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On Tauber's Theorem

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It seems to be a general principle that theorems which are „Tauberian" in the sense of Hardy and Littlewood are mere corollaries of universal inequalities, which are valid for arbitrary (rather than just for convergent) series and contain, therefore, absolute constants. In the present note the corresponding refinement of Tauber's own theorem [3] will be deduced.

Tauber's theorem states that, in order that a series $a_1 + a_2 + \cdots$ be convergent, its (A)-summability and the Cauchy-Kronecker condition

$$a_1 + \cdots + na_n = o(n) \tag{1}$$

are not only necessary but sufficient as well. This will be refined as follows:

There exists an absolute constant, τ, having the property that

$$\limsup_{r \to 1-0} \left| \sum_{n=1}^{\infty} a_n r^n - \sum_{n \leqq -1/\log r} a_n \right| \leqq \tau \limsup_{n \to \infty} \left| a_1 + \cdots + na_n \right| / n$$

holds for every power series,

$$f(r) = \sum_{n=1}^{\infty} a_n r^n , \tag{2}$$

which converges for $r < 1$.

It is understood that the lim sup can be ∞ on either side of the inequality which, however, is then trivial.

A corollary is that, if the Cauchy-Kronecker condition is assumed, then, since the expression on the right of the inequality becomes 0, the deviation of the function (2) from the $[-1/\log r]$-th partial sum of the series $a_1 + a_2 + \cdots$ must tend to 0 as $r \to 1 - 0$, whether the series $a_1 + a_2 + \cdots$ be convergent or divergent. Since Tauber's theorem assumes for (2) the existence of a limit $f(1 - 0)$, it is equivalent to the first of the two cases of this corollary.

Another corollary is that if (1) is relaxed to

$$a_1 + \cdots + na_n = O(n) , \tag{3}$$

the asymptotic behavior of the partial sums of $a_1 + a_2 + \cdots$ can be obtained from that of the function (2) as $r \to 1 - 0$:

$$\sum_{n=1}^{x} a_n = f(e^{-1/x}) + O(1) \quad \text{as} \quad x \to \infty \tag{4}$$

(needless to say, (3) implies the convergence of (2) for $r < 1$).

The full content of the theorem is that, if c and C denote the greatest lower bounds of the constants, $c + \varepsilon$ and $C + \varepsilon$, which are admissible as factors absorbed in the O of (3) and in the O of (4), respectively, then

$$C \leqq \tau c\ , \tag{5}$$

where τ is an absolute constant. For the latter, the proof of the theorem will supply only the estimate

$$\tau \leqq 3 + \int_1^\infty x^{-1} e^{-x} dx \qquad (< 3 + \tfrac{1}{2})\ . \tag{6}$$

The integral occurring in (6) will be obtained from an expression connected with the harmonic series, Σn^{-1}, whereas the 3 will result from three dependent sources (hence, very roughly), as $1 + 1 + 1$, one of the latter being supplied by the fact that, as easily verified by differentiation,

$$0 < x^{-1}(1 - e^{-x}) < 1 \quad \text{if} \quad 0 < x < 1\ . \tag{7}$$

The determination of the true value of τ (that is, of the least absolute constant) seems to be hard. The lower estimate

$$\tau \geqslant 1\ , \tag{8}$$

which is quite far from the upper estimate (6), is trivial. In fact, if $a_n = (-1)^n$, then

$$\Big| \sum_{n=1}^{x} n a_n \Big| / x \to \tfrac{1}{2} \text{ and } \sum_{n=1}^{x} a_n = -\tfrac{1}{2} \pm \tfrac{1}{2}\ ;$$

so that, since (2) becomes $-f(r)/r = (1 + r)^{-1} \to \frac{1}{2}$, the inequality (5) gives

$$\left| \tfrac{1}{2} - (\tfrac{1}{2} \pm \tfrac{1}{2}) \right| \leqq \tau \tfrac{1}{2}\ ,$$

which is (8).

There exists an absolute constant, say τ^*, having the property that

$$\limsup_{r \to 1-0} \Big| f(r) - \sum_{n \leqq -1/\log r} a_n \Big| \leqq \tau^* \limsup_{n \to \infty} | n a_n | \tag{9}$$

holds for every power series (2) which is convergent for $r < 1$. In fact, if τ and τ^* have their least values, then

$$\tau^* \leqq \tau \ , \tag{10}$$

since τ^* belongs to the restriction

$$na_n = O(1) \tag{11}$$

in the same way as τ belongs to (3), a generalization of (11). The existence of τ^* (which, in view of (10), is implied by the existence of τ) has been pointed out by Hadwiger [1] (actually, he considers another constant, for which he proves the estimate

$$0.4858\ldots \leqq \varrho \leqq 1.0160\ldots \ , \tag{12}$$

and for which the inequality

$$\varrho \leqq \tau^* \tag{13}$$

is clear from the definitions).

