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Autor: Nanakorn, Pruettha / Petcherdchoo, Aruz
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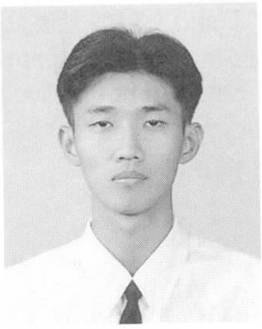
Analysis of Cracking Localization Using the Smeared Crack Approach

Pruettha NANAKORN
Associate Professor
Sirindhorn Int. Inst. of Tech.
Pathumthani, THAILAND

Pruettha Nanakorn, born 1968, received his D. Eng. degree from The University of Tokyo in 1994. He is currently a deputy chairperson of Department of Civil Engineering, Sirindhorn International Institute of Technology, Thammasat University, Thailand

Aruz PETCHERDCHOO
Graduate Student
Sirindhorn Int. Inst. of Tech.
Pathumthani, THAILAND



Aruz Petcherdchoo, born 1974, received his B. Eng. degree from Sirindhorn International Institute of Technology in 1997. He is currently pursuing his graduate study in the same Institute.

Summary

In order to consider cracking localization in the finite element analysis, it is more suitable to have an energy expression written in terms of discrete irreversible variables, which will allow differentiation of the energy expression with respect to those irreversible variables. This implies that the discrete crack approach should be more appropriate to this kind of analysis than the smeared crack approach. However, the discrete crack approach in the finite element analysis may not be the best solution for problems with many cracks, which are unavoidable for the analysis of cracking localization. To avoid the drawbacks in both approaches, a special treatment on the smeared crack finite element analysis to allow the consideration of cracking localization is proposed. In the method, a discrete irreversible variable related to crack strain is introduced, and cracking localization is investigated, based on this discrete irreversible variable.

1. Introduction

It is commonly known that cracking localization prior to the failure plays a very important role in the fracture behavior of quasi-brittle materials such as concrete. In order to capture the ultimate capacity of such materials in structures, it is necessary to consider not only formation and propagation of cracks but also localization of them. In the analysis of cracking localization, consideration of stability and bifurcation of equilibrium states is one of the tasks that have to be done. To this end, the stationary condition of the energy of the system can be examined. This requires expression of the energy in terms of discrete irreversible variables. Having the energy written as a function of discrete irreversible variables allows differentiation of the energy expression with respect to these irreversible variables, which is necessary for the consideration of the stability condition. This fact implies that the discrete crack approach in the finite element method may be more suitable for the cracking localization analysis than the smeared crack approach. In the discrete crack approach, the irreversible variables that have to be considered are the crack opening displacements. These crack opening displacement variables are usually discretized along crack paths and treated as the degrees of freedom in the analysis. Therefore, the energy of the system can be expressed in terms of these degrees of freedom. Computing the first and second variations of the energy with respect to the crack opening displacement degrees of freedom can be done easily. On the contrary, if the smeared crack approach is employed, the energy of the system will be expressed in terms of irreversible crack strain components. These crack strain components are functions of position. To compute the first and second variations of the energy with respect to these crack strain functions, complex mathematics involving the calculus of variations must be employed.

Nevertheless, the discrete crack approach may not perform best when there are many cracks. In the cracking localization analysis, there will be many cracks in the domain. Having many cracks in the domain leads to more degrees of freedom, and the mesh topology of the problems may have to be changed drastically. Moreover, the singularity problem of the system stiffness equation may also appear. These problems can be mostly avoided if the smeared crack approach is employed. In the smeared crack model, no increase in the degrees of freedom or change in the mesh topology is required. Although the smeared crack model may also face the singularity problem of the system in case of softening materials, the problem is less serious than that of the discrete crack model.

In this study, a special consideration on the smeared crack finite element analysis is proposed. The proposed consideration will make it possible to consider cracking localization even when the smeared crack model is used. In the proposed method, a discrete irreversible variable related to the crack strain is introduced in the smeared crack model. This discrete variable will allow the consideration of stability and bifurcation of the equilibrated solution to be done easily by considering the variations of the energy with respect to the proposed discrete variable.

