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Relations between cohomology groups in a complex

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In a recent paper [4] the author and S. MacLane have established a rather peculiar isomorphism in the cohomology theory of groups. The isomorphism asserts that

$$H^n(Q, G^*) \approx H^{n+2}(Q, G) , \qquad n > 0 \qquad (*)$$

where Q is any (multiplicative) group, G is an abelian group with Q as operators. The group G^* is defined, in terms of a representation $Q = F/R$ where F is a free group and R is an invariant subgroup, as the group of all homomorphisms $R \to G$ with suitable operators of Q on G^*. Next, both F and R are factored by the commutator group of R and there results a representation $Q = F_0/R_0$ with R_0 abelian. This is a group extension which defines a cohomology class $f_0 \in H^2(Q, R_0)$ with Q suitably operating on R_0. The groups R_0 and G^* are suitably paired to G, and the isomorphism (*) is then defined for each $f \in H^n(Q, G^*)$ as the cup-product $f_0 \cup f \in H^{n+2}(Q, G)$.

Let now K be a simplicial complex and $Q = \pi_1(K)$ the fundamental group of K. If suitable homotopy groups of K vanish then the cohomology groups of Q are isomorphic with the cohomology groups of K (see [3], [1], [2]). Since Q operates on the coefficient groups, local coefficient systems have to be used in K. Thus (*) becomes an isomorphism of the cohomology groups of K.

In this paper we show how this isomorphism can be set up intrinsically in the complex K using the cup-product with a suitable 2-dimensional cocycle in K. The proof of the isomorphism still utilizes (*). It would be desirable to make this proof intrinsic.

The particular 2-dimensional cocycle in K used, is one of a sequence of characteristic cocycles that we define in this paper and that should find other applications.

1. Local systems of groups in a simplicial complex.

This section gives a brief review of Steenrod's local coefficient theory [5] for simplicial complexes, with minor modifications.

Let K be a connected simplicial complex. A local system $\boldsymbol{G} = \{G_A, \gamma_{AB}\}$ of groups in K consists of two functions; the first assigns to each vertex A of K a group G_A, the second to each pair A, B of vertices of K, which are contained in a simplex of K, an isomorphism

$$\gamma_{AB} : G_B \rightarrow G_A$$

subject to the condition

$$\gamma_{AB}\, \gamma_{BC} = \gamma_{AC}$$

for A, B, C in a simplex of K. It follows that

$$\gamma_{AA} = \text{identity}, \qquad \gamma_{AB} = \gamma_{BA}^{-1}$$

A homomorphism Ψ of the system $\boldsymbol{G} = \{G_A, \gamma_{AB}\}$ into the system $\boldsymbol{G}' = \{G'_A, \gamma'_{AB}\}$ is a family of homomorphisms

$$\psi_A : G_A \rightarrow G'_A$$

defined for each vertex of K, such that in the diagram

$$\begin{array}{ccc} G_B & \xrightarrow{\gamma_{AB}} & G_A \\ {\scriptstyle \psi_B}\downarrow & & \downarrow{\scriptstyle \psi_A} \\ G'_B & \xrightarrow[\gamma'_{AB}]{} & G'_A \end{array}$$

the commutativity relation

$$\psi_A\, \gamma_{AB} = \gamma'_{AB}\, \psi_B$$

holds. If each ψ_A is an isomorphism (onto) then Ψ is called an isomorphism

$$\Psi : \boldsymbol{G} \approx \boldsymbol{G}' .$$

A typical example of a local system is obtained by taking G_A to be the n-th homotopy group $\pi_n(K, A)$ relative to the vertex A as base point. The isomorphisms γ_{AB} are then the well known isomorphisms ϱ^n_{AB} used to prove that "the homotopy group is independent of the base point". The resulting local system will be denoted by $\Pi_n(K) = \{\pi_n(K, A), \varrho^n_{AB}\}$.

An edge-path (Kantenzug) in K is a sequence

$$P = A_0, \ldots, A_r$$

of vertices of K such that each pair A_{i-1} and A_i is in a simplex of K for $i = 1, \ldots, r$. If $G = \{G_A, \gamma_{AB}\}$ is a local system in K, then

$$\gamma_P = \gamma_{A_0 A_1} \cdots \gamma_{A_{r-1} A_r}$$

is an isomorphism

$$\gamma_P : G_{A_r} \to G_{A_0} .$$

In particular if $A_0 = A_r = A$ then γ_P is an automorphism of G_A. This leads to the definition of the automorphisms

$$\gamma_\alpha : G_A \to G_A$$

for each $\alpha \in \pi_1(K, A)$. These automorphisms have the following properties

$$\gamma_{\alpha_1 \alpha_2} = \gamma_{\alpha_1} \gamma_{\alpha_2} , \qquad \gamma_1 = \text{identity}$$

$$\gamma_{AB} \gamma_{\alpha'} = \gamma_\alpha \quad \text{if} \quad \alpha' \in \pi_1(K, A) \quad \text{and} \quad \varrho^1_{AB} \alpha' = \alpha .$$

In view of these properties we say that the local system $\Pi_1(K)$ operates on the local system G. If $\Psi = \{\psi_A\}$ is a homomorphism $G \to G'$ then each $\psi_A : G_A \to G'_A$ is an operator homomorphism. Conversely any operator homomorphism $\psi_A : G_A \to G'_A$ given for a particular vertex A of K extends uniquely to a homomorphism $\Psi : G \to G'$.

If for a vertex A of K, $\gamma_\alpha =$ identity for all $\alpha \in \pi_1(K, A)$, then the same holds at all the vertices of K and the system G is called *simple*. A simple system is isomorphic with a local system $G' = \{G'_A, \gamma'_{AB}\}$ in which the groups G'_A are all equal and $\gamma'_{AB} =$ identity. Thus a simple local system may be treated as a single group.

