psi_p^2 \subset C^2(L)$ be the dual co-chains. Let

$$g\, e_i^2 = \sum_{\alpha=1}^{p} g_i^\alpha e_\alpha'^2\,, \qquad g^* \psi_\alpha^2 = \sum_{i=1}^{n} g_i^\alpha \varphi_i^2 \ .$$

Let $e'_{\alpha\beta} \,\varepsilon\, i\,\pi_3(L^2)$ be defined in the same way as e_{ij}. Then it follows from (11.6) and (14.1) that, calculating mod. m,

$$f_0 \gamma c = \sum_{\alpha \leqslant \beta} e'_{\alpha\beta} \sum_{i=1}^{n} \sum_{j=1}^{n} g_i^\alpha g_j^\beta (\varphi_i^2 \varphi_j^2) c$$

$$= \sum_{\alpha \leqslant \beta} e'_{\alpha\beta} \sum_{i=1}^{n} \sum_{j=1}^{n} (g_i^\alpha \varphi_i^2)(g_j^\beta \varphi_j^2) c$$

$$= \sum_{\alpha \leqslant \beta} e'_{\alpha\beta} (g^* \psi_\alpha^2)(g^* \psi_\beta^2) c \ . \tag{14.2}$$

Let $\delta\psi_1^2 = \tau_1 \psi_1^3, \ldots, \delta\psi_q^2 = \tau_q \psi_q^3, \delta\psi_{q+1}^2 = \cdots = \delta\psi_p^2 = 0$, and let $\tau_{\alpha\beta} = (\tau_\alpha, \tau_\beta)(\alpha \neq \beta)$, $\tau_{\alpha\alpha} = (\tau_\alpha^2, 2\tau_\alpha)$, with $\alpha, \beta = 1, \ldots, p$, $\tau_{q+1} = \ldots = \tau_p = 0$ and $(0,0) = 0$. Then $\tau_{\alpha\beta} e'_{\alpha\beta} = 0$ in $i\,\pi_3(L^2)$, as in (11.2).

I say that
$$g^*(\psi_\alpha^2 \psi_\beta^2) \sim (g^* \psi_\alpha^2)(g^* \psi_\beta^2) \qquad \text{mod. } \tau_{\alpha\beta} \ . \tag{14.3}$$

For let j_m mean the same as in Lemma 4, in § 2. Then (3.2) may be written in the form

$$(j_r \psi)(j_s \psi') = j_{(r,s)}(\psi \psi') \ ,$$

where ψ and ψ' are co-cycles mod. r and mod. s, respectively. Also

$$\mathfrak{p}_{2r} j_{2r} \psi = j_{4r} \mathfrak{p} \psi \ ,$$

if ψ is a co-cycle mod. $2r$. As proved in § 10, $\mathfrak{p}\psi = \psi\psi$ if $\psi \varepsilon C^2(P)$, where P is a reduced, simple, 4-dimensional complex. Finally $j g^* = f^* j$, since g^* realizes f^*. Therefore, if $\tau = \tau_\alpha$ is even, whence $\tau_{\alpha\alpha} = 2\tau$, we have, writing ψ for ψ_α^2,

$$\begin{aligned} j_{2\tau} g^*(\psi\psi) &= j_{2\tau} g^* \mathfrak{p} \psi = f^* j_{2\tau} \mathfrak{p} \psi \\ &= f^* \mathfrak{p}_\tau j_\tau \psi = \mathfrak{p}_\tau f^* j_\tau \psi \\ &= \mathfrak{p}_\tau j_\tau g^* \psi = j_{2\tau} \mathfrak{p} g^* \psi \\ &= j_{2\tau} (g^* \psi)(g^* \psi) \ . \end{aligned}$$

This establishes (14.3) in case $\alpha = \beta$ and τ_α is even. If $\alpha \neq \beta$ or if $\alpha = \beta$ and τ_α is odd it follows from a similar argument. Therefore there are co-chains $u_{\alpha\beta} \varepsilon C^4(K)$, $v_{\alpha\beta} \varepsilon C^3(K)$ such that

$$(g^* \psi_\alpha^2)(g^* \psi_\beta^2) = g^*(\psi_\alpha^2 \psi_\beta^2) + \tau_{\alpha\beta} u_{\alpha\beta} + \delta v_{\alpha\beta} \ .$$