Needless to say, what the existence of τ^* reduces to absolute terms is that particular case of Tauber's theorem according to which the o-form of Littlewood's condition (11), that is, the strengthening of (1) to

$$na_n = o(1) \ , \tag{14}$$

is sufficient for the convergence of an (A)-summable series $a_1 + a_2 + \cdots$. Thus it is clear that the existence of τ^*, in contrast to the existence of τ, does not imply Tauber's theorem; simply because (1), hence (3), is necessary, but (11), hence (14), is not necessary, for the convergence of $a_1 + a_2 + \cdots$.

Tauber's own proof [3] of the sufficiency of the necessary condition (1) (in order that an (A)-summable series $a_1 + a_2 + \cdots$ be convergent) first establishes the sufficiency of condition (14), which is not a necessary condition, and then passes from (14) to the true condition, (1), by additional steps. This detour to the final theorem is followed by all the textbooks consulted (Hobson, Knopp, Landau, Widder), even though it just complicates the proof of Tauber's theorem. A shorter approach can be read off from a paper of Hardy [2], appearing some time ago. Hardy is concerned in [2] with a Laplace integral, which he writes as a Lebesgue integral, but his proof, which avoids the detour just mentioned, is valid, of course, for Stieltjes integrals as well, and so for power series (or Dirichlet series) also. This possibility of avoiding the detour, and thus simplifying the traditional approach, will be utilized in the following proof.

Let

$$\alpha(x) = \sum_{n=1}^{x} a_n \quad \text{and} \quad \beta(x) = \sum_{n=1}^{x} n a_n , \tag{15}$$

where x is a continuous variable (the summations are thought to be arrested at $n = [x]$). If $r = e^{-s}$, then $r \to 1 - 0$ goes over into $s \to +0$, and the series (2) becomes the integral ·

$$F(s) = \int_0^\infty e^{-sx}\, d\alpha(x) \qquad (s > 0) , \tag{16}$$

where $F(s)$ denotes $f(e^{-s})$, the function $\alpha(x)$ is 0 when $0 \leqq x < 1$, and

$$\beta(x) = \int_0^x t d\alpha(t) , \tag{17}$$

by (15). It can, of course, be assumed that

$$\alpha(x) = \alpha(1+0) = 0, \text{ hence } \beta(x) = \beta(1+0) = 0, \text{ if } 0 \leqq x \leqq 1 . \tag{18}$$

It is clear from (15) that the theorem to be proved, that italicized after (1), is equivalent to the assertion that

$$\limsup_{s \to +0} |\, F(s) - \alpha(s^{-1})\, | \leqq \tau \limsup_{x \to \infty} |\, \beta(x)\, | \,/\, x . \tag{19}$$

Actually, only (18), (17) and the convergence of (16) for $s > 0$ will be used in the proof of (19); so that the existence of some absolute constant will be proved for the case of Laplace-Stieltjes integrals also (but this is not of course the point, every „generalization“ of this kind being automatic indeed).

First, (17) and (18) show that (16) can be written in the form

$$F(s) = \int_1^\infty x^{-1} e^{-sx}\, d\beta(x) ,$$

where $s > 0$. In view of (18), and since

$$e^{sx} d(x^{-1} e^{-sx}) = -(x^{-2} + s x^{-1})\, dx ,$$

a partial integration of this integral gives

$$F(s) = \int_1^\infty x^{-2} \beta(x) e^{-sx} dx + sA(s) , \tag{20}$$

if $A(s)$ is an abbreviation for

$$A(s) = \int_1^\infty x^{-1} \beta(x) e^{-sx} dx . \tag{20_0}$$

On the other hand, again by partial integration,

$$\int_1^{1/s} x^{-1} d\beta(x) = s\beta(1/s) - \beta(1) + \int_1^{1/s} x^{-2}\beta(x)dx \ .$$