Let $G = \{G_A, \gamma_{AB}\}$ be a local system of *abelian* groups in K. A q-cochain φ of K over G is a function which to every sequence $A^0, \ldots, A^q$ of vertices of K, all contained is a simplex of K, assigns an element $\varphi(A^0, \ldots, A^q)$ of G_{A^0}. The q-cochains form an abelian group $C^q(K, G)$. The coboundary $\delta\varphi$ is a $(q+1)$-cochain defined by

$$(\delta\varphi)(A^0, \ldots, A^{q+1}) = \gamma_{A^0 A^1} \varphi(A^1, \ldots, A^{q+1})$$

$$+ \sum_{i=1}^{q=1} (-1)^i \varphi(A^0, \ldots, \hat{A^i}, \ldots, A^{q+1})$$

It is easy to verify that $\delta\delta\varphi = 0$. The group of cocycles $Z^q(K, G)$ is then defined as the kernel of $\delta : C^q \to C^{q+1}$ while the group of coboundaries $B^q(K, \boldsymbol{G})$ is the image group of $\delta : C^{q-1} \to C^q$. The q-th cohomology group of K over $\boldsymbol{G}$ is

$$H^q(K, \boldsymbol{G}) = Z^q(K, \boldsymbol{G}) / B^q(K, \boldsymbol{G}) .$$

If $\Psi = \{\psi_A\}$ is a homomorphism of the local system $\boldsymbol{G}$ into the local system $\boldsymbol{G}'$ then

$$\varphi'(A^0, \ldots, A^q) = \psi_{A^0} \varphi(A^0, \ldots, A^q)$$

yields a homomorphism $C^q(K, \boldsymbol{G}) \to C^q(K, \boldsymbol{G}')$ which commutes with δ. This in turn induces a homomorphism

$$\Psi^q : H^q(K, \boldsymbol{G}) \to H^q(K, \boldsymbol{G}') .$$

If Ψ is an isomorphism $\boldsymbol{G} \approx \boldsymbol{G}'$ then Ψ^q also is an isomorphism onto. Consequently if $\boldsymbol{G}$ is simple $H^q(K, \boldsymbol{G})$ is isomorphic with an ordinary cohomology group $H^q(K, G)$, with a single coefficient group G.

An alternative description of the cohomology groups of K over $\boldsymbol{G}$ can be obtained by passing to the universal covering complex $\tilde{K}$ of K. The fundamental group $\pi_1(K)$ (relative to the base point V) acts on $\tilde{K}$ as a group of transformations (covering transformations) and also acts as a group of operators on the group $G = G_V$. The equivariant cohomology group $H^q_e(\tilde{K}, G)$ is then defined and is isomorphic with $H^q(K, \boldsymbol{G})$, as is shown in detail in [2].

Let $T : K_1 \to K$ be a simplicial mapping. Given any local system of groups $\boldsymbol{G} = \{G_A, \gamma_{AB}\}$ in K define a local system $T^* \boldsymbol{G}$ of groups in K_1 as follows

$$T^* \boldsymbol{G} = \{G_{T(A)}, \gamma_{T(A)T(B)}\}$$

for vertices A, B in K_1.

If $\boldsymbol{G}$ is abelian and $\varphi \in C^q(K, \boldsymbol{G})$ then setting

$$T^* \varphi(A^0, \ldots, A^q) = \varphi\big(T(A^0), \ldots, T(A^q)\big)$$

we find that $T^* \varphi \in C^q(K_1, T^* \boldsymbol{G})$. Clearly $\delta T^* \varphi = T^* \delta\varphi$ so that a homomorphism

$$T^* : H^q(K, \boldsymbol{G}) \to H^q(K_1, T^* \boldsymbol{G})$$

is obtained.

Let $\boldsymbol{G} = \{G_A, \gamma_{AB}\}$, $\boldsymbol{G}' = \{G'_A, \gamma'_{AB}\}$ and $\boldsymbol{G}'' = \{G''_A, \gamma''_{AB}\}$ be three local systems of abelian groups in K. We shall say that the systems $\mathbf{G}'$ and $\mathbf{G}''$ are *paired* to the system $\mathbf{G}$ provided for every $g' \in G'_A$ and $g'' \in G''_A$ an element of $g' \cup g'' \in G_A$ is defined such that

$$(g'_1 + g'_2) \cup g'' = g' \cup g'' + g'_2 \cup g'' , \quad g' \cup (g''_1 + g''_2) = g' \cup g''_1 + g' \cup g''_2$$

$$\gamma_{AB} (g' \cup g'') = (\gamma'_{AB} g') \cup (\gamma''_{AB} g'') .$$

A pairing of the groups $C^p(K, \mathbf{G}')$ and $C^q(K, \mathbf{G}'')$ to the group $C^{p+q}(K, \mathbf{G})$ is then defined as follows

$$(\varphi' \cup \varphi'')(A^0, \ldots, A^{p+q}) = \varphi'(A^0, \ldots, A^p) \cup \gamma''_{A^0 A^p} \varphi''(A^p, \ldots, A^{p+q})$$

for $\varphi' \in C^p(K, \mathbf{G}')$, $\varphi'' \in C^q(K, \mathbf{G}'')$. The usual coboundary formula

$$\delta(\varphi' \cup \varphi'') = (\delta\varphi') \cup \varphi'' + (-1)^p \varphi' \cup \delta\varphi''$$

follows by computation. It follows that, if φ' and φ'' are cocycles, $\varphi' \cup \varphi''$ is a cocycle, and if in addition either φ' or φ'' is a coboundary then $\varphi' \cup \varphi''$ is a coboundary. There results a pairing of the cohomology groups $H^p(K, \mathbf{G}')$ and $H^q(K, \mathbf{G}'')$ to the group $H^{p+q}(K, \mathbf{G})$.

If $T: K_1 \to K$ is simplicial and $\mathbf{G}'$, $\mathbf{G}''$, $\mathbf{G}$ are abelian local systems in K with $\mathbf{G}'$, $\mathbf{G}''$ paired to $\mathbf{G}$ then $T^*\mathbf{G}'$, $T^*\mathbf{G}''$ are paired to $T^*\mathbf{G}$ and

$$T^* \varphi' \cup T^* \varphi'' = T^*(\varphi' \cup \varphi'') .$$

2. The effect of the fundamental group on higher cohomology groups.

In this section we state the theorem on the influence of the fundamental group upon the higher cohomology groups. In the case of simple coefficients the theorem was first proved by Eilenberg and MacLane [3], and independently by Eckmann [1]. For local coefficients the theorem was established by the author [2]. The statement of the relationship has been altered formally to suit the applications in the later sections.

Let a vertex V of K be selected as base point. We shall write G and $\pi_n(K)$ instead of G_V and $\pi_n(K, V)$.

