A partition formula connected with Abelian groups

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Let p be a given prime. The object of this note is to prove the following rather curious result.

The sum of the reciprocals of the orders of all the Abelian groups of order a power of p is equal to the sum of the reciprocals of the orders of their groups of automorphisms.

It is well known that the Abelian groups of order p^n stand in $(1-1)$ correspondence with the $\omega(n)$ unrestricted partitions of n, the partition corresponding to a given Abelian group being called its *type*.

Thus the sum of the reciprocals of the orders of all the Abelian groups of order a power of p is equal to

$$\sum_{n=0}^{\infty} \frac{\omega(n)}{p^n} . \tag{1}$$

Writing

$$f_n(x) \equiv (1-x)(1-x^2)(1-x^3)\dots(1-x^n) , \qquad f_0(x) = 1 , \tag{2}$$

and

$$\varrho = \frac{1}{p} , \tag{3}$$

the value of the sum (1) is easily seen to be

$$\frac{1}{f_\infty(\varrho)} . \tag{4}$$

But, by an identity due to Euler, this is the same as

$$\sum_{n=0}^{\infty} \frac{\varrho^n}{f_n(\varrho)} . \tag{5}$$

And the theorem mentioned above will accordingly follow once we have shown that *the sum of the reciprocals of the orders of the groups of automorphisms of the $\omega(n)$ Abelian groups of order p^n is equal to $\varrho^n | f_n(\varrho)$.*

For partitions we use the notation of Macmahon. Thus, the Abelian group G of order p^n and type

$$(1^{\lambda_1}\ 2^{\lambda_2}\ 3^{\lambda_3} \cdots) \tag{6}$$

is the direct product of cyclic groups, λ_1 of which are of order p, λ_2 of order p^2, λ_3 of order p^3, and so on. Clearly,

$$n = \lambda_1 + 2\,\lambda_2 + 3\,\lambda_3 + \cdots . \tag{7}$$

The partition of n which is *associated* with the partition (6) has the parts $\mu_1, \mu_2, \ldots$ given by

$$\mu_i = \lambda_i + \lambda_{i+1} + \lambda_{i+2} + \cdots \qquad (i = 1, 2, \ldots) \ . \tag{8}$$

Thus, since $\lambda_i \geqslant 0$ for each i, we have

$$\mu_1 \geqslant \mu_2 \geqslant \cdots \qquad \geqslant 0 \ . \tag{9}$$

And plainly, from (7) and (8),

$$\mu_1 + \mu_2 + \cdots \qquad = n \ . \tag{10}$$

Conversely, given any partition of n in the form (9), (10), the associated partition (6) is obtained at once by the rule that

$$\lambda_i = \mu_i - \mu_{i+1} \qquad (i = 1, 2, \ldots) \ . \tag{11}$$

The associated partition has a simple meaning for the group G. Let G_k denote the characteristic subgroup of G which consists of all elements of G of order p^k or less. Then

$$1 = G_0 < G_1 < G_2 < \cdots < G_m = G \ ,$$

where m is the largest of the type invariants[1]) of G, and *the order of* $G_k \mid G_{k-1}$ *is precisely* p^{μ_k}.

Now the order of the group of automorphisms of G can be expressed very simply in terms of the "associated invariants" μ_k. It is[2])

$$\frac{f_{\mu_1-\mu_2}(\varrho)\, f_{\mu_2-\mu_3}(\varrho) \cdots}{\varrho^{\mu_1^2 + \mu_2^2 + \cdots}} \ . \tag{12}$$

And the result we require to prove is the case $x = \varrho$ of the identity

$$\frac{x^n}{f_n(x)} = \sum_{(\mu)} \frac{x^{\mu_1^2 + \mu_2^2 + \cdots}}{f_{\mu_1-\mu_2}(x)\, f_{\mu_2-\mu_3}(x) \cdots} \ , \tag{13}$$

the sum being taken over all $\omega(n)$ partitions (9), (10) of the number n.

The various terms of (13) may be regarded as the generating functions of partitions or compositions of certain definite kinds. For example, the coefficient of x^N on the left of (13) is equal to the number of partitions of N for which the greatest part is exactly n. As a first step in the proof of the identity, we shall connect every such partition of N with a particular

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1) I. e. $\lambda_m > 0$, $\lambda_{m+1} = \lambda_{m+2} = \cdots = 0$.

2) Cf. *A. Speiser*, Theorie der Gruppen von endlicher Ordnung, 3er. Aufl., § 43, Satz 114.

one of the $\omega(n)$ partitions (9), (10) of n, and thereby with a particular one of the $\omega(n)$ summands on the right of (13).

