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A criterion for divisibility of n-gons into k-gons

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1. Introduction. In this paper (§ 5) a simple criterion is derived by means of which it can at once be decided, for given n and k, whether a convex n-gon may or may not be divided into convex k-gons. If at all, then the partition can be effectuated so that no vertex of a k-gon is situated on a side (without end-points) of another k-gon.

The condition of convexity can be replaced by a topological requirement. If this condition is entirely dropped the problem becomes trivial: an n-gon can be divided into triangles, and every triangle by a broken line into two k-gons.

Similar problems in three dimensions seem very difficult to treat. However a few initial results are given by *Lennes*, *Hayashi* and *Schönhardt*[1]).

The problem of the divisibility of a polygon has been dealt with by *Mahlo*[2]). The partial results obtained by Mahlo are specified in footnotes 5, 8, 9 of this paper.

2. Definitions and auxiliary relations. We consider a convex n-gon P, $n \geqq 3$[3]), which is divided into $m (\geqq 2)$ convex k-gons $P_1, P_2, \ldots, P_m$ by a finite number of closed straight segments.

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[1]) *N. J. Lennes*, Theorems on the simple finite polygon and polyhedron, Amer. Journ. of Math. 33 (1911) pp. 37—62, esp. pp. 55—62.

T. Hayashi, On division of space, Tôhoku Math. Journ. 24 (1925) pp. 277—286 (Japanese).

E. Schönhardt, Über die Zerlegung von Dreieckspolyedern in Tetraeder, Math. Ann. 98 (1928) pp. 309—312.

[2]) *P. Mahlo*, Topologische Untersuchungen über Zerlegung in ebene und sphärische Polygone, Diss. (Halle) 1908.

Cf. also *B. Bernheim*, Partitions of convex polygons into pentagons, Riveon Lematematika 1 (1947) pp. 95—98 (Hebrew).

Several authors have treated the question of finding the number of different partitions of a polygon by means of non-intersecting diagonal lines; cf. *Th. Motzkin*, Relations between hypersurface cross-ratios, and a combinatorial formula for partitions of a polygon, for permanent preponderance, and for non-associative products, Bull. Am. Soc. Math. vol. 54 (1948) pp. 352—360, and the references given there.

[3]) For the spherical case $n = 2$ and the topological cases $n = 1$ and $n = 0$, see *B. Bernheim*, Riveon Lematematika, l. c.

The set of these segments (which do not belong to P) consists of t $(1 \leqq t \leqq m - 1)$ maximal connected parts called *inner continua.*

Those P_i which have no segment, though perhaps some vertices, in common with P are called *nuclei* and their number is denoted by m'. The sides of a nucleus belong to the same inner continuum.

The set of all the sides of the k-gons is called *net.*

There are five possible kinds of vertices within a net:

1. *Bivalent vertices,* i. e., vertices of P where only two sides meet.

2. *Proper exterior vertices,* i.e., vertices of P and at least two k-gons P_i.

3. *Improper exterior vertices,* i. e., points on an open side (without endpoints) of P, that are vertices of k-gons P_i.

4. *Proper interior vertices,* i. e., vertices not on P, that do not lie on an open side of any P_i.

5. *Improper interior vertices,* i. e., vertices not on P, that lie on an open side of some P_i.

The number of vertices of the different kinds are denoted by a, $\bar{b}$, $b - \bar{b}$, $\bar{c}$, $c - \bar{c}$ respectively. Then $n = a + \bar{b}$; we put $\bar{n} = a + b$.

The number of segments meeting at a vertex is its *valency.* The valency of a vertex minus 3 is its *tetravalency*; it equals 1 for points of valency 4 and -1 for bivalent vertices.

The sum of the tetravalencies of all the vertices in a net is denoted by f, while that of the positive teravalencies alone is denoted by $\bar{f} = f + a$.

The *pentagonal equivalent* of a polygon is 6 minus the number of its own vertices and of the improper vertices on its sides.

Lemma 1. $m = (\bar{n} + c + \bar{c} - 2) / (k - 2)$.

This is the Euler-Descartes theorem for nets of the kind considered[4].

Theorem 1. $$m \geqq \left[\frac{n + k - 5}{k - 2}\right] .^{5)}$$

($[x]$ means the integral part of x.)

Proof. By $c + \bar{c} \geqq 0$, $\bar{n} \geqq n$ and Lemma 1 we have $m \geqq (n-2)/(k-2)$ and since m is an integer $m \geqq [(n + k - 5)/(k - 2)]$.

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[4]) This relation occurs in Mahlo's paper (p. 53) and was again found (for $c = \bar{c}$) by Hayashi and extended by Kubota. See *T. Kubota,* Partitioning of the plane by polygons, Tôhoku Math. Journ. 24 (1925) pp. 273—276. For an analogous relation in space see *Hayashi,* l. c. footnote 1.

[5]) Mahlo had this result in the form $m \geqq (n - 2)/(k - 2)$. It can be shown that, for every n and $k > 5$ allowing partition, $[(n + k - 5)/(k - 2)]$ is a possible value of m.

Lemma 2. *The sum of the pentagonal equivalents of P and all the k-gons P_i in a net is*

$$12 + 2f .$$

(Since the tetravalency of bivalent vertices is -1 and each of them belongs exactly to two polygons, omission of the bivalent vertices in the definition of the pentagonal equivalent would give the sum $12 + 2\bar{f}$.)

The proof is a slight variation (because of the bivalent vertices) of the proof of the analogous theorem for polyhedra [6]).

From

$$(6 - \bar{n}) + (6 - k)\, m - (c - \bar{c}) = 12 + 2f$$

follows by $\bar{f} = f + a$, $\bar{n} = a + b$, $n = a + \bar{b}$

Lemma 3. $n = 6 + 2\bar{f} + (k - 6)\, m + c - \bar{c} + \bar{b} + b$.

Hence by application of Lemma 5

Lemma 4. $n = 6 + 2\bar{f} + (k - 5)\, m + c - \bar{c} + \bar{b} + t - m' - 1$.

Lemma 5. $m - m' = b - t + 1$.

Proof. The number of k-gons that are not nuclei is $m - m'$. The number of connected parts of P that belong to these k-gons is b. It is seen without difficulty that the latter number exceeds the former by $t - 1$.

3. The minimal value of n for nets with $k > 5$ that contain nuclei.

Lemma 6. *If $k > 5$ then $a > k$.*

Proof. By Lemma 3 and because of $a = n - \bar{b}$, $\bar{f} \geqq 0$, $m > 1$, $b > 0$, $c - \bar{c} \geqq 0$ we have $a = 6 + 2\bar{f} + (k - 6)\, m + c - \bar{c} + b > 6 + k - 6 = k$.

Lemma 7. *If the vertices of a sum of disjoint finite trees are divided into two classes $A_1, \ldots, A_g$ and $B_1, \ldots, B_{h-g}$, then $u \geqq v - 2(g - 1)$, where u is the number of the univalent vertices among the B_i and v is the sum of valencies of the vertices A_i.*

Obviously the validity of the inequality for a single tree entails its validity (even without the sign of equality) for a sum of at least two disjoint trees.

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[6]) E. g. *Sainte-Laguë*, Géométrie de situation et jeux, Mémorial des sc. math., fasc. 41 (1929) p. 7.

Proof for a single tree. If $h = 2$ then $u = 2 - g = g - 2(g - 1) = v - 2(g - 1)$. Assuming that the lemma holds for $h - 1$ we prove it for h. In the given tree there necessarily exists a univalent vertex. If $u > 0$, then by deleting a univalent vertex B_j from the given tree we obtain a tree with $h - 1$ vertices. If B_j was connected with a vertex A_i then u and v decrease by 1, while g remains unchanged. If B_j was connected with a vertex B_i, then v and g remain unchanged, while u may decrease by 1. If $u = 0$, delete a univalent vertex A_j. In case A_j was connected with a vertex B_i, v decreases by 1, but $-2(g - 1)$ increases by 2, whereas u may decrease by 1. If A_j was connected with a vertex A_i, u remains unchanged, while v and $2(g - 1)$ decrease by 2. In each of the four possible cases, the induction is thus justified.

The set of sides of the m' nuclei in a net is called the *derived net.* It consists of maximal parts called *groups of nuclei* which may still be connected at single vertices.

Lemma 8. *If* $k > 5$ *and* $m' > 0$ *then* $b \geqq k$.

Proof. We construct a graph G as follows.