For each vertex A of K select an edge-path (= Kantenzug) $P(A)$ leading from V to A, with $P(V)$ being the identity path. If A and B are vertices in a simplex of K then

$$W(A, B) = P(A)\, AB\, P(B)^{-1}$$

is an edge-path from V to V and determines an element of $\pi_1(K)$ that we shall denote by $\omega(A, B)$. It is easy to see that

$$\omega(A, B)\,\omega(B, C) = \omega(A, C) \tag{2.1}$$

for vertices A, B, C in a simplex of K.

Let $\mathbf{G} = \{G_A, \gamma_{AB}\}$ be a local system of abelian groups in K. Since $\pi_1(K)$ acts as a group of operators on $G = G_V$ we may consider the cohomology theory of $\pi_1(K)$ with coefficients in G. We shall use the non-homogenous description of cochains [4]. Given a cochain

$$f \in C^q(\pi_1(K), G)$$

for the group $\pi_1(K)$, consider the cochain

$$\varkappa f \in C^q(K, \mathbf{G})$$

for the complex K, defined by

$$\varkappa f(A^0, \ldots, A^q) = \gamma^{-1}_{P(A^0)} f\big(\omega(A^0, A^1), \ldots, \omega(A^{q-1}, A^q)\big)\ . \tag{2.2}$$

Since for $x_1, \ldots, x_{q+1} \in \pi_1(K)$

$$\begin{aligned}(\delta f)(x_1, \ldots, x_{q+1}) &= \gamma_{x_1} f(x_2, \ldots, x_{q+1})\\ &+ \sum_{l=1}^{q} (-1)^i f(x_1, \ldots, x_i\, x_{i+1}, \ldots, x_{q+1})\\ &+ (-1)^{q+1} f(x_1, \ldots, x_q)\end{aligned}$$

and since $\gamma^{-1}_{P(A^0)}\,\gamma_{\omega(A^0, A^1)} = \gamma_{A^0A^1}\,\gamma^{-1}_{P(A^1)}$ it follows from (2.2) and (2.1) that

$$\begin{aligned}(\varkappa\,\delta f)(A^0, \ldots, A^{q+1}) &= \gamma_{A^0A^1}\,\varkappa f(A^1, \ldots, A^{q+1})\\ &+ \sum_{l=1}^{q+1} (-1)^i \varkappa f(A^0, \ldots, \hat{A}^i, \ldots, A^{q+1}) = (\delta \varkappa f)(A^0, \ldots, A^{q+1})\ .\end{aligned}$$

Thus $\varkappa\delta = \delta\varkappa$ and consequently a homomorphism

$$\varkappa : H^q(\pi_1(K), G) \to H^q(K, \mathbf{G}) \tag{2.3}$$

is defined.

Although the definition of $\varkappa$ for cochains depends upon the choice of the path system $\{P(A)\}$ we shall show that the homomorphism (2.3) is independent of this choice. In fact let $\{\bar{P}(A)\}$ be another path system and let $\bar{\omega}$, $\bar{\varkappa}$ be defined from $\{\bar{P}(A)\}$ in the same way as ω and $\varkappa$ were defined from $\{P(A)\}$. For every vertex A of K, let $\tau(A)$ be the element of $\pi_1(K)$ determined by the closed path $P(A)\bar{P}(A)^{-1}$. Then

$$\omega(A, B)\,\tau(B) = \tau(A)\,\bar{\omega}(A, B) \ . \tag{2.4}$$

For every cochain

$$f \in C^{q+1}(\pi_1(K), G)$$

define the cochain

$$D\,f \in C^q(K, \mathbf{G})$$

by setting

$$(D\,f)\,(A^0, \ldots, A^q)$$

$$= \sum_{i=0}^{q} (-1)^i \gamma_{P(A^0)}^{-1} f\big(\omega(A^0, A^1), \ldots, \omega(A^{i-1}, A^i), \tau(A^i), \bar{\omega}(A^i, A^{i+1}), \ldots, \bar{\omega}(A^{q-1}, A^q)\big) \ .$$

Using (2.4), we find by a straightforward computation that

$$\delta\,Df = \varkappa f - \bar{\varkappa} f - D\delta f \ .$$

Hence $\varkappa f - \bar{\varkappa} f = \delta\,Df$ if f is a cocycle and thus

$$\varkappa \boldsymbol{f} = \bar{\varkappa} \boldsymbol{f} \quad \text{for} \quad \boldsymbol{f} \in H^q(\pi_1(K), G) \ .$$

A cocycle $f \in Z^q(K, \mathbf{G})$ will be called a *spherical annihilator* if for every simplicial map $T : S^q \to K$, where S^q is any simplicial division of the q-sphere, the cocycle $T^* f \in Z^q(S^q, T^* \mathbf{G})$ is a coboundary. The spherical annihilators form a subgroup of $Z^q(K, \mathbf{G})$ containing the group of coboundaries $B^q(K, \mathbf{G})$. The subgroup of $H^q(K, \mathbf{G})$ determined by the spherical annihilators is denoted by $\Lambda^q(K, \mathbf{G})$.

We shall now prove that

$$\varkappa\big(H^q(\pi_1(K), G)\big) \subset \Lambda^q(K, \mathbf{G}) \qquad \text{for } q \geqq 2 \ . \tag{2.5}$$

Indeed let $f \in Z^q(\pi_1(K), G)$: it was proved in [4], § 6 that we may assume that f is normalized, i. e. that $f(x_1, \ldots, x_q) = 0$ if at least one of the arguments $x_1, \ldots, x_q$ is 1. Let $T : S^q \to K$ be simplicial; without loss of generality we may assume that the base point V of K is of the form $V = T(U)$ where U is a vertex of S^q. Select the path system $\{P(A)\}$ in K in such a way that if $A \in T(S^q)$ then $P(A)$ is the

T-image of a path in S^q (with U as base point). Then for any vertices A^0, A^1 in a simplex of S^q the path

$$W\big(T(A^0),\,T(A^1)\big) = P\big(T(A^0)\big)\,T(A^0)\,T(A^1)\,P\big(T(A^1)\big)^{-1}$$

is the T-image of a closed path in S^q. Since $q \geqq 2$ we have $\pi_1(S^q) = 0$ so that W is nullhomotopic. Thus

$$\omega\big(T(A^0),\,T(A^1)\big) = 1\ .$$

Now

$$\begin{aligned}(T^* \varkappa f)\,(A^0,\ldots,A^q) &= \varkappa f\big(T(A^0),\ldots,T(A^q)\big)\\ &= \gamma^{-1}_{T(A^0)} f\big(\omega\,(T\,(A^0)\,,T\,(A^1))\,,\ldots,\omega\,(T\,(A^{q-1})\,,T\,(A^q))\big)\\ &= \gamma^{-1}_{T(A^0)} f\,(1,\ldots,1) = 0\end{aligned}$$

Hence $T^* \varkappa f = 0$ and $\varkappa f$ is a spherical annihilator.