Since $\tau_{\alpha\beta} e'_{\alpha\beta} = 0$ and $(\delta v_{\alpha\beta}) c = v_{\alpha\beta}(\partial c) = 0$, mod. m, it follows from (14.2) that

$$\begin{aligned} f_0 \gamma c &= \sum_{\alpha \leqslant \beta} e'_{\alpha\beta} g^*(\psi_\alpha^2 \psi_\beta^2) c \qquad (\text{mod. } m) \\ &= \sum_{\alpha \leqslant \beta} e'_{\alpha\beta} (\psi_\alpha^2 \psi_\beta^2) g c \ . \end{aligned}$$

Hence it follows from the analogue of (14.1) in L, with c replaced by gc, that $f_0 \gamma c = \gamma g c$, mod. m, or that $(f_0 \gamma - \gamma g) c = 0$, mod. m. Therefore $f_0 \gamma - \gamma g \sim 0$ and the proof is complete.

As an example of the applications of this theorem let P be a simple, 4-dimensional complex, in which there is no 2-dimensional or 3-dimensional torsion. Then P is of the same homotopy type as a reduced complex, K, without any bounded 3-cells (i. e. $t = 0$) and without any 4-cells which are bounded in the sense of homology. The homotopy type of such a complex is completely determined by its Betti numbers and the "mixed tensor" which is the equivalence class of the set of components γ_λ^{ij} under transformations of the form

$$\eta_\mu^{pq} = a_i^p a_j^q \gamma_\lambda^{ij} b_\mu^\lambda ,$$

where $i, j, p, q = 1, \dots, n$, $\lambda, \mu = 1, \dots, r$, repeated indices imply summation and $|| a_i^p ||$, $|| b_\mu^\lambda ||$ are unimodular matrices of integers. The components γ_λ^{ij} are the coefficients of the trilinear form $xyv = (x \cup y) \cap v$, defined by Hassler Whitney[41]), where $x, y \subset H^2(K)$, $v \varepsilon H_4(K)$. Thus the homotopy type of P is completely determined by its Betti numbers and this tri-linear form. That is to say, two such complexes, P and P', are of the same homotopy type if, and only if, they have the same Betti numbers and if there are isomorphisms (onto), $a : H^2(P') \to H^2(P)$, $b : H_4(P) \to H_4(P')$, such that

$$(a\, x')(a\, y')\, v = x'\, y'(b\, v)$$

for all elements $x', y' \subset H^2(P')$, $v \varepsilon H_4(P)$. In particular the problem of classifying the homotopy types of complexes of this nature, whose fourth Betti number is unity, is reduced to the classification of quadratic forms, with integral coefficients, under unimodular transformations, on the understanding that a quadratic form and its negative belong to the same class.

15. Proof of Lemma 8. We have to prove that any simple, 4-dimensional complex, P, is of the same homotopy type as a reduced complex. First let $P = P^3$ and let K^3 be a reduced 3-dimensional complex, whose homology groups are isomorphic to those of P^3. Then it follows from Lemmas 3 and 4, interpreted in terms of homology rather than co-homology, that there is a chain map $g : C_n(P^3) \to C_n(K^3)$, which induces isomorphisms of the homology groups of P^3 onto those of K^3. Clearly $g | C_2(P^2)$ can be realized by a map $f_0 : P^2 \to K^2$, such that $f_0 P^1 = K^0$.

[41]) [9].

Because of the natural isomorphisms $\pi_2(P^2) \approx H_2(P^2)$ and $\pi_2(K^2) \approx H_2(K^2)$ it follows easily enough from Lemma 7, in § 9, that f_0 can be extended to a map $f: P^3 \to K^3$, which realizes g. Then f induces isomorphisms of the homology groups of P^3 onto those of K^3 and it follows from [12] that f is a homotopy equivalence.