As vertices we consider:

1) the groups of nuclei of the derived net;

2) the vertices that belong to the inner continua containing the nuclei, but not to the derived net;

3) the vertices of the groups of nuclei that are situated upon P.

We connect the following pairs of vertices in G:

1) two vertices of the net that are connected by a segment;

2) a vertex on P and a group of nuclei to which it belongs;

3) a group of nuclei and a vertex connected with this group by a segment;

4) two groups of nuclei with a common vertex [7]) or connected by a segment.

The maximal connected parts of G are finite trees since otherwise there would exist other nuclei than those considered. We may therefore apply Lemma 7 to G, with the g groups of nuclei as A_i and all the other vertices as B_i. The sum v of valencies of the A_i is at least $g \cdot k$ since each group of nuclei has by Lemma 6 (and if it consists of a single nucleus, trivially) at least k bivalent vertices and since every such vertex is either

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[7]) The vertex may, in one of the groups, be on a side (without end-points) of a nucleus.

on P or on another group of nuclei or connected by a segment with some other vertex not belonging to the same group of nuclei. By Lemma 7 we therefore have $u \geqq g\,k - 2(g-1) = k + (g-1)(k-2) \geqq k$. Now it is easily seen that each of the u univalent vertices B_i of G represents a vertex on P. These vertices are thus part of the b proper and improper exterior vertices. Hence $b \geqq u \geqq k$, Q. E. D.

Theorem 2. *If* $k > 5$ *then* $m' > 0$ *implies* $n \geqq k(k-4)$.

Proof. If $m' > 0$ then $b \geqq k$ (Lemma 8) and $m \geqq k+1$ since there are at least k k-gons adjacent to a nucleus. By Lemma 3 and $\bar{f} \geqq 0$, $c \geqq \bar{c}$, $\bar{b} \geqq 0$ follows $n = 6 + 2\bar{f} - (6-k)\,m + c - \bar{c} + \bar{b} + b \geqq 6 + (k+1)(k-6) + k = k(k-4)$.

Remark. Equality holds for a net with $c = \bar{c}$, $\bar{f} = 0$, $\bar{b} = 0$, $m' = 1$, $m = k+1$, $b = k$; and, as verified without difficulty, only then.

Theorem 3. *If* $k > 5$ *and* $n < k(k-4)$ *then*

$$m = \left[\frac{n-6}{k-5}\right].$$ [8]

Proof. It follows from Theorem 2 that $m' = 0$. Since $\bar{f} \geqq 0$, $c \geqq \bar{c}$, $\bar{b} \geqq 0$, $t \geqq 1$, Lemma 4 implies $(n-6)/(k-5) \geqq m$.

4. Normal partitions. We now consider special types of partitions of polygons, characterized by certain inner continua and values of $\bar{b}$.

Partitions are called *normal* if the following conditions are satisfied:

1. On P there are the non-bivalent vertices $D_1, \ldots, D_t$, $E_1, \ldots, E_s$, D'_t, $E'_{s'}$, $E'_{s'-1}, \ldots, E'_1$, D'_{t-1}, $D'_{t-2}, \ldots, D'_1$, in this order, where

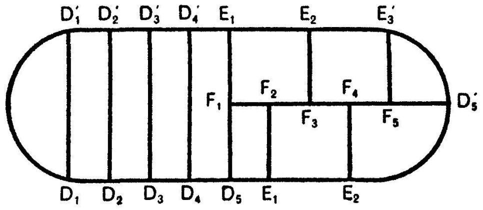

Figure 1

[8]) It can be shown that if there exists at least one partition then there exists a partition with this maximal value of m. For $n \geqq k(k-4)$, explicit formulae for the maximum of m, and constructions of the corresponding partitions, have been obtained by Bernheim and shall be published elsewhere; there is however not a single formula for all values of n. For every n the Euler-Descartes theorem yields easily $m = (n-6)/(k-6)$; this, and the immediate consequence that $n > k$ for $k > 5$, was already noted by Mahlo.

$s \geqq 0$ and s' equals either s or $s+1$. These vertices are connected by inner continua as follows. The vertices D_t and D'_t are joined by a simple broken line with the inner vertices $F_1, \ldots, F_{s+s'}$ (in this order from D_t to D'_t). Every vertex D_i, $i = 1, \ldots, t-1$, is connected with D'_i by a single segment, similarly every F_{2i-1}, $i = 1, \ldots, s'$, with E'_i and F_{2i}, $i = 1, \ldots, s$, with E_i. (See fig. 1, with $t = 5$, $s + 1 = s' = 3$, where P is indicated by a convex curve.)