2. Smeared Crack Finite Element Analysis for Cracking Localization

The fundamental scheme of the smeared crack model is the decomposition of the total strain increment $\Delta\varepsilon$ into an elastic strain increment $\Delta\varepsilon_e$ and a crack strain increment $\Delta\varepsilon_{cr}$, i.e.,

$$\Delta\varepsilon = \Delta\varepsilon_e + \Delta\varepsilon_{cr}. \tag{1}$$

Therefore, the total energy increment can be written as

$$\begin{aligned}\Delta U &= \Delta U^m + \Delta U^d \\ &= \left[\frac{1}{2}\int_V \Delta\boldsymbol{\varepsilon}_e^T \mathbf{D}_e \Delta\boldsymbol{\varepsilon}_e dV - \int_S \Delta\mathbf{u}^T \Delta\mathbf{t} dS - \int_V \Delta\mathbf{u}^T \Delta\mathbf{f} dV\right] + \left[\frac{1}{2}\int_V \Delta\boldsymbol{\varepsilon}_{cr}^T \hat{\mathbf{D}}_{cr} \Delta\boldsymbol{\varepsilon}_{cr} dV\right]\end{aligned} \tag{2}$$

where ΔU^m and ΔU^d represent the mechanical potential energy increment and the dissipated energy increment, respectively [1, 2]. Here, $\Delta\mathbf{t}$ and $\Delta\mathbf{f}$ denote the surface traction increment vector and the body force increment vector, respectively. In addition, $\mathbf{D}_e$ denotes the elastic constitutive matrix for the uncracked solid, i.e.,

$$\sigma = \mathbf{D}_e \Delta\varepsilon_e, \tag{3}$$

and $\hat{\mathbf{D}}_{cr}$ is a matrix defined as

$$\hat{\mathbf{D}}_{cr} = \hat{\mathbf{N}}^T \mathbf{D}_{cr} \hat{\mathbf{N}} \tag{4}$$

where $\mathbf{D}_{cr}$ is the constitutive matrix defining the relation between the crack strain increment $\Delta\mathbf{e}_{cr}$ and the crack stress increment $\Delta\mathbf{s}_{cr}$ in the crack local coordinate system [3], i.e.,

$$\Delta\mathbf{s}_{cr} = \mathbf{D}_{cr} \Delta\mathbf{e}_{cr}, \tag{5}$$

and $\hat{\mathbf{N}}$ is the transformation matrix defined as

$$\Delta\mathbf{e}_{cr} = \hat{\mathbf{N}} \Delta\varepsilon_{cr}. \tag{6}$$

In the expression for the total energy increment in Eq. (2), the irreversible variable that has to be considered in the stability analysis is the crack strain increment $\Delta\varepsilon_{cr}$. The first and second variations of the total energy increment with respect to this crack strain increment must be obtained in order to get the equilibrium path and the stability condition of the equilibrium path. Since the total energy increment is a functional of the crack strain increment function, the calculus of variations is required. To avoid this difficulty, we introduce a crack displacement increment vector $\Delta\mathbf{u}_{cr}$ defined as

$$\Delta\mathbf{u} = \Delta\mathbf{u}_e + \Delta\mathbf{u}_{cr} \tag{7}$$

where $\Delta\mathbf{u}$ and $\Delta\mathbf{u}_e$ are the total displacement increment vector and the displacement increment vector corresponding to the elastic strain, respectively.

Consider the i^{th} element in the finite element analysis. The element is assumed to be a cracked element. Interpolate these three displacement increments from nodal quantities, i.e.,

$$\Delta\mathbf{u} = \mathbf{N}\Delta\mathbf{U}, \quad \Delta\mathbf{u}_e = \mathbf{N}\Delta\mathbf{U}_e, \quad \Delta\mathbf{u}_{cr} = \mathbf{N}\Delta^i\mathbf{U}_{cr}, \quad \Delta\mathbf{U} = \Delta\mathbf{U}_e + \Delta^i\mathbf{U}_{cr} \tag{8}$$

where $\Delta\mathbf{U}$ is the nodal total displacement increment vector, $\Delta\mathbf{U}_e$ is the nodal displacement increment vector corresponding to the elastic strain, and $\Delta^i\mathbf{U}_{cr}$ is the nodal crack displacement increment vector. Here, $\mathbf{N}$ is the shape function matrix. Note that the superscript i for the i^{th} element is necessary for the nodal crack displacement increment vector $\Delta^i\mathbf{U}_{cr}$ because the nodal crack displacement increments of the same node for different elements can be different. This is natural because, in the smeared crack approach, cracking in each element is completely independent of each other.

Computing strains from Eq. (7), we obtain Eq. (1), i.e.,

$$\Delta\varepsilon = \Delta\varepsilon_e + \Delta^i\varepsilon_{cr} \tag{9}$$

where

$$\Delta\varepsilon = \mathbf{B}\Delta\mathbf{U}\,,\ \Delta\varepsilon_e = \mathbf{B}\Delta\mathbf{U}_e,\ \text{and}\ \Delta^i\varepsilon_{cr} = \mathbf{B}\Delta^i\mathbf{U}_{cr} \tag{10}$$

in which **B** is the strain-displacement matrix.

Rewrite Eq. (2) as

$$\Delta U = \frac{1}{2}\int_V \left(\Delta\varepsilon - \Delta^i\varepsilon_{cr}\right)^T \mathbf{D}_e \left(\Delta\varepsilon - \Delta^i\varepsilon_{cr}\right) dV + \frac{1}{2}\int_V \Delta^i\varepsilon_{cr}^T \hat{\mathbf{D}}_{cr} \Delta^i\varepsilon_{cr} dV - \int_V \Delta\mathbf{u}^T \Delta\mathbf{f} dV - \int_S \Delta\mathbf{u}^T \Delta\mathbf{t} dS \tag{11}$$

which yields

$$\begin{aligned} \Delta U = {} & \frac{1}{2}\int_V \Delta\varepsilon^T \mathbf{D}_e \Delta\varepsilon dV - \frac{1}{2}\int_V \Delta\varepsilon^T \mathbf{D}_e \Delta^i\varepsilon_{cr} dV - \frac{1}{2}\int_V \Delta^i\varepsilon_{cr}^T \mathbf{D}_e \Delta\varepsilon dV + \\ & \frac{1}{2}\int_V \Delta^i\varepsilon_{cr}^T \mathbf{D}_e \Delta^i\varepsilon_{cr} dV + \frac{1}{2}\int_V \Delta^i\varepsilon_{cr}^T \hat{\mathbf{D}}_{cr} \Delta^i\varepsilon_{cr} dV - \int_V \Delta\mathbf{u}^T \Delta\mathbf{f} dV - \int_S \Delta\mathbf{u}^T \Delta\mathbf{t} dS. \end{aligned} \tag{12}$$

Substituting Eqs. (8) and (10) into Eq. (12), and applying the stationary condition $\delta(\Delta U) = 0$ give

$$\begin{bmatrix} \int_V \mathbf{B}^T \mathbf{D}_e \mathbf{B} dV & -\int_V \mathbf{B}^T \mathbf{D}_e \mathbf{B} dV \\ -\int_V \mathbf{B}^T \mathbf{D}_e \mathbf{B} dV & \int_V \mathbf{B}^T \mathbf{D}_e \mathbf{B} dV + \int_V \mathbf{B}^T \hat{\mathbf{D}}_{cr} \mathbf{B} dV \end{bmatrix} \begin{Bmatrix} \Delta\mathbf{U} \\ \Delta^i\mathbf{U}_{cr} \end{Bmatrix} = \begin{Bmatrix} \int_V \mathbf{N}^T \Delta\mathbf{f} dV + \int_S \mathbf{N}^T \Delta\mathbf{t} dS \\ \mathbf{0} \end{Bmatrix} \tag{13}$$

which is the element stiffness equation for the i^{th} element. After assembling all element stiffness equations and applying prescribed displacements and forces, the static condensation can be used to remove the nodal total displacement increment from the obtained global matrix equation. Therefore, the equation can be written in the following form, i.e.,

$$\mathbf{K}_{cr}\Delta\mathbf{U}_{cr} = \Delta\mathbf{R}_{cr} \tag{14}$$

where $\mathbf{K}_{cr}$ and $\Delta\mathbf{R}_{cr}$ are the stiffness matrix and the right-hand-side force increment vector after the static condensation.