Now let P be any simple, 4-dimensional complex and let K^3 be a reduced complex, which is of the same homotopy type as P^3. Let $f: P^3 \to K^3$ be a homotopy equivalence, which is a cellular map. Assuming that K^3 does not meet P, we identify each point $p \varepsilon P^3$ with $f p \varepsilon K^3$, thus forming a space K, whose points are those of K^3 and of $P - P^3$. Let $\varphi: P \to K$ be the map, which is given by $\varphi | P^3 = f$, $\varphi | (P - P^3) = 1$. Let e^4 be any 4-cell in P and let $h: \sigma^4 \to P$ be a characteristic map for e^4. Then $\varphi h: \sigma^4 \to K$ is a map such that $\varphi h | (\sigma^4 - \dot{\sigma}^4)$ is a homeomorphism onto e^4 and $\varphi h \dot{\sigma}^4 \subset K^3$. Therefore K is a complex, whose cells are the cells in K^3 and the 4-cells in P, the map $\varphi h: \sigma^4 \to K$ being a characteristic map for e^4 in K. Since $f: P^3 \to K^3$ is a homotopy equivalence so is[42] $\varphi: P \to K$.

The complex K is not necessarily reduced, since the frontier of a 4-cell may meet one or more of the bounded 3-cells, $e_1^3, \ldots, e_t^3 \subset K^3$. Let

$$K_0^3 = K^2 + S_1^3 + \cdots + S_l^3, \qquad K_1^3 = K^2 + e_1^3 + \cdots + e_t^3 ,$$

the notations being the same as in §§ 10, 11. As in § 11 we write

$$\begin{aligned} \pi_3(K^3) &= \pi_3(K_1^3) \quad + \pi_3(S_1^3 + \cdots + S_l^3) \\ &= i\,\pi_3(K^2) + \pi_3(S_1^3 + \cdots + S_l^3) \\ &= i\,\pi_3(K_0^3) \ , \end{aligned}$$

where $i: \pi_3(K^2) \to \pi_3(K_1^3)$, $i: \pi_3(K_0^3) \to \pi_3(K^3)$ are the natural homomorphisms. Therefore every map of the form $\dot{E}^4 \to K^3$ is homotopic in K^3 to a map of the form $\dot{E}^4 \to K_0^3$. Therefore it follows from reiterated applications of Lemma 5, in § 7, that each 4-cell in K may be replaced by one which is attached to K^3 by a map of the form $\dot{E}^4 \to K_0^3$, without altering the homotopy type. This last operation transforms K into a reduced complex and the proof is complete.

16. Note on cell complexes [43]. It is clear from the definition of a (finite) cell complex, K, that a sub-set of the cells in K constitutes a sub-

[42]) [15], Theorem 2.

[43]) The complexes referred to in this section need not have simplicial sub-divisions.

complex if, and only if, the union of these cells is a closed sub-set of the space K. Thus the union[44]) and intersection of any set of sub-complexes are themselves sub-complexes. If X is any set of points in K we shall denote by $K(X)$ the intersection of all the sub-complexes of K, which contain X. Thus $K(X) \subset L$ if $L \subset K$ is any sub-complex which contains X. Clearly $K(\bar{X}) = K(X)$, where $\bar{X}$ is the closure of X.

We shall describe a homotopy, $f_t : P \to K$, where P, K are complexes, as *restricted* if, and only if,

$$f_t P_0 \subset K(f_0 P_0) \tag{16.1}$$

for every sub-complex $P_0 \subset P$. Let $f_t, g_t : P \to K$ be restricted homotopies such that $g_0 = f_1$. Then we have:

Lemma 9. *The homotopy which consists of f_t followed by g_t is also restricted.*

It follows from (16.1) that

$$K(f_t P_0) \subset K(f_0 P_0) \ ,$$

whence

$$g_t P_0 \subset K(g_0 P_0) = K(f_1 P_0) \subset K(f_0 P_0) \ ,$$

which proves the lemma.