2. $c = \bar{c}$.

3. $\bar{b} = 0$ if $s' \neq 0$ and $0 \leqq \bar{b} \leqq 2(m-1) = b$ if $s = s' = 0$.

Since each of its polygons has at least one side on P and at most five proper vertices, every normal partition can be considered as consisting of k-gons for every given value of $k \geqq 5$. It is easily seen that if a particular n-gon has a certain normal partition then every other convex n-gon with the same n can be divided in the same way.

To the normal partitions with a given k, there belong certain values of n, say $n_1 < n_2 < n_3 < \ldots$. Obviously $n_1 > k+1$ if $k > 5$, and there may be *gaps* (differences greater than 1) between consecutive n_i. For large i these gaps disappear and every sufficiently great value of n even belongs to different normal partitions.

Denoting by n^* the value n_i that follows immediately upon the last gap, we have

Theorem 4. $n^* = (k-5)[k/3] + 6$, *and for* n^* *there exists a normal partition with* $m = m^* = [k/3]$.

Proof. It follows from Lemma 4 (since $\bar{f} = 0$, $c - \bar{c} = 0$, $m' = 0$) that of two normal partitions with the same m and k, the one that has the larger t and (or) larger $\bar{b}$, also has the larger n. The smallest n for a given m, $n_{\min}(m; k)$, is thus obtained with $t = 1$, $\bar{b} = 0$ and by Lemma 4 therefore $n_{\min}(m; k) = m(k-5) + 6$. On the other hand the largest value of n, $n_{\max}(m; k)$, is obtained when all inner continua are single segments, i. e. $t = m - 1$ and $\bar{b} = b = 2(m-1)$. By Lemma 4 $n_{\max}(m; k) = m(k-5) + 5 + 3(m-1)$. Every n between $n_{\min}(m; k)$ and $n_{\max}(m; k)$ has a normal partition with the same m and k. There are gaps between the n_i as long, and only as long, as $n_{\min}(m; k) - n_{\max}(m-1; k) \geqq 2$, i. e., $m(k-5) + 6 - \big((m-1)(k-5) + 5 + 3(m-1-1)\big) = k + 2 - 3m \geqq 2$ or $m \leqq k/3$. Hence $m^* = [k/3]$ and $n^* = n_{\min}(m^*; k) = m^*(k-5) + 6 = [k/3](k-5) + 6$.

Corollary 1. *For* $n \geqq (k-5)[k/3]+6$ *a convex n-gon is divisible into convex k-gons* ($k \geqq 5$).

Hence

Corollary 2. *If* $n \geqq k(k-4)$ *and* $k \geqq 6$ *a convex n-gon is divisible into convex k-gons.*

Proof. If $k \geqq 6$ then $0<(2k+5)(k-6)+12$ whence $k^2-5k+18<3k^2-12k$ or $(k-5)(k/3)+6<k(k-4)$.

5. Criterion of divisibility.

Theorem 5. *A convex n-gon can be divided into convex k-gons*

1) *for* $k \geqq 6$, *if and only if*

$$[(n-6)/(k-5)] \geqq [(n+k-5)/(k-2)] \ ;$$

2) for $k<6$, always [9]).

(The undivided n-gon is not considered as a partition.)

Proof. 1. The case $k=3$ is trivial. So are the cases $k=4$ and $k=5$: divide first into triangles and each triangle into 3 quadrangles or 9 pentagons respectively as shown in fig. 2 [10]).

2. $k \geqq 6$.

For $n \geqq k(k-4)$ we have $n \geqq (k^2-4k+13)/3$ if $k \geqq 6$ whence $(n-6)/(k-5) \geqq (n+k-5)/(k-2)$. On the other hand we saw in

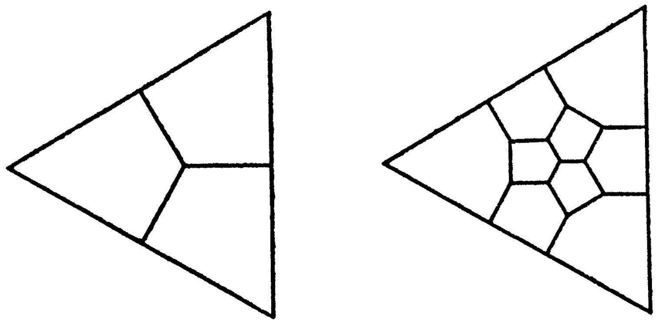

Figure 2

Corollary 2 to Theorem 4 that, for every $n \geqq k(k-4)$, partitions do exist.