It must be noted that Eq. (14) is a singular equation because $\Delta\mathbf{U}_{cr}$ contains the rigid-body crack displacements. To avoid these rigid-body crack displacements, constraints to remove them must be applied to the equation (see the next section). This leads to a modified equation, i.e.,

$$\hat{\mathbf{K}}_{cr}\Delta\hat{\mathbf{U}}_{cr} = \Delta\hat{\mathbf{R}}_{cr}. \tag{15}$$

The stability condition can be obtained by checking the eigenvalues of $\hat{\mathbf{K}}_{cr}$. If all the eigenvalues are positive, it means the equilibrium is stable and there is no bifurcation.

3. Results and Discussion

To show the validity of the method, a simple one-dimensional cracking localization problem is considered. The problem is shown in Fig. 1. It is an axial bar with one fixed support. At the other end, the displacement is controlled. The total length of the bar is $4L$ and the area is A. The material is assumed to be elastic with Young's modulus and Poisson's ratio equal to E and 0, respectively. The bar is discretized into 4 axial two-noded elements, each of which has the length of L. Assume that there are two existing cracks in the elements ***2*** and ***3***. Further assume that the ratio between the incremental transmitted axial stress and the incremental axial crack strain, $\Delta\sigma / \Delta\varepsilon_{cr}$, for the current incremental step is equal to H for both cracks.

The linear shape function is used for all elements. Therefore, for the elastic bar elements without cracks ***1*** and ***4***, the element stiffness equations will be the normal ones. However, for the cracked elements ***2*** and ***3***, the stiffness equations are written as

$$\begin{bmatrix} \frac{AE}{L} & -\frac{AE}{L} & -\frac{AE}{L} & \frac{AE}{L} \\ -\frac{AE}{L} & \frac{AE}{L} & \frac{AE}{L} & -\frac{AE}{L} \\ -\frac{AE}{L} & \frac{AE}{L} & \frac{A(E+H)}{L} & -\frac{A(E+H)}{L} \\ \frac{AE}{L} & -\frac{AE}{L} & -\frac{A(E+H)}{L} & \frac{A(E+H)}{L} \end{bmatrix} \begin{bmatrix} \Delta U^j \\ \Delta U^k \\ \Delta^i U^j_{cr} \\ \Delta^i U^k_{cr} \end{bmatrix} = \begin{bmatrix} \Delta^i R^j \\ \Delta^i R^k \\ 0 \\ 0 \end{bmatrix} \tag{16}$$

where $i = 2, j = 2, k = 3$ for element ***2*** and $i = 3, j = 3, k = 4$ for element ***3***. The superscript before a variable denotes the element number while the one after denotes the node number.

Assembling all the element stiffness equations, applying the prescribed conditions and removing the total displacement increment degrees of freedom by the static condensation give

$$\begin{bmatrix} \frac{A(E+4H)}{4L} & -\frac{A(E+4H)}{4L} & \frac{AE}{4L} & -\frac{AE}{4L} \\ -\frac{A(E+4H)}{4L} & \frac{A(E+4H)}{4L} & -\frac{AE}{4L} & \frac{AE}{4L} \\ \frac{AE}{4L} & -\frac{AE}{4L} & \frac{A(E+4H)}{4L} & -\frac{A(E+4H)}{4L} \\ -\frac{AE}{4L} & \frac{AE}{4L} & -\frac{A(E+4H)}{4L} & \frac{A(E+4H)}{4L} \end{bmatrix} \begin{bmatrix} \Delta^2 U^2_{cr} \\ \Delta^2 U^3_{cr} \\ \Delta^3 U^3_{cr} \\ \Delta^3 U^4_{cr} \end{bmatrix} = \begin{bmatrix} -\frac{AE}{4L}\Delta\bar{u} \\ \frac{AE}{4L}\Delta\bar{u} \\ -\frac{AE}{4L}\Delta\bar{u} \\ \frac{AE}{4L}\Delta\bar{u} \end{bmatrix} \tag{17}$$

