Zeitschrift: IABSE reports = Rapports AIPC = IVBH Berichte

Band: 70 (1993)

Artikel: Limit analysis of block masonry shell structures

Autor: D'Asdia, Piero / D'Ayala, Dina

DOI: https://doi.org/10.5169/seals-53317

Limit Analysis of Block Masonry Shell Structures

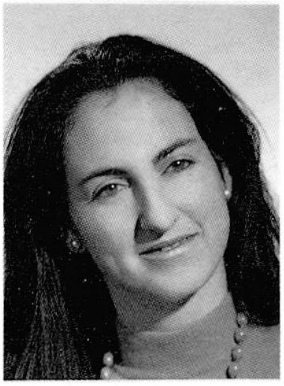
Analyse limite des dôme en pierres de taille

Grenzwertanalyse von Gewölbestrukturen aus Quadersteinen

Piero D'ASDIA
Assoc. Professor
Univ. of Rome 'La Sapienza'
Rome, Italy

P. D'Asdia, born 1946, got his civil eng. degree in 1971 at the Univ. of Rome 'La Sapienza', where he worked as a researcher. In 1980, he became assist. prof. and is now assoc. prof. Since 1971, he has been involved in research in several fields of structural engineering.

Dina D'AYALA
PhD Student
Univ. of Rome 'La Sapienza'
Rome, Italy



D. D'Ayala, born1962, graduated in civil engineering at 'La Sapienza' in 1988. She is now working on a PhD at the Structural Engineering Dep. and part of it has been spent at the Cambridge Univ. Engineering Dept. She has been involved in a number of restoration projects.

SUMMARY

The paper presents the limit-state analysis of three-dimensional masonry structures with special interest focusing on domes. A lower bound approach using linear programming techniques has been developed and improved with respect to the 'simplex method' formulation. The duality theorem and an algorithm have been applied to handle three-dimensional mechanisms resulting from sliding and rotating, or both, at the blocks interface. The results are compared with those obtained from experimental work carried out on a masonry dome model in order to study the crack pattern.

RÉSUMÉ

Ce document décrit l'analyse limite de structures tridimensionnelles en maçonnerie, en particulier des dômes. On a développé une méthