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A partition formula connected with Abelian groups

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Let p be a given prime. The object of this note is to prove the following rather curious result.

The sum of the reciprocals of the orders of all the Abelian groups of order a power of p is equal to the sum of the reciprocals of the orders of their groups of automorphisms.

It is well known that the Abelian groups of order p^n stand in (1 — 1) correspondence with the $\omega(n)$ unrestricted partitions of n , the partition corresponding to a given Abelian group being called its *type*.

Thus the sum of the reciprocals of the orders of all the Abelian groups of order a power of p is equal to

$$\sum_{n=0}^{\infty} \frac{\omega(n)}{p^n} . \quad (1)$$

Writing

$$f_n(x) \equiv (1 - x)(1 - x^2)(1 - x^3) \dots (1 - x^n) , \quad f_0(x) = 1 , \quad (2)$$

and

$$\varrho = \frac{1}{p} , \quad (3)$$

the value of the sum (1) is easily seen to be

$$\frac{1}{f_{\infty}(\varrho)} . \quad (4)$$

But, by an identity due to Euler, this is the same as

$$\sum_{n=0}^{\infty} \frac{\varrho^n}{f_n(\varrho)} . \quad (5)$$

And the theorem mentioned above will accordingly follow once we have shown that *the sum of the reciprocals of the orders of the groups of automorphisms of the $\omega(n)$ Abelian groups of order p^n is equal to $\varrho^n | f_n(\varrho)$.*

For partitions we use the notation of Macmahon. Thus, the Abelian group G of order p^n and type

$$(1^{\lambda_1} 2^{\lambda_2} 3^{\lambda_3} \dots) \quad (6)$$

is the direct product of cyclic groups, λ_1 of which are of order p , λ_2 of order p^2 , λ_3 of order p^3 , and so on. Clearly,

$$n = \lambda_1 + 2 \lambda_2 + 3 \lambda_3 + \dots . \quad (7)$$

The partition of n which is *associated* with the partition (6) has the parts μ_1, μ_2, \dots given by

$$\mu_i = \lambda_i + \lambda_{i+1} + \lambda_{i+2} + \dots \quad (i = 1, 2, \dots) . \quad (8)$$

Thus, since $\lambda_i \geq 0$ for each i , we have

$$\mu_1 \geq \mu_2 \geq \dots \geq 0 . \quad (9)$$

And plainly, from (7) and (8),

$$\mu_1 + \mu_2 + \dots = n . \quad (10)$$

Conversely, given any partition of n in the form (9), (10), the associated partition (6) is obtained at once by the rule that

$$\lambda_i = \mu_i - \mu_{i+1} \quad (i = 1, 2, \dots) . \quad (11)$$

The associated partition has a simple meaning for the group G . Let G_k denote the characteristic subgroup of G which consists of all elements of G of order p^k or less. Then

$$1 = G_0 < G_1 < G_2 < \dots < G_m = G ,$$

where m is the largest of the type invariants¹⁾ of G , and *the order of $G_k \mid G_{k-1}$ is precisely p^{μ_k}* .

Now the order of the group of automorphisms of G can be expressed very simply in terms of the "associated invariants" μ_k . It is²⁾

$$\frac{f_{\mu_1 - \mu_2}(\varrho) f_{\mu_2 - \mu_3}(\varrho) \dots}{\varrho^{\mu_1^2 + \mu_2^2 + \dots}} . \quad (12)$$

And the result we require to prove is the case $x = \varrho$ of the identity

$$\frac{x^n}{f_n(x)} = \sum_{(\mu)} \frac{x^{\mu_1^2 + \mu_2^2 + \dots}}{f_{\mu_1 - \mu_2}(x) f_{\mu_2 - \mu_3}(x) \dots} , \quad (13)$$

the sum being taken over all $\omega(n)$ partitions (9), (10) of the number n .

The various terms of (13) may be regarded as the generating functions of partitions or compositions of certain definite kinds. For example, the coefficient of x^N on the left of (13) is equal to the number of partitions of N for which the greatest part is exactly n . As a first step in the proof of the identity, we shall connect every such partition of N with a particular

¹⁾ I. e. $\lambda_m > 0$, $\lambda_{m+1} = \lambda_{m+2} = \dots = 0$.

²⁾ Cf. *A. Speiser, Theorie der Gruppen von endlicher Ordnung*, 3er. Aufl., § 43, Satz 114.

one of the $\omega(n)$ partitions (9), (10) of n , and thereby with a particular one of the $\omega(n)$ summands on the right of (13).