Let $f_0 : P \to K$ be a given map and let $g_t : Q \to K$ be a restricted deformation of the map $g_0 = f_0 | Q$, where Q is a sub-complex of P. Then we have:

Lemma 10. *There is a restricted homotopy, $f_t : P \to K$, such that $f_t | Q = g_t$.*

If $P = Q$ there is nothing to prove and we proceed by induction on the number of cells in $P - Q$. Let $P = Q' + e^n$, where e^n is a principal cell in $P - Q$, and assume that g_t has been extended to a restricted homotopy $g'_t : Q' \to K$ $(g'_0 = f_0 | Q')$. Let $h : (\sigma^n, \dot{\sigma}^n) \to (P, Q')$ be a characteristic map for e^n and let the homotopy $g'_t h | \dot{\sigma}^n$ be extended to a homotopy, $\xi_t : \sigma^n \to K$, of $\xi_0 = f_0 h : \sigma^n \to K$, by the method used on p. 501 of [7]. Then it follows from an argument, which is similar to one used in the proof of Lemma 7, in § 9 above, that a (single-valued and continuous) homotopy, $f_t : P \to K$, is defined by $f_t | Q' = g'_t$, $f_t | \bar{e}^n = \xi_t h^{-1} | \bar{e}^n$. Let $P_0 \subset P$ be any sub-complex. Then $f_t P_0 \subset K(f_0 P_0)$ if $P_0 \subset Q'$, since $f_t | Q' = g'_t$ is restricted. Let $P_0 = Q'_0 + e^n$, where $Q'_0 \subset Q'$. Then h is of the form $h : (\sigma^n, \dot{\sigma}^n) \to (P_0, Q'_0)$ and

$$g'_t \, h \, \dot{\sigma}^n \subset g'_t \, Q'_0 \subset K(g'_0 \, Q'_0) \subset K(f_0 P_0) \ .$$

[44]) N. B. there are but a finite number of sub-complexes of a (finite) complex.

Also it follows from the formula at the bottom of p. 501 in [7] that the set of points covered by $\xi_t \sigma^n$ consists of $\xi_0 \sigma^n = f_0 h \sigma^n$ together with the set covered by $g'_t h \dot{\sigma}^n \subset K(f_0 P_0)$. Therefore

$$f_t e^n \subset \xi_t \sigma^n \subset \xi_0 \sigma^n \cup K(f_0 P_0) = f_0 \bar{e}^n \cup K(f_0 P_0) = K(f_0 P_0) .$$

Since $f_t | Q'$ is restricted it follows that

$$f_t P_0 = f_t Q' \cup f_t e^n \subset K(f_0 Q'_0) \cup K(f_0 P_0) = K(f_0 P_0) .$$

Therefore f_t is restricted and the Lemma is proved.

Let $f_0 : P \to K$ be a given map of a complex P in a complex K and let $f_0 | Q$ be cellular, where Q is a sub-complex of P. Then we have:

Theorem 6. *There is a restricted homotopy, $f_t : P \to K$, such that $f_t = f_0$ in Q and f_1 is cellular.*

This will follow from Lemmas 9 and 10 and induction on the numer of cells in $P - Q$ when we have proved it in case $P = Q + e^n$. Let $P = Q + e^n$ and let $L = K(f_0 e^n)$. Let $h : \sigma^n \to \bar{e}^n$ be a characteristic map for e^n. Then $h \dot{\sigma}^n \subset Q^{n-1}$ and since $f_0 | Q$ is cellular it follows that $f_0 h$ is of the form $\xi_0 = f_0 h : (\sigma^n, \dot{\sigma}^n) \to (L, L^{n-1})$. Assume that there is a homotopy, $\xi_t : \sigma^n \to L$, such that $\xi_t = \xi_0$ in $\dot{\sigma}^n$, $\xi_1 \sigma^n \subset L^n$, and let f_t be defined by $f_t = f_0$ in Q, $f_t | \bar{e}^n = \xi_t h^{-1} | \bar{e}^n$. Then $f_1 e^n \subset \xi_1 \sigma^n \subset K^n$ and $f_1 | Q = f_0 | Q$, which is cellular. Therefore f_1 is cellular. Let $P_0 \subset P$ be any sub-complex. If $P_0 \subset Q$, then $f_t P_0 = f_0 P_0 \subset K(f_0 P_0)$, and if $P_0 = Q_0 + e^n$, with $Q_0 \subset Q$, then