The main theorem on the effect of $\pi_1(K)$ upon the cohomology groups of K can now be formulated.

Theorem I. If K is a connected simplicial complex such that

$$\pi_i(K) = 0 \quad \text{for} \quad 1 < i < q$$

then for every local system $\boldsymbol{G} = \{G_A\,,\gamma_{AB}\}$ of abelian groups in K the following isomorphisms hold

$$\varkappa : H^i\,\big(\pi_1(K),G\big) \approx H^i\,(K,\,\boldsymbol{G}) \qquad \text{for } i < q$$

$$\varkappa : H^q\,\big(\pi_1(K),G\big) \approx A^q\,(K,\,\boldsymbol{G})$$

where $G = G_V$, V is the base point of K, and the operators of $\pi_1(K)$ on G are defined by the local system $\boldsymbol{G}$.

If the local systems $\boldsymbol{G}' = \{G'_A\,,\gamma'_{AB}\}$ and $\boldsymbol{G}'' = \{G''_A\,,\gamma''_{AB}\}$ are paired to $\boldsymbol{G}$ then the groups $G' = G'_V$ and $G'' = G''_V$ are operator paired to $G = G_V$, i. e.

$$\gamma_\alpha(g' \cup g'') = (\gamma'_\alpha\,g') \cup (\gamma''_\alpha\,g'')$$

for every $\alpha \in \pi_1(K)$. A pairing of the groups $C^p\big(\pi_1(K),G'\big)$ and $C^q\big(\pi_1(K),G''\big)$ to the group $C^{p+q}\big(\pi_1(K),G\big)$ is then defined as follows (see [4])

$$(f' \cup f'')(x_1,\ldots,x_{p+q}) = f'(x_1,\ldots,x_p) \cup \gamma''_{x_1\ldots x_p} f''(x_{p+1},\ldots,x_{p+q})\ .$$

The usual formula

$$\delta(f' \cup f'') = (\delta f') \cup f'' + (-1)^p f' \cup \delta f''$$

is then valid and therefore a pairing of the cohomology groups $H^p(\pi_1(K), G')$ and $H^q(\pi_1(K), G'')$ to the group $H^{p+q}(\pi_1(K), G)$ is obtained.

From the definitions it follows that

$$\begin{aligned}
&\varkappa(f' \cup f'')(A^0, \ldots, A^{p+q}) \\
&= \gamma_{P(A^0)}^{-1}(f' \cup f'')\big(\omega(A^0, A^1), \ldots, \omega(A^{p+q-1}, A^{p+q})\big) \\
&= \gamma_{P(A^0)}'^{-1} f'\big(\omega(A^0, A^1), \ldots, \omega(A^{p-1}, A^p)\big) \cup \gamma_{P(A^0)}''^{-1} \gamma''_{\omega(A^0, A^p)} f''\big(\omega(A^p, A^{p+1}), \ldots, \\
&\qquad \omega(A^{p+q-1}, A^{p+q})\big) = \varkappa f'(A^0, \ldots, A^p) \cup \gamma''_{A^0 A^p} \varkappa f''(A^p, \ldots, A^{p+q}) \\
&= (\varkappa f' \cup \varkappa f'')(A^0, \ldots, A^{p+q})
\end{aligned}$$

thus

$$\varkappa(f' \cup f'') = \varkappa f' \cup \varkappa f'' \tag{2.5}$$

i. e. $\varkappa$ preserves the cup-product.

3. Characteristic cocycles χ^{n+1} $(n > 1)$.

Let K^n $(n > 0)$ be the n-dimensional skeleton of the complex K. For each vertex A of K the homotopy groups of K, of K^n, and the relative homotopy groups of K and K^n, form a sequence

$$\begin{aligned}
&\pi_1(K, A) \leftarrow \pi_1(K^n, A) \overset{\partial}{\leftarrow} \pi_2(K, K^n, A) \leftarrow \cdots \\
&\quad \leftarrow \pi_r(K, K^n, A) \leftarrow \pi_r(K, A) \leftarrow \pi_r(K^n, A) \overset{\partial}{\leftarrow} \pi_{r+1}(K, K^n, A) \leftarrow \cdots
\end{aligned}$$

which is exact, in the sense that the kernel of each homomorphism is the image of the next one. If A and B are vertices in a simplex of K then the isomorphism ϱ_{AB} of the homotopy sequence at B onto the homotopy sequence at A is defined in the usual way, and satisfies

$$\varrho_{AA} = \text{identity}, \qquad \varrho_{AB} = \varrho_{BA}^{-1}$$

Now assume that $n > 1$. If A, B, C are vertices in a simplex of K, then A, B, C are also vertices of a simplex of K^n and therefore

$$\varrho_{AB}\, \varrho_{BC} = \varrho_{AC}\,.$$

Thus the groups of the homotopy sequence are local systems of groups in K. There results an exact sequence of local systems

$$\Pi_1(K) \leftarrow \Pi_1(K^n) \overset{\partial}{\leftarrow} \Pi_2(K, K^n) \leftarrow \cdots$$
$$\leftarrow \Pi_r(K, K^n) \leftarrow \Pi_r(K) \leftarrow \Pi_r(K^n) \leftarrow \Pi_{r+1}(K, K^n) \leftarrow \cdots$$

Observe that $\Pi_r(K, K^n) = 0$ for $r<n$ and therefore $\Pi_r(K) \approx \Pi_r(K^n)$.

We shall be particularly interested in the homomorphisms

$$\Pi_{n+1}(K, K^n) \overset{\partial}{\rightarrow} \Pi_n(K^n) \rightarrow \Pi_n(K) \ .$$

The image system under ∂ will be denoted by $\theta_n = \{\vartheta_n(A)\}$.

Let $A^0, \ldots, A^{n+1}$ be vertices of K contained in a simplex of K. Let Δ^{n+1} be an $(n+1)$-dimensional simplex with vertices $d_0, \ldots, d_{n+1}$ and let $L : \Delta^{n+1} \rightarrow K$ be a simplicial map such that $L(d_i) = A^i$, $i = 0, \ldots, n+1$. The boundary $\dot{\Delta}^{n+1}$ of Δ^{n+1} is then mapped into K^n. Thus L is a map of triples

$$L : (\Delta^{n+1}, \dot{\Delta}^{n+1}, d_0) \rightarrow (K, K^n, A^0) \ .$$

With Δ^{n+1} oriented by the ordering of its vertices, this map determines an element

$$\chi^{n+1}(A^0, \ldots, A^{n+1}) \in \pi_{n+1}(K, K^n, A^0)$$

and we may regard χ^{n+1} as a cochain of K over the local system $\Pi_{n+1}(K, K^n)$.

Theorem II$_n$. χ^{n+1} is a cocycle

$$\chi^{n+1} \in Z^{n+1}\big(K, \Pi_{n+1}(K, K^n)\big)$$

its cohomology class will be denoted by χ^{n+1}.

Indeed, let $A^0, \ldots, A^{n+2}$ be vertices of K contained in a simplex of K and let $L : \Delta^{n+2} \rightarrow K$ be a simplicial map of an $(n+2)$-simplex Δ^{n+2} with vertices $d_0, \ldots, d_{n+2}$ such that $L(d_i) = A^i$, $i = 0, \ldots n+2$. Let Δ_i be the face of Δ^{n+2} with vertices $d_0, \ldots, \hat{d}_i, \ldots, d_{n+2}$. The maps

$$L : (\Delta_i, \dot{\Delta}_i, d_0) \rightarrow (K, K^n, A^0) \ , \qquad i = 1, \ldots, n+2$$

determine the elements

$$\chi^{n+1}(A^0, \ldots, \hat{A}^i, \ldots, A^{n+2}) \in \Pi_{n+1}(K, K^n, A^0)$$

while the map
$$L : (\Delta_0, \dot{\Delta}_0, d_1) \to (K, K^n, A^1)$$
determines the element
$$\chi^{n+1}(A^1, \ldots, A^{n+2}) \in \Pi_{n+1}(K, K^n, A^1) \ .$$
Also the map
$$L : (\dot{\Delta}^{n+2}, A^0, A^0) \to (K, K^n, K^0)$$
determines an element $\alpha \in \pi_{n+1}(K, K^n, A^0)$. By a general additivity theorem in homotopy theory
$$\alpha = \varrho_{A^0A^1}\, \chi^{n+1}(A^1, \ldots, A^{n+2}) + \sum_{i=1}^{n+2} (-1)^i \chi^{n+1}(A^0, \ldots, \hat{A}^i, \ldots, A^{n+2})$$
$$= \delta\chi^{n+1}(A^0, \ldots, A^{n+2}) \ .$$
However $\alpha = 0$ since L is defined throughout all of Δ^{n+2} and thus $\delta\chi^{n+1} = 0$.

Combining the cocycle χ^{n+1} with the map
$$\partial : \Pi_{n+1}(K, K^n) \to \theta_n$$
yields a cocycle
$$\xi^{n+1} = \partial\chi^{n+1} \in Z^{n+1}(K, \theta_n)$$
whose cohomology class will be denoted by ξ^{n+1}.

Assume now that $\pi_1(K) = 0$. The local systems $\Pi_{n+1}(K, K^n)$ and θ_n are then simple and can be replaced by the single groups $\pi_{n+1}(K, K^n)$ and ϑ_n. Thus χ^{n+1} and ξ^{n+1} become cocycles in the ordinary sense
$$\chi^{n+1} \in Z^{n+1}\big(K, \pi_{n+1}(K, K^n)\big) \ , \qquad \xi^{n+1} \in Z^{n+1}(K, \vartheta_n) \ .$$
Assume further that
$$\pi_i(K) = 0 \qquad \text{for} \qquad i = 1, \ldots, n-1 \ .$$
It follows that
$$\pi_i(K^n) = 0 \qquad \text{for} \qquad i = 1, \ldots, n-1$$
and from the exactness of the sequence
$$\ldots \to \pi_{n+1}(K, K^n) \xrightarrow{\partial} \pi_n(K^n) \to \pi_n(K) \to \pi_n(K, K^n) = 0$$
it follows that $\pi_n(K^n)$ maps onto $\pi_n(K)$ with kernel ϑ_n. Further from a theorem of Hurewicz
$$\pi_n(K) \approx H_n(K)$$
$$\pi_n(K^n) \approx H_n(K^n) = Z_n(K^n) = Z_n(K) \ .$$

The map $\pi_n(K^n) \to \pi_n(K)$ may thus be replaced by the natural homomorphism $Z_n(K) \to H_n(K)$ and under this substitution the group ϑ_n is replaced by the group $B_n(K)$ of boundaries. Thus ξ^{n+1} may be regarded as a cocycle

$$^h\xi^{n+1} \in Z^{n+1}\big(K, B_n(K)\big) .$$

This last cocycle could of course be defined directly in terms of homology, without any assumption on K and without local systems.

Indeed, consider the boundary homomorphism

$$\partial : C_{n+1}(K) \to B_n(K)$$

which may be regarded as a cochain

$$\partial \in C^{n+1}\big(K, B_n(K)\big) .$$

Then ∂ is a cocycle and is equal to $^h\xi^{n+1}$.

We may observe that the cocycle ξ^{n+1} may be regarded as the „obstruction" against retracting K^{n+1} to K^n, and χ^{n+1} as the „obstruction" against retracting K^{n+1} to K^n by deformation.

4. The cocycles χ^2 and ξ^2.

We now turn to the discussion of the case $n = 1$. The relation $\varrho_{AB}\, \varrho_{BC} = \varrho_{AC}$ is then not always valid and the groups involved will have to be modified before they form local systems.