This may be done most conveniently by means of the *graph*[3]) of the partition of N in question. Let the parts of this partition, arranged in descending order of magnitude be $N_1, N_2, \ldots$, so that we have

$$\begin{aligned} n = N_1 \geqslant N_2 \geqslant \cdots , \\ N_1 + N_2 + \cdots = N . \end{aligned} \tag{14}$$

Then its graph may be defined to consist of a set of N coplanar lattice-points, viz. all those points whose Cartesian coordinates (x, y) are positive integers satisfying

$$x \leqslant N_y . \tag{15}$$

(When y exceeds the number of parts of (14), we take $N_y = 0$.)

We are now able to define, successively, the numbers $\mu_1, \mu_2, \ldots$, which correspond to the partition (14).

We take μ_1 to be the greatest integer such that the point (μ_1, μ_1) belongs to the graph (15). Next, supposing that $\mu_1, \mu_2, \ldots, \mu_{i-1}$ have already been defined, and that their sum is less than n, we define μ_i to be the greatest integer such that $(\mu_1 + \mu_2 + \cdots + \mu_i, \mu_i)$ is a point of the graph.

It follows at once, from (14) and (15), that the numbers μ_i so defined satisfy (9) and (10). Plainly, also, the square of μ_i^2 lattice-points having for opposite corners the points $(\mu_1 + \cdots + \mu_{i-1} + 1, 1)$ and $(\mu_1 + \cdots + \mu_i, \mu_i)$ belongs entirely to the graph. Thus, if we write

$$M = N - \mu_1^2 - \mu_2^2 - \cdots , \tag{16}$$

there remain, outside the squares just mentioned, precisely M further points of the graph.

We divide these M remaining points into sets, according to the values of their x-coordinates. Let the number of them which lie in the strip $0 < x \leqslant \mu_1$ be M_1. And, for any $i > 1$, let the number which lie in the strip $\mu_{i-1} < x \leqslant \mu_i$ be M_i. If the number of μ's is r, we obtain in this way a definite composition[4]) of M,

$$M = M_1 + M_2 + \cdots + M_r , \tag{17}$$

into r non-negative integers, this composition, like the partition (9), (10), being uniquely determined by the original partition (14) of N.

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[3]) *P. A. Macmahon*, Combinatory Analysis, II, 3. Our graph reads upwards, not downwards as in Macmahon.

[4]) A composition is a partition in which the order of the summands is important.

As a final consequence of (14), (15), we remark that (for each $i = 1, 2, \ldots, r$) the M_i points of the i-th strip constitute, a translation apart, the graph of a certain partition P_i of M_i, these partitions P_i being, just as much as the numbers M_i themselves, uniquely determined by (14). Further, for each i, the greatest part of P_i is not greater than μ_i. And, for each $i > 1$, the number of parts of P_i is not greater that $\mu_{i-1} - \mu_i$.

But, conversely, suppose that we choose any set of positive integers μ_i satisfying (9) and (10), the sum of whose squares does not exceed N, and define M by (16); then choose any composition (17) of M, one part M_i corresponding to each μ_i, taking care that

$$M_i \leqslant \mu_i (\mu_{i-1} - \mu_i) \qquad (i > 1) \ ;$$

and finally, for each M_i, choose arbitrarily a partition P_i having its greatest part not greater than μ_i and having (for $i > 1$) not more than $\mu_{i-1} - \mu_i$ parts.

Then it is obvious that we can reverse our former construction at every step, and arrive at a definite partition (14) of N, which has n as its greatest part, and for which the corresponding μ's, M's and P's are precisely the ones we have chosen.

If, then, we denote by $\psi_{a,b}(x)$ the generating function for the partitions of N into at most a parts none of which exceed b, we have proved the identity

$$\frac{x^n}{f_n(x)} = \sum_{(\mu)} \psi_{\infty, \mu_1}(x)\, \psi_{\mu_1 - \mu_2, \mu_2}(x)\, \psi_{\mu_2 - \mu_3, \mu_3}(x) \ldots x^{\mu_1^2 + \mu_2^2 + \cdots} , \qquad (18)$$

the sum being taken over all partitions (9), (10) of n. But it is known[1]) that, for finite a and b,

$$\psi_{a,b}(x) = \frac{f_{a+b}(x)}{f_a(x)\, f_b(x)} ,$$

while

$$\psi_{\infty, b}(x) = \frac{1}{f_b(x)} .$$

Substituting these values in (18), we obtain the required identity (13). This concludes the proof.

(Eingegangen den 17. September 1938.)

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[5]) *P. A. Macmahon*, loc. cit., 5.