This may be done most conveniently by means of the *graph*³⁾ of the partition of N in question. Let the parts of this partition, arranged in descending order of magnitude be N_1, N_2, \dots , so that we have

$$\begin{aligned} n &= N_1 \geq N_2 \geq \dots, \\ N_1 + N_2 + \dots &= N. \end{aligned} \tag{14}$$

Then its graph may be defined to consist of a set of N coplanar lattice-points, viz. all those points whose Cartesian coordinates (x, y) are positive integers satisfying

$$x \leq N_y. \tag{15}$$

(When y exceeds the number of parts of (14), we take $N_y = 0$.)

We are now able to define, successively, the numbers μ_1, μ_2, \dots , which correspond to the partition (14).

We take μ_1 to be the greatest integer such that the point (μ_1, μ_1) belongs to the graph (15). Next, supposing that $\mu_1, \mu_2, \dots, \mu_{i-1}$ have already been defined, and that their sum is less than n , we define μ_i to be the greatest integer such that $(\mu_1 + \mu_2 + \dots + \mu_i, \mu_i)$ is a point of the graph.

It follows at once, from (14) and (15), that the numbers μ_i so defined satisfy (9) and (10). Plainly, also, the square of μ_i^2 lattice-points having for opposite corners the points $(\mu_1 + \dots + \mu_{i-1} + 1, 1)$ and $(\mu_1 + \dots + \mu_i, \mu_i)$ belongs entirely to the graph. Thus, if we write

$$M = N - \mu_1^2 - \mu_2^2 - \dots, \tag{16}$$

there remain, outside the squares just mentioned, precisely M further points of the graph.

We divide these M remaining points into sets, according to the values of their x -coordinates. Let the number of them which lie in the strip $0 < x \leq \mu_1$ be M_1 . And, for any $i > 1$, let the number which lie in the strip $\mu_{i-1} < x \leq \mu_i$ be M_i . If the number of μ 's is r , we obtain in this way a definite composition⁴⁾ of M ,

$$M = M_1 + M_2 + \dots + M_r, \tag{17}$$

into r non-negative integers, this composition, like the partition (9), (10), being uniquely determined by the original partition (14) of N .

³⁾ *P. A. Macmahon*, *Combinatory Analysis*, II, 3. Our graph reads upwards, not downwards as in Macmahon.

⁴⁾ A composition is a partition in which the order of the summands is important.

As a final consequence of (14), (15), we remark that (for each $i = 1, 2, \dots, r$) the M_i points of the i -th strip constitute, a translation apart, the graph of a certain partition P_i of M_i , these partitions P_i being, just as much as the numbers M_i themselves, uniquely determined by (14). Further, for each i , the greatest part of P_i is not greater than μ_i . And, for each $i > 1$, the number of parts of P_i is not greater than $\mu_{i-1} - \mu_i$.

But, conversely, suppose that we choose any set of positive integers μ_i satisfying (9) and (10), the sum of whose squares does not exceed N , and define M by (16); then choose any composition (17) of M , one part M_i corresponding to each μ_i , taking care that

$$M_i \leq \mu_i (\mu_{i-1} - \mu_i) \quad (i > 1) ;$$

and finally, for each M_i , choose arbitrarily a partition P_i having its greatest part not greater than μ_i and having (for $i > 1$) not more than $\mu_{i-1} - \mu_i$ parts.

Then it is obvious that we can reverse our former construction at every step, and arrive at a definite partition (14) of N , which has n as its greatest part, and for which the corresponding μ 's, M 's and P 's are precisely the ones we have chosen.

If, then, we denote by $\psi_{a,b}(x)$ the generating function for the partitions of N into at most a parts none of which exceed b , we have proved the identity

$$\frac{x^n}{f_n(x)} = \sum_{(\mu)} \psi_{\infty, \mu_1}(x) \psi_{\mu_1 - \mu_2, \mu_2}(x) \psi_{\mu_2 - \mu_3, \mu_3}(x) \dots x^{\mu_1^2 + \mu_2^2 + \dots}, \quad (18)$$

the sum being taken over all partitions (9), (10) of n . But it is known¹⁾ that, for finite a and b ,

$$\psi_{a,b}(x) = \frac{f_{a+b}(x)}{f_a(x) f_b(x)},$$

while

$$\psi_{\infty,b}(x) = \frac{1}{f_b(x)}.$$

Substituting these values in (18), we obtain the required identity (13). This concludes the proof.

(Eingegangen den 17. September 1938.)

¹⁾ *P. A. Macmahon*, loc. cit., 5.