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# A partition formula connected with Abelian groups

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Let  $p$  be a given prime. The object of this note is to prove the following rather curious result.

*The sum of the reciprocals of the orders of all the Abelian groups of order a power of  $p$  is equal to the sum of the reciprocals of the orders of their groups of automorphisms.*

It is well known that the Abelian groups of order  $p^n$  stand in (1 — 1) correspondence with the  $\omega(n)$  unrestricted partitions of  $n$ , the partition corresponding to a given Abelian group being called its *type*.

Thus the sum of the reciprocals of the orders of all the Abelian groups of order a power of  $p$  is equal to

$$\sum_{n=0}^{\infty} \frac{\omega(n)}{p^n} . \quad (1)$$

Writing

$$f_n(x) \equiv (1 - x)(1 - x^2)(1 - x^3) \dots (1 - x^n) , \quad f_0(x) = 1 , \quad (2)$$

and

$$\varrho = \frac{1}{p} , \quad (3)$$

the value of the sum (1) is easily seen to be

$$\frac{1}{f_{\infty}(\varrho)} . \quad (4)$$

But, by an identity due to Euler, this is the same as

$$\sum_{n=0}^{\infty} \frac{\varrho^n}{f_n(\varrho)} . \quad (5)$$

And the theorem mentioned above will accordingly follow once we have shown that *the sum of the reciprocals of the orders of the groups of automorphisms of the  $\omega(n)$  Abelian groups of order  $p^n$  is equal to  $\varrho^n | f_n(\varrho)$* .

For partitions we use the notation of Macmahon. Thus, the Abelian group  $G$  of order  $p^n$  and type

$$(1^{\lambda_1} 2^{\lambda_2} 3^{\lambda_3} \dots) \quad (6)$$

is the direct product of cyclic groups,  $\lambda_1$  of which are of order  $p$ ,  $\lambda_2$  of order  $p^2$ ,  $\lambda_3$  of order  $p^3$ , and so on. Clearly,

$$n = \lambda_1 + 2 \lambda_2 + 3 \lambda_3 + \dots . \quad (7)$$

The partition of  $n$  which is *associated* with the partition (6) has the parts  $\mu_1, \mu_2, \dots$  given by

$$\mu_i = \lambda_i + \lambda_{i+1} + \lambda_{i+2} + \dots \quad (i = 1, 2, \dots) . \quad (8)$$

Thus, since  $\lambda_i \geq 0$  for each  $i$ , we have

$$\mu_1 \geq \mu_2 \geq \dots \geq 0 . \quad (9)$$

And plainly, from (7) and (8),

$$\mu_1 + \mu_2 + \dots = n . \quad (10)$$

Conversely, given any partition of  $n$  in the form (9), (10), the associated partition (6) is obtained at once by the rule that

$$\lambda_i = \mu_i - \mu_{i+1} \quad (i = 1, 2, \dots) . \quad (11)$$

The associated partition has a simple meaning for the group  $G$ . Let  $G_k$  denote the characteristic subgroup of  $G$  which consists of all elements of  $G$  of order  $p^k$  or less. Then

$$1 = G_0 < G_1 < G_2 < \dots < G_m = G ,$$

where  $m$  is the largest of the type invariants<sup>1)</sup> of  $G$ , and the order of  $G_k | G_{k-1}$  is precisely  $p^{\mu_k}$ .

Now the order of the group of automorphisms of  $G$  can be expressed very simply in terms of the “associated invariants”  $\mu_k$ . It is<sup>2)</sup>

$$\frac{f_{\mu_1 - \mu_2}(\varrho) f_{\mu_2 - \mu_3}(\varrho) \dots}{\varrho^{\mu_1^2 + \mu_2^2 + \dots}} . \quad (12)$$

And the result we require to prove is the case  $x = \varrho$  of the identity

$$\frac{x^n}{f_n(x)} = \sum_{(\mu)} \frac{x^{\mu_1^2 + \mu_2^2 + \dots}}{f_{\mu_1 - \mu_2}(x) f_{\mu_2 - \mu_3}(x) \dots} , \quad (13)$$

the sum being taken over all  $\omega(n)$  partitions (9), (10) of the number  $n$ .

The various terms of (13) may be regarded as the generating functions of partitions or compositions of certain definite kinds. For example, the coefficient of  $x^N$  on the left of (13) is equal to the number of partitions of  $N$  for which the greatest part is exactly  $n$ . As a first step in the proof of the identity, we shall connect every such partition of  $N$  with a particular

<sup>1)</sup> I. e.  $\lambda_m > 0$ ,  $\lambda_{m+1} = \lambda_{m+2} = \dots = 0$ .

<sup>2)</sup> Cf. A. Speiser, Theorie der Gruppen von endlicher Ordnung, 3er. Aufl., § 43, Satz 114.

one of the  $\omega(n)$  partitions (9), (10) of  $n$ , and thereby with a particular one of the  $\omega(n)$  summands on the right of (13).