In view of (17) and (18), this relation can be written in the form

$$\alpha(1/s) = s\beta(1/s) + \int_1^{1/s} x^{-2}\beta(x)dx \ .$$

If this is subtracted from (20), it follows that

$$F(s) - \alpha(1/s) = sA(s) + B(s) - D(s) - s\beta(1/s) \ , \tag{21}$$

where

$$B(s) = \int_{1/s}^{\infty} x^{-2}\beta(x)e^{-sx}dx \tag{22}$$

and

$$D(s) = \int_1^{1/s} x^{-2}\beta(x)(1 - e^{-sx})dx \ . \tag{23}$$

It is seen from (21) that, if

$$\limsup_{x\to\infty} |\beta(x)| / x \tag{24}$$

is assumed to have the value 1, both (19) and (6) will be proved if it is shown that, on the one hand, the upper limit, as $s \to +0$, of none of the three functions

$$(25_1) \quad s\,|\beta(1/s)| \ ; \qquad (25_2) \quad s\,|A(s)| \ ; \qquad (25_3) \quad |D(s)|$$

can exceed 1, and, on the other hand,

$$\limsup_{s\to+0} |B(s)| \leq \int_1^{\infty} x^{-1}e^{-x}dx \ . \tag{26}$$

But the assumption that the value of (24) is 1 does not involve a loss of generality. For, if (19) is true when (24) has the value 1, then, for reasons of distributivity, (19) is true if (24) has any value distinct from 0 and ∞, and so, again for reasons of distributivity, if (24) has any value distinct from ∞; and (19) is trivial if (24) is ∞. Accordingly, it can be assumed that (24) is 1, i. e., that there belongs to every $\varepsilon>0$ an R such that

$$|\beta(x)| < (1+\varepsilon)x \quad \text{if} \quad x > R = R_\varepsilon \ . \tag{27}$$

Ad (25_1). The upper limit, as $s \to +0$, of (25), is (24), which is 1.

Ad (25_2). According to (27), the contribution of the range $R<x<\infty$ to the integral (20_0) is majorized by

$$(1+\varepsilon)\int_R^\infty e^{-sx}dx<(1+\varepsilon)\int_0^\infty e^{-sx}dx=(1+\varepsilon)/s\ .$$

Since s times the contribution of the complementary range, $0\leqq x\leqq R$, where $R=R_\varepsilon$, tends to 0 as $s\to+0$ when ε is fixed, it follows that the upper limit of (25_2) cannot exceed $1+\varepsilon$ and is, therefore, not greater than 1.

Ad (25_3). It is clear from (23) and (7) that

$$|\,D(s)\,|\leqq\int_1^{1/s}x^{-2}\,|\,\beta(x)\,|\,sx\,dx=s\int_1^{1/s}x^{-1}\,|\,\beta(x)\,|\,dx\ .$$

Hence, from (27),

$$|\,D(s)\,|<s\int_1^R M\,dx+s\int_R^{1/s}(1+\varepsilon)\,dx\qquad\text{if}\qquad 1/s>R\ ,$$

where M and R depend on ε only. Consequently, the upper limit of (25_3), as $s\to+0$, cannot exceed that of

$$s\int_R^{1/s}(1+\varepsilon)\,dx<s\int_0^{1/s}(1+\varepsilon)\,dx=1+\varepsilon$$

and is, therefore, not greater than 1.

Ad (26). According to (22) and (27),

$$|\,B(s)\,|<\int_{1/s}^\infty x^{-2}(1+\varepsilon)\,xe^{-sx}dx\qquad\text{if}\qquad 0<s<1/R_\varepsilon\ .$$

Hence, in order to prove (26), it is sufficient to ascertain the inequality

$$\limsup_{s\to+0}\int_{1/s}^\infty x^{-1}\,e^{-sx}\,dx\leqq\int_1^\infty x^{-1}\,e^{-x}dx\ .$$

But this inequality actually is an equality, since the value of the integral on the left is independent of s.

It remains undecided whether or not the (best values of the) absolute constants τ, τ^* remain the same if they apply to arbitrary Laplace integrals (16), rather than just to power series (2), and whether the sign of equality does or does not hold in (10) (in either case). Even the estimate $\tau^* \geqq 1$, corresponding to the trivial inequality (8), is problematic. All that is clear is that τ^* cannot be less than

$$\lim_{x \to \infty} \left(\sum_{n=1}^{x} n^{-1} - \log x \right) = 0.57\ldots$$

(in either case). In fact, if $na_n = 1$, then the power series (2) becomes $-\log(1 - r)$, and so, if r in (9) is replaced by $e^{-1/x}$,

$$\limsup_{x \to \infty} \left| \log(1 - e^{-1/x}) + \sum_{n=1}^{x} n^{-1} \right| \leqq \tau^* .$$

On the other hand,

$$\lim_{x \to \infty} \left| \log(1 - e^{-1/x}) - (-\log x) \right| = \lim_{\varepsilon \to 0} \log(1 - \varepsilon) = \log 1 = 0 .$$

Clearly, the last three formula lines imply that $\tau^* \geqq 0.57\ldots$.

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