From Theorems 1 and 3 follows for $n<k(k-4)$, $k \geqq 6$, that $[(n-6)/(k-5)] \geqq m \geqq [(n+k-5)/(k-2]$ whenever a partition exists.

That partitions always exist when the inequality of Theorem 5 holds,

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[9]) The cases $k=3,4,5$ were obtained by Mahlo.

[10]) The value of m obtained by this method is in general much greater than the minimal m.

will be proved by showing that within the gaps and for $n < n_1$ always $[(n-6)/(k-5)] < [(n+k-5)/(k-2)]$.

We have seen in the proof of Theorem 4 that for any value of m, say m_0, $n_0 = n_{\max}(m_0; k)$ has a normal partition with $m_0 - 1$ inner continua consisting of a single segment each and $\bar{b} = b = 2t = 2(m_0 - 1)$. Since $c + \bar{c} = 0$, $n = \bar{n}$ we have by Lemma 1, $m_0 = (n_0 - 2)/(k-2)$ $= [(n_0 + k - 5)/(k-2)]$. By Lemma 4 we have $m_0 = (n_0 - 6 - (3t-1))/(k-5)$ since $\bar{f} = 0$, $c - \bar{c} = 0$, $m' = 0$ and $\bar{b} = 2t$. If $m_0 \leqq m^* - 1$ then $t = m_0 - 1 \leqq m^* - 2 \leqq k/3 - 2$ (Theorem 4) and hence $3t - 1 \leqq k - 7 < k - 5$ whence $m_0 = (n_0 - 6 - (3t-1))/(k-5) = [(n_0 - 6)/(k-5)]$. Therefore $m_0 = [(n_0 - 6)/(k-5)] = [(n_0 + k - 5)/(k-2)]$ if $m_0 \leqq m^* - 1$ and $n_0 = n_{\max}(m_0; k)$.

From $m_0 = (n_0 - 2)/(k-2)$ follows $(n_0 + k - 5)/(k-2) - [(n_0 + k - 5)/(k-2)] = (k-3)/(k-2)$ whence $[((n_0+1) + k - 5)/(k-2)] > [(n_0 + k - 5)/(k-2)]$. But $[((n_0 + 1) - 6)/(k-5)] = [(n_0 - 6)/(k-5)]$ since $(n_0 - 6)/(k-5) - [(n_0-6)/(k-5)] = (3t-1)/(k-5) < 1$ for $m_0 < m^*$. It follows that for $n = n_0 + 1$ (where a gap begins) $[(n-6)/(k-5)] < [(n+k-5)/(k-2)]$. The function $[(n-6)/(k-5)]$ of n only increases again at the end of the gap since for $n = n_{\min}(m_0 + 1; k)$ there exists a normal partition with $\bar{f} = 0$, $c - \bar{c} = 0$, $\bar{b} = 0$, $t = 1$, $m' = 0$ and hence, by Lemma 4, $[(n-6)/(k-5)] = (n-6)/(k-5) = m_0 + 1$. Thus indeed $[(n-6)/(k-5)] < [(n+k-5)/(k-2)]$ for every n within a gap.

If $n \leqq k$ then $[(n-6)/(k-5)] = 0$ but $[(n+k-5)/(k-2)] = 1$ since $n \geqq 3$; and if $k < n < n_1 = n_{\min}(2; k) = 2(k-5) + 6 = 2k - 4$, then $[(n-6)/(k-5)] = 1$ but $[(n+k-5)/(k-2)] = 2$. Hence $[(n-6)/(k-5)] < [(n+k-5)/(k-2)]$ if $n < n_1$.