$$f_t(Q_0 + e^n) = f_t Q_0 \cup f_t e^n \subset f_0 Q_0 \cup \xi_t \sigma^n \subset f_0 Q_0 \cup K(f_0 e^n) \subset K(f_0 P_0) .$$

Therefore f_t is restricted and the theorem follows.

It remains to prove the existence of ξ_t, which we do by induction on the number of cells in $L - L^n$. If $L = L^n$ we may take $\xi_t - \xi_0$. Otherwise let e^m be an m-cell in L, where $m = \dim L > n \geqslant 0$, and let $g : \sigma^m \to \bar{e}^m$ be a characteristic map for e^m. Let p_0 be the centroid of σ^m and let $\varrho_t : \sigma^m - p_0 \to \sigma^m - p_0$ be the radial deformation, which is given in polar coordinates by $\varrho_t(r, p) = \{t + (1-t)r, p\}$ $(p \in \dot{\sigma}^m)$. Then the homotopy $g \varrho_t g^{-1} : \bar{e}^m - q_0 \to \bar{e}^m - q_0$ $(q_0 = g\, p_0)$ is single-valued, and hence continuous [24]). Also $g \varrho_t g^{-1} | g \dot{\sigma}^m = 1$. Therefore a homotopy $\theta_t : L - q_0 \to L - q_0$ is defined by $\theta_t | L - e^m = 1, \theta_t | \bar{e}^m - q_0 = g \varrho_t g^{-1}$. If $\xi_0 \sigma^n \subset L - q_0$ we define $\xi_t = \theta_t \xi_0$.

Let $q_0 \in \xi_0 \sigma^n$. Then the theorem will follow from the preceding paragraph when we have proved that there is a homotopy, $\xi'_t : \sigma^n \to L$

$(\xi_0' 0 = \xi_0)$ such that $\xi_t' = \xi_0$ in $\dot{\sigma}^n$ and $\xi_1' \sigma^n \subset L - q_0$. Let E be a triangulation of σ^n, whose mesh is so small that $\xi_0\, \sigma \subset e^m$ if $q_0 \varepsilon\, \xi_0\, \sigma$, where σ is any (closed) simplex of E. Let $A \subset E$ be the sub-complex consisting of all the closed simplexes which meet $\xi_0^{-1} q_0$. Then $\xi_0 A \subset e^m$. Let $B = \overline{E - A}$. Each point of $\xi_0^{-1} q_0$ is obviously an inner point of A, whence $\xi_0\, B \subset L - q_0$. Therefore $\xi_0(A \cap B) \subset e^m - q_0$. Since $e^m - q_0$ is arcwise connected $(m > 0)$ and $\pi_k(e^m - q_0) = 0$ if $1 \leqslant k \leqslant n - 1 < m - 1$ the map $\xi_0 | A \cap B$ can be extended to a map $\xi' : A \rightarrow (e^m - q_0)$. Let e^m be given a Euclidean geometry and let $\xi_t' a$ $(a \,\varepsilon\, A)$ be the point which divides the linear segment $(\xi_0 a)(\xi' a)$ in the ratio $t : (1-t)$. Then $\xi_t' : A \rightarrow e^m$ is a deformation of ξ_0 into ξ', such that $\xi_t' = \xi_0$ in $A \cap B$. We extend ξ_t' throughout E by taking $\xi_t' = \xi_0$ in B. Then ξ_t', thus extended, is a homotopy of ξ_0 such that $\xi_t' = \xi_0$ in $\dot{\sigma}^n$, $\xi_1' \sigma^n \subset L - q_0$. This completes the proof.

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