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On finite groups which are necessarily commutative

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It is known that every group of prime order is Abelian. We want to show some further cases in which the commutativity of the group is a consequence of arithmetical properties of its order. Indeed, $p_1, \ldots, p_n$ denoting different primes, we have the following

Theorem 1. *Every group of order* $p_1 \ldots p_n$ *is commutative provided that* $p_i \not\equiv 1 \pmod{p_k}$ *for* $i \neq k,\ i, k = 1, \ldots, n$.

With the same trouble, we prove the following more general

Theorem 2. *Every solvable group G of an order* $m = p_1^{\alpha_1} \ldots p_n^{\alpha_n}$ *is commutative provided its Sylow-groups are commutative and, for* $i = 1, \ldots, n$, $\alpha_i < \gamma_i$, γ_i *denoting the least positive integer for which* $p_i^{\gamma_i} \equiv 1 \pmod{p_k}$ *for some* $k = (1, \ldots, n)$.

For $n = 1$, our assertion is trivial. Let $n \geqq 2$ and suppose, the theorem holds for $n - 1$ (instead of n). G being solvable, it has, by a theorem of Hall[1]), a subgroup H of order $m p_n^{-\alpha_n}$. By hypothesis, H is commutative. Denote P the subgroup of order $p_1^{\alpha_1}$ of H (which is a Sylow-group of G), further, denote N the normalizer of P. As $H \subseteqq N$, we have for the order ν of N:

$$m p_n^{-\alpha_n} \mid \nu \,, \qquad \nu \mid m \,.$$

Decomposing G according to the modul P, P:

$$G = P + PA_2P + PA_3P + \cdots \,,$$

[1]) *P. Hall*, A characteristic property of soluble groups, Journal London Math. Soc. **12** (1937), pp. 198—200.

we denote the number of right-side cosets of P in the terms right-hand side by $a_1(=1)$, $a_2, a_3, \ldots$ respectively. Then we have

$$m p_1^{-\alpha_1} = \sum a_i \ .$$

Each a_i is a divisor of the order $p_1^{\alpha_1}$ of P and the number of the terms with $a_i = 1$ equals the index $\nu p_1^{-\alpha_1}$ of P in N[2]), and therefore

$$m p_1^{-\alpha_1} \equiv \nu p_1^{-\alpha_1} \pmod{p_1} \ .$$

Hence $\frac{m}{\nu} \equiv 1 \pmod{p_1}$. Here the left-hand side is a power $\leqq p_n^{\alpha_n} < p_n^{\gamma_n}$ of p_n, and so we have necessarily $\frac{m}{\nu} = 1$, $\nu = m$, $N = G$. Thus we have got that P is normal in G, and this holds for every Sylow-group of G. Since any two of these Sylow-groups have relatively prime orders, they are commutable element by element[3]). Moreover, they are Abelian and so the same holds for their product G, as stated.

Remark. Owing to Dirichlet's theorem, for each n, there is an infinite set of numbers $p_1, \ldots, p_n$ for which theorem 1 applies.

(Eingegangen den 5. Januar 1947.)

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[2]) *A. Speiser*, Theorie der Gruppen von endlicher Ordnung, 2nd Edition (Berlin 1937), theorems 64, 66.

[3]) *A. Speiser*, loc. cit., theorem 17.