Since the proof of Theorem 5 shows that if a convex n-gon can be divided into convex k-gons, $k > 5$, then this can be done by a normal partition, we have

Corollary 1. *Whenever a convex n-gon can be divided into convex k-gons with $k > 5$, this can be done without nuclei.*

The same is easily seen to be true for $k = 3$ and $k = 4$, and, by use of normal partitions, for $k = 5$, $n > k$. On the other hand, by Lemma 4 and because of $\bar{f} \geqq 0$, $c - \bar{c} \geqq 0$, $\bar{b} \geqq 0$, $t \geqq 1$, nuclei are needed for $k = 5$ and $n \leqq k$[11]).

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[11]) Mahlo states that he tried unsuccessfully to prove the latter fact which he needs in the proof of his theorem on the minimal m for $k = 5$, $n \leqq k$.

Similarly the proof of Theorem 5 implies:

Corollary 2. *Whenever a convex n-gon can be divided into convex k-gons, this can be done so that* $c - \bar{c} = 0$.

At the same time we may require that $\bar{f} = 0$.

It can be shown that: a) in addition to partitions without nuclei there exist for $n \geqq k(k-4)$, $k > 5$, partitions with nuclei; b) for given n and $k > 5$, and for $c - \bar{c} = 0$, the maximal number of nuclei appears in all partitions with maximal m; c) for given n and $k > 5$, and for $c - \bar{c} = 0$, all values of m between its maximum and minimum may be attained, and the same holds for m'.

The part of Theorem 5 that concerns $k > 5$ can also be stated as follows.

Corollary 3. *For* $k > 5$, *a convex n-gon can be divided into convex k-gons if and only if* $r \leqq 3q$, *where* $n - 3 = (q+1)(k-2) - r$, $0 < r \leqq k - 2$.

Thus, for a given $k > 5$, the values of n allowing partition are

$$2k - 4 \leqq n \leqq 2k - 2 \,,$$

$$3k - 9 \leqq n \leqq 3k - 4 \,,$$

$$4k - 14 \leqq n \leqq 4k - 6 \,,$$

$$\cdots\cdot$$

$$(q+1)k - 5q + 1 \leqq n \leqq (q+1)k - 2q \,,$$

$$\cdots\cdot \;.$$

For a given n, the number $l = l(n)$ of values of k allowing partition is finite (in particular $l(n) = 3$ for $n < 8$). For $n > 5$, the number $l(n)$ can be determined in the following way. For $k - 2 \leqq 3q$ the criterion of Corollary 3 is obviously fulfilled. Let k_0 be the smallest value of k for which $k - 2 > 3q$, with $q = q_0$ and $r = r_0$.

Every $q \leqq q_0$ belongs to some k. The values of r corresponding to consecutive k that belong to the same q differ by $q + 1$. For the smallest of these values we have $1 \leqq r \leqq q + 1$. Hence $r \leqq 3q$. Further $r + q + 1 \leqq 3q$ except if $q = 1$, $r = 2$. Finally $r + 2(q+1) \leqq 3q$ except if $r = q + 4 - p$, $p = 3, 4, 5$. The next value $r + 3(q+1)$ is certainly greater than $3q$. As long as $r \leqq 3q$, r belongs to the same value of q since $3q < k - 2$.

Thus in general every $q \leqq q_0$ occurs three times. To find the number d of exceptions, note that $n - p = (k - 3)(q + 1)$. Hence d is the number of divisors of $n - 3$, $n - 4$ and $n - 5$ that are greater than 1 and smaller than $q_0 + 2$. For odd n, 2 is to be counted twice.

To exemplify the method take $n = 1000$. Here $k_0 = 57$, $q_0 = 18$, $r_0 = 48$. To q_0 there belong also $k_0 - 1$ and $k_0 - 2$. The value d is in this case 6 since 997 has no divisors smaller than $q_0 + 2$, while 996 has the divisors $2, 3, 4, 6, 12$, and 995 has the divisor 5. Thus altogether $l(1000) = 55 + 3 \cdot 18 - 3 - 6 = 100$.

Theorem 6. *The number $l(n)$ of values k, for which a partition of a convex n-gon into convex k-gons exists, is asymptotically equal to $(12n)^{\frac{1}{2}}$.*

Proof. If q_0 belongs to e values less or equal to k_0 ($e = 1, 2, 3$) then, for $n > 5$, $l(n) = (k_0 - 2) + 3q_0 - e - d$. Hence $l(n)/(3n)^{\frac{1}{2}} \to 1 + 1 + 0 = 2$.

(Eingegangen den 16. Oktober 1947.)