This may be done most conveniently by means of the *graph*<sup>3)</sup> of the partition of  $N$  in question. Let the parts of this partition, arranged in descending order of magnitude be  $N_1, N_2, \dots$ , so that we have

$$\begin{aligned} n &= N_1 \geq N_2 \geq \dots, \\ N_1 + N_2 + \dots &= N. \end{aligned} \tag{14}$$

Then its graph may be defined to consist of a set of  $N$  coplanar lattice-points, viz. all those points whose Cartesian coordinates  $(x, y)$  are positive integers satisfying

$$x \leq N_y. \tag{15}$$

(When  $y$  exceeds the number of parts of (14), we take  $N_y = 0$ .)

We are now able to define, successively, the numbers  $\mu_1, \mu_2, \dots$ , which correspond to the partition (14).

We take  $\mu_1$  to be the greatest integer such that the point  $(\mu_1, \mu_1)$  belongs to the graph (15). Next, supposing that  $\mu_1, \mu_2, \dots, \mu_{i-1}$  have already been defined, and that their sum is less than  $n$ , we define  $\mu_i$  to be the greatest integer such that  $(\mu_1 + \mu_2 + \dots + \mu_i, \mu_i)$  is a point of the graph.

It follows at once, from (14) and (15), that the numbers  $\mu_i$  so defined satisfy (9) and (10). Plainly, also, the square of  $\mu_i^2$  lattice-points having for opposite corners the points  $(\mu_1 + \dots + \mu_{i-1} + 1, 1)$  and  $(\mu_1 + \dots + \mu_i, \mu_i)$  belongs entirely to the graph. Thus, if we write

$$M = N - \mu_1^2 - \mu_2^2 - \dots, \tag{16}$$

there remain, outside the squares just mentioned, precisely  $M$  further points of the graph.

We divide these  $M$  remaining points into sets, according to the values of their  $x$ -coordinates. Let the number of them which lie in the strip  $0 < x \leq \mu_1$  be  $M_1$ . And, for any  $i > 1$ , let the number which lie in the strip  $\mu_{i-1} < x \leq \mu_i$  be  $M_i$ . If the number of  $\mu$ 's is  $r$ , we obtain in this way a definite composition<sup>4)</sup> of  $M$ ,

$$M = M_1 + M_2 + \dots + M_r, \tag{17}$$

into  $r$  non-negative integers, this composition, like the partition (9), (10), being uniquely determined by the original partition (14) of  $N$ .

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<sup>3)</sup> *P. A. MacMahon, Combinatory Analysis, II, 3.* Our graph reads upwards, not downwards as in MacMahon.

<sup>4)</sup> A composition is a partition in which the order of the summands is important.

As a final consequence of (14), (15), we remark that (for each  $i = 1, 2, \dots, r$ ) the  $M_i$  points of the  $i$ -th strip constitute, a translation apart, the graph of a certain partition  $P_i$  of  $M_i$ , these partitions  $P_i$  being, just as much as the numbers  $M_i$  themselves, uniquely determined by (14). Further, for each  $i$ , the greatest part of  $P_i$  is not greater than  $\mu_i$ . And, for each  $i > 1$ , the number of parts of  $P_i$  is not greater than  $\mu_{i-1} - \mu_i$ .

But, conversely, suppose that we choose any set of positive integers  $\mu_i$  satisfying (9) and (10), the sum of whose squares does not exceed  $N$ , and define  $M$  by (16); then choose any composition (17) of  $M$ , one part  $M_i$  corresponding to each  $\mu_i$ , taking care that

$$M_i \leq \mu_i (\mu_{i-1} - \mu_i) \quad (i > 1) ;$$

and finally, for each  $M_i$ , choose arbitrarily a partition  $P_i$  having its greatest part not greater than  $\mu_i$  and having (for  $i > 1$ ) not more than  $\mu_{i-1} - \mu_i$  parts.

Then it is obvious that we can reverse our former construction at every step, and arrive at a definite partition (14) of  $N$ , which has  $n$  as its greatest part, and for which the corresponding  $\mu$ 's,  $M$ 's and  $P$ 's are precisely the ones we have chosen.

If, then, we denote by  $\psi_{a,b}(x)$  the generating function for the partitions of  $N$  into at most  $a$  parts none of which exceed  $b$ , we have proved the identity

$$\frac{x^n}{f_n(x)} = \sum_{(\mu)} \psi_{\infty, \mu_1}(x) \psi_{\mu_1 - \mu_2, \mu_2}(x) \psi_{\mu_2 - \mu_3, \mu_3}(x) \dots x^{\mu_1^2 + \mu_2^2 + \dots}, \quad (18)$$

the sum being taken over all partitions (9), (10) of  $n$ . But it is known<sup>1)</sup> that, for finite  $a$  and  $b$ ,

$$\psi_{a,b}(x) = \frac{f_{a+b}(x)}{f_a(x) f_b(x)} ,$$

while

$$\psi_{\infty, b}(x) = \frac{1}{f_b(x)} .$$

Substituting these values in (18), we obtain the required identity (13). This concludes the proof.

(Eingegangen den 17. September 1938.)

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<sup>1)</sup> *P. A. MacMahon*, loc. cit., 5.