Let $\gamma_2(A)$ be the commutator subgroup of $\pi_2(K, K^1, A)$ and let $\varkappa_1(A)$ be the kernel of the homomorphism

$$i_1 : \pi_1(K^1, A) \to \pi_1(K, A) . \tag{4.1}$$

Since i_1 is a mapping onto, $\varkappa_1(A)$ is an invariant subgroup of $\pi_1(K^1, A)$ and by exactness, $\varkappa_1(A) = \partial\pi_2(K, K^1, A)$. It follows that

$$\partial\gamma_2(A) = [\varkappa_1(A), \varkappa_1(A)]$$

is the commutator subgroup of $\varkappa_1(A)$ and is an invariant subgroup of both $\varkappa_1(A)$ and $\pi_1(K^1, A)$. Define

$$\tilde{\pi}_2(K, K^1, A) = \pi_2(K, K^1, A) \,/\, \gamma_2(A) \tag{4.2}$$

$$\tilde{\pi}_1(K^1, A) \qquad = \pi_1(K^1, A) \,/\, [\varkappa_1(A), \varkappa_1(A)] \tag{4.3}$$

$$\vartheta_1(A) \qquad = \partial\tilde{\pi}_2(K, K^1, A) = \varkappa_1(A) \,/\, [\varkappa_1(A), \varkappa_1(A)] \tag{4.4}$$

and consider the sequence

$$\vartheta_1(A) \overset{\partial}{\leftarrow} \tilde{\pi}_2(K, K^1, A) \leftarrow \pi_2(K, A) \leftarrow \cdots \tag{4.5}$$

The definition of the isomorphisms ϱ_{AB} carries over to the groups of the sequence (4.5). We shall show that the groups of the sequence (4.5) form a sequence of local systems

$$\theta_1 \overset{\partial}{\leftarrow} \tilde{\Pi}_2(K, K^1) \leftarrow \Pi_2(K) \leftarrow \cdots$$

Indeed, let A, B, C be vertices in a simplex of K and let α be the element of $\pi_1(K^1, A)$ determined by the path $ABCA$. The automorphism $\varrho_{AB}\,\varrho_{BC}\,\varrho_{CA}$ is then equivalent with operating with the element α on the groups of (4.5). Since α maps into the unit element of $\pi_1(K, A)$ it follows from exactness that $\alpha = \partial\beta$ for some $\beta \in \pi_2(K, K^1, A)$. It is known that operating with α on $\pi_2(K, K^1, A)$ is equivalent with conjugation by β, while operating with α on the higher groups of the sequence is trivial. Thus α operates trivially on $\tilde{\pi}_2(K, K^1, A)$ and therefore also on $\vartheta_1(A)$. It follows that $\varrho_{AB}\,\varrho_{BC} = \varrho_{AC}$.

The cocycle χ^2 is then defined exactly as in the previous section except that $\chi^2(A^0, A^1, A^2)$ is considered as an element of $\tilde{\pi}_2(K, K^1, A^0)$. The same proof as before yields

Theorem II$_1$. χ^2 is a cocycle

$$\chi^2 \in Z^2\big(K, \tilde{\Pi}_2(K, K^1)\big)$$

its cohomology class will be denoted by χ^2.

Combining χ^2 with the map $\partial : \tilde{\Pi}_2(K, K^1) \to \theta_1$ we obtain a cocycle

$$\xi^2 = \partial\chi^2 \in Z^2(K, \theta_1)$$

whose cohomology class will be denoted by ξ^2.

Consider now the homomorphism

$$\Phi : \tilde{\pi}_1(K^1) \to \pi_1(K)$$

(both groups taken at the base point V) induced by (4.1). Φ maps $\tilde{\pi}_1(K^1)$ onto $\pi_1(K)$ with the abelian group ϑ_1 as kernel. Thus the pair $(\tilde{\pi}_1(K^1), \Phi)$ is a group extension of $\pi_1(K)$ by ϑ_1. This extension induces operators of $\pi_1(K)$ on ϑ_1. We shall show that these operators agree with the operators resulting from the local system θ_1. What needs to be proved is that for $\alpha \in \tilde{\pi}_1$, $\beta \in \vartheta_1$

$$\varrho_{\Phi(\alpha)}\,\beta = \alpha\,\beta\,\alpha^{-1}\,. \tag{4.6}$$

If P and Q are closed edge-paths (about the base point V) representing the elements α and β of $\tilde{\pi}_1$ then $\varrho_{\Phi(\alpha)}\beta = \varrho_P\beta$ and from the definition of ϱ_P it follows directly that $\varrho_P\beta$ is represented by the path PQP^{-1}. This proves (4.6).

The extension $(\pi_1(K^1), \Phi)$ determines a cohomology class

$$f_0 \in H^2(\pi_1(K), \vartheta_1)$$

Theorem III. $\varkappa f_0 = \xi^2$.

Proof. Let $\{P(A)\}$ be the path system used in defining $\varkappa$. For each pair of vertices A, B of a simplex of K let

$$\omega(A, B) \in \pi_1(K) , \qquad \Omega(A, B) \in \tilde{\pi}_1(K^1)$$

be the elements determined by the closed path

$$W(A, B) = P(A)ABP(B)^{-1} .$$

Clearly

$$\Phi\,\Omega(A, B) = \omega(A, B) . \tag{4.7}$$

Let A^0, A^1, A^2 be vertices of K in a simplex of K. From the definition of $\chi^2(A^0, A^1, A^2) \in \tilde{\pi}_2(K, K^1, A^0)$ and from the definition of the map $\partial : \tilde{\pi}_2(K, K^1, A) \to \vartheta_1(A)$ it follows that $\xi^2(A^0, A^1, A^2)$ is the element of $\vartheta_1(A_0)$ determined by the closed path $A^0\,A^1\,A^2\,A^0$. Thus the element $\varrho_{P(A^0)}\,\xi^2(A^0, A^1, A^2)$ of ϑ_1 is determined by the path

$$P(A^0)\,A^0\,A^1\,A^2\,A^0\,P(A^0)^{-1} .$$

Since this path is homotopic in K^1 with the path

$$W(A^0, A^1)\,W(A^1, A^2)\,W(A^0, A^2)^{-1} .$$

It follows that

$$\Omega(A^0, A^1)\,\Omega(A^1, A^2) = \varrho_{P(A^0)}\,\xi^2(A^0, A^1, A^2)\,\Omega(A^0, A^2) . \tag{4.8}$$

Now select for each $x \in \pi_1(K)$ a representative $u(x) \in \tilde{\pi}_1(K^1)$ so that

$$\Phi\,u(x) = x .$$

Then for $x, y \in \pi_1(K)$

$$u(x)\,u(y) = f_0(x, y)\,u(xy)$$

where the factor set f_0 is a cocycle $f_0 \in Z^2(\pi_1(K), \vartheta_1)$ in the cohomology class $\boldsymbol{f}_0$.

From (4.7) we deduce that

$$\Omega(A, B) = h(A, B)\, u(\omega(A, B))$$

for some $h(A, B) \in \vartheta_1$. The left hand side of (4.8) then yields

$$\begin{aligned}
&\Omega(A^0, A^1)\,\Omega(A^1, A^2) = h(A^0, A^1)\, u(\omega(A^0, A^1))\, h(A^1, A^2)\, u(\omega(A^1, A^2)) \\
&= h(A^0, A^1)[\varrho_{\omega(A^0, A^1)} h(A^1, A^2)]\, u(\omega(A^0, A^1))\, u(\omega(A^1, A^2)) \\
&= h(A^0, A^1)\,[\varrho_{\omega(A^0, A^1)} h(A^1, A^2)]\, f_0(\omega(A^0, A^1), \omega(A^1, A^2))\, u(\omega(A^0, A^2))
\end{aligned}$$

while the right hand side gives

$$\varrho_{P(A^0)}\, \xi^2(A^0, A^1, A^2)\, h(A^0, A^2)\, u(\omega(A^0, A^2))$$

and thus

$$\begin{aligned}
&h(A^0, A^1)\,[\varrho_{\omega(A^0, A^1)} h(A^1, A^2)]\, f_0(\omega(A^0, A^1), \omega(A^1, A^2)) \\
&= \varrho_{P(A^0)}\, \xi^2(A^0, A^1, A^2)\, h(A^0, A^2)\ .
\end{aligned}$$

Since all the elements involved are in ϑ_1 we can pass to additive notation. Applying $\varrho^{-1}_{P(A^0)}$ will place all the elements in the group $\vartheta_1(A^0)$:

$$\begin{aligned}
&\varrho^{-1}_{P(A^0)} h\,(A^0, A^1) + \varrho^{-1}_{P(A^0)}\, \varrho_{\omega(A^0, A^1)} h\,(A^1, A^2) \\
&+ \varrho^{-1}_{P(A^0)} f_0\,(\omega\,(A^0, A^1), \omega\,(A^1, A^2)) = \xi^2\,(A^0, A^1, A^2) + \varrho^{-1}_{P(A^0)} h\,(A^0, A^2)\ .
\end{aligned}$$

Now observe that

$$\varrho^{-1}_{P(A^0)}\, \varrho_{\omega(A^0, A^1)} = \varrho_{A^0A^1}\, \varrho^{-1}_{P(A^1)}$$

and by (2.2)

$$\varrho^{-1}_{P(A^0)} f_0(\omega(A^0, A^1), \omega(A^1, A^2)) = \varkappa\, f_0(A^0, A^1, A^2)\ .$$

Define the cochain $\varphi \in C^1(K, \theta_1)$ by setting

$$\varphi(A^0, A^1) = \varrho^{-1}_{P(A^0)} h(A^0, A^1) \in \vartheta_1(A^0)\ .$$

Then

$$\begin{aligned}
&\varphi(A^0, A^1) + \varrho_{A^0A^1}\, \varphi(A^1, A^2) + \varkappa\, f_0(A^0, A^1, A^2) \\
&= \xi^2(A^0, A^1, A^2) + \varphi(A^0, A^2)
\end{aligned}$$

or

$$\xi^2 = \varkappa\, f_0 + \delta\varphi$$

which proves the theorem.

From Theorems I and III we deduce

Corollary IV. $\xi^2 \in \Lambda^2(K, \theta_1)$.

This fact could easily be established directly.

5. The reduction theorems.

Given two local systems $\boldsymbol{G} = \{G_A, \gamma_{AB}\}$, $\boldsymbol{H} = \{H_A, \eta_{AB}\}$ of abelian groups in K define a new system

$$\boldsymbol{G}^{\boldsymbol{H}} = \{T_A, \tau_{AB}\}$$

where T_A is the group of all homomorphisms of H_A into G_A, while τ_{AB} is defined for each $t \in T_B$ as

$$\tau_{AB}\, t = \gamma_{AB}\, t\, \eta_{AB}^{-1} .$$

Setting

$$h \cup t = t(h)$$

for $h \in H_B$, $t \in T_B$ we find

$$\gamma_{AB}(h \cup t) = \gamma_{AB}\, t\,(h) = \gamma_{AB}\, t\,(\eta_{AB}^{-1}\, \eta_{AB}\, h)$$
$$= (\tau_{AB}\, t)\,(\eta_{AB}\, h) = \eta_{AB}\, h \cup \tau_{AB}\, t$$

so that $\boldsymbol{H}$ and $\boldsymbol{G}^{\boldsymbol{H}}$ are paired to $\boldsymbol{G}$.

In particular if we consider the local system $\boldsymbol{G}^{\theta_1}$ then the mapping $\varphi \to \xi^2 \cup \varphi$ for $\varphi \in H^n(K, \boldsymbol{G}^{\theta_1})$ defines a homomorphism

$$\xi^2 \cup : H^n(K, \boldsymbol{G}^{\theta_1}) \to H^{n+2}(K, \boldsymbol{G}) . \qquad (5.1)$$

Let G^{ϑ_1} be the group of the system $\boldsymbol{G}^{\theta_1}$ at the base point V. Then G^{ϑ_1} is the group of all homomorphisms $t : \vartheta_1 \to G$ with operators

$$\tau_\alpha\, t = \gamma_\alpha\, t\, \varrho_\alpha^{-1} .$$

Then the correspondence $f \to f_0 \cup f$ for $f \in H^n(\pi_1(K), G^{\vartheta_1})$, $\alpha \in \pi_1(K)$ yields a homomorphism

$$f_0 \cup : H^n(\pi_1(K), G^{\vartheta_1}) \to H^{n+2}(\pi_1(K), G) .$$

This is precisely the homomorphism used in [4], § 10 to establish the cup product reduction theorem. Indeed setting

$$F = \pi_1(K^1) \qquad R = \partial\pi_2(K, K^1)$$

we find that F is free and that F is mapped onto $\pi_1(K)$ with R as kernel. Thus

$$\pi_1(K) = F/R ,$$

Further by (4.3) and (4.4)

$$R_0 = R/[R, R] = \vartheta_1$$
$$F_0 = F/[R, R] = \tilde{\pi}_1$$

so that $$\pi_1(K) = \pi_1/\vartheta_1 = F_0/R_0 \tag{5.2}$$

and f_0 is the cohomology class in $H^2(\pi_1(K), \vartheta_1) = H^2(\pi_1(K), R_0)$ determined by the extension (5.2). Hence the reduction theorem of [4] implies that

$$f_0 \cup : H^n(\pi_1(K), G^{\delta_1}) \approx H^{n+2}(\pi_1(K), G)\ , \qquad \text{for } n>0. \tag{5.3}$$

Further from (2.5) and Theorem III we deduce that

$$\varkappa(f_0 \cup f) = \varkappa f_0 \cup \varkappa f = \xi^2 \cup \varkappa f\ . \tag{5.4}$$

Combining (5.3), (5.4) and Theorem I yields

Theorem V. If $q>0$ and

$$\pi_i(K) = 0 \quad \text{for} \quad 1<i<q+2$$

then the following isomorphisms hold

$$\xi^2 \cup : H^n(K, \mathbf{G}^{\theta_1}) \approx H^{n+2}(K, \mathbf{G}) \quad \text{for } 0<i<q$$

$$\xi^2 \cup : H^q(K, \mathbf{G}^{\theta_1}) \approx \Lambda^{q+2}(K, \mathbf{G})\ .$$

Since in this theorem it is assumed that $\pi_2(K) = 0$ it follows that the local systems θ_1 and $\tilde{\Pi}_2(K, K^1)$ are isomorphic under ∂ and therefore in the theorem ξ^2 can be replaced by χ^2 with $\mathbf{G}^{\theta_1}$ replaced by $\mathbf{G}^{\tilde{\Pi}_2}$.

Before we proceed with a discussion of the homomorphism (5.1) for $n=0$, we must examine more closely the groups $H^0(K, \mathbf{G})$ and $H^0(\pi_1(K), G)$. Let $\varphi \in C^0(K, \mathbf{G})$ be any 0-cochain of K over $\mathbf{G}$. Then $\delta\varphi(A^0, A^1) = \gamma_{A^0A^1}\varphi(A^0) - \varphi(A^1)$. Hence φ is a cocycle if and only if

$$\varphi(A^0) = \gamma_{A^0A^1}\varphi(A^1)\ .$$

This implies that if P is any edge-path joining vertices A and B of K then

$$\varphi(A) = \gamma_P\, \varphi(B)\ .$$

Since K is connected it follows that the value $\varphi(V)$ of φ at the base point V, determines the cocycle φ. Moreover the element $\varphi(V) \in G$ is invariant under the operators γ_α, $\alpha \in \pi_1(K)$. Let G_A^0 denote the subgroup of G_A consisting of the elements invariant under the operators. The groups G_A^0 form a simple subsystem $\mathbf{G}^0$ of $\mathbf{G}$. Thus $\mathbf{G}^0$ may be identified with the group G^0 (at the base point) and

$$H^0(K, \mathbf{G}) = H^0(K, \mathbf{G}^0) = G^0\ .$$

Now consider $f \in C^0(\pi_1(K), G)$. Then $f \in G$ and $(\delta f)(x) = \gamma_x f - f$ for every $x \in \pi_1(K)$. Hence f is a cocycle if and only if f is in the subgroup G^0. Thus

$$H^0(\pi_1(K), G) = H^0(\pi_1(K), G^0) = G^0 .$$

With these conventions the mapping $\varkappa : H^0(\pi_1(K), G) \to H^0(K, \mathbf{G})$ becomes the identity map of G^0 onto G^0.

Now consider a 0-cocycle $\varphi \in Z^0(K, \mathbf{G}^{\theta_1})$. The coboundary formula then gives $\varphi(A) = \tau_{AB}\, \varphi(B) = \gamma_{AB}\, \varphi(B)\, \varrho_{AB}^{-1}$, or

$$\varphi(A)\, \varrho_{AB} = \gamma_{AB}\, \varphi(B) .$$

Thus φ is simply a homomorphism $\theta_1 \to \mathbf{G}$ of the local system θ_1 into $\mathbf{G}$. This homomorphism is determined by an operator homomorphism

$$\varphi(V) : \vartheta_1 \to G .$$

Further

$$(\xi^2 \cup \varphi)(A^0, A^1, A^2) = \xi^2(A^0, A^1, A^2) \cup \tau_{A^0A^2}\, \varphi(A^2)$$

$$= \xi^2(A^0, A^1, A^2) \cup \varphi(A^0) = \varphi(A^0)\big(\xi^2(A^0, A^1, A^2)\big) .$$

Thus $\xi^2 \cup \varphi$ is the image of the cohomology class ξ^2 under the homomorphism

$$H^2(K, \theta_1) \to H^2(K, \mathbf{G})$$

induced by the homomorphism

$$\varphi : \theta_1 \to \mathbf{G} .$$

Similarly if we consider $\varphi(V)$ as a cohomology class in $H^0(\pi_1(K), G^{\vartheta_1})$ then $f_0 \cup \varphi(V)$ is the image of f_0 under the homomorphism

$$H^2(\pi_1(K), \vartheta_1) \to H^2(\pi_1(K), G)$$

induced by the operator homomorphism

$$\varphi(V) : \vartheta_1 \to G .$$

Since $\varkappa$ maps $H^2(\pi_1(K), G)$ isomorphically onto $\Lambda^2(K, \mathbf{G})$. Theorem 13.1 of [4] yields

Theorem VI. Let Hom (θ_1, G) be the group of all homomorphisms φ of the local system θ_1 into the local system G. The mapping

$$\mathrm{Hom}(\theta_1, G) \to H^2(K, G) \qquad (5.5)$$

which to each φ assigns the image of ξ^2 under the induced homomorphism

$$H^2(K, \theta_1) \to H^2(K, G)$$

maps $\mathrm{Hom}(\theta_1, G)$ onto $\Lambda^2(K, G)$. The kernel of (5.5) consists of those homomorphisms φ, for which the operator homomorphism $\varphi(V) : \vartheta_1 \to G$ can be extended to a crossed homomorphism $\bar{\varphi} : \tilde{\pi}_1(K^1) \to G$.

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