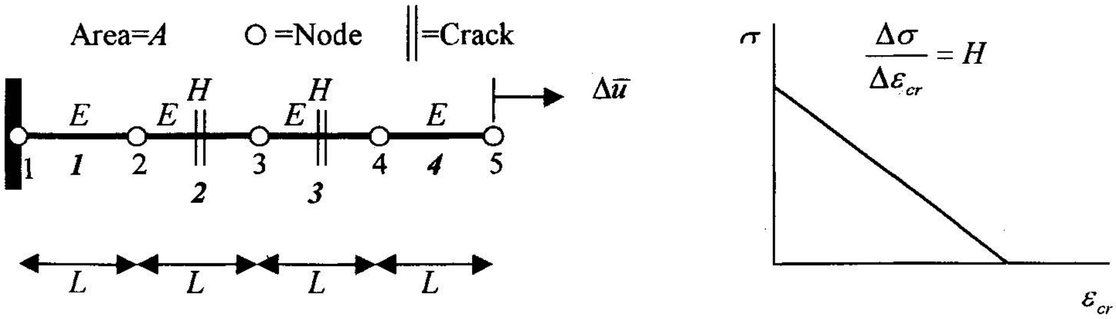


Fig. 1 One-dimensional problem with two cracks

As discussed earlier, the equation obtained above is singular due to the rigid-body crack displacements. To constrain the rigid-body crack displacement, the crack displacements at the centers of the cracked elements are set to zero. From the linear interpolation, we get

$$\Delta^2 u_{cr}(0)=\frac{1}{2}\Delta^2 U_{cr}^2+\frac{1}{2}\Delta^2 U_{cr}^3=0,\ \Delta^3 u_{cr}(0)=\frac{1}{2}\Delta^3 U_{cr}^3+\frac{1}{2}\Delta^3 U_{cr}^4=0. \tag{18}$$

Using Eqs. (18) in Eq. (17) leads to

$$\begin{bmatrix} \frac{A(E+4H)}{2L} & -\frac{AE}{2L} \\ -\frac{AE}{2L} & \frac{A(E+4H)}{2L} \end{bmatrix} \begin{bmatrix} \Delta^2 U_{cr}^2 \\ \Delta^3 U_{cr}^4 \end{bmatrix} = \begin{bmatrix} -\frac{AE}{4L}\Delta\bar{u} \\ \frac{AE}{4L}\Delta\bar{u} \end{bmatrix}. \tag{19}$$

The eigenvalues of the stiffness in Eq. (19) are used to obtain the stability of the equilibrium state. The eigenvalues are equal to $2AH/L$ and $A(E+2H)/L$. If $H>0$, two eigenvalues are positive and the equilibrium state is stable. This solution represents cases where hardening behavior occurs after cracking. This kind of hardening behavior is unrealistic for quasi-brittle materials. In this case, both cracks will open and there will be no localization. If $-E/2<H<0$, one eigenvalue is negative and the equilibrium state is unstable. This solution represents cases where softening behavior occurs after cracking. This softening behavior is common for quasi-brittle materials. The result means that the solution with two opening cracks is unstable. Actually, with further investigation, it can be shown that only one crack will open and the other crack will close. This behavior represents the localization. If $H<-E/2$, both eigenvalues are negative. This also corresponds to unstable equilibrium where both cracks will open in an unstable manner. These results agree with those obtained by Brocca [1] who investigated the same problem analytically.

4. Conclusion

The smeared crack approach can be used in the analysis of cracking localization by introducing an appropriate discrete irreversible variable related to the crack strain. In this study, a crack displacement variable is introduced in the smeared crack finite element analysis for this purpose. The relationship between the proposed variable and the crack strain follows the ordinary strain-displacement relationship. This newly introduced variable allows the consideration of stability of equilibrium states and bifurcation. The scheme is tried with a simple axial problem. The results obtained show promising capability of the method in analyzing problems with cracking localization. Therefore, using the discrete crack finite element, which is not suitable for problems with many cracks, in cracking localization problems can be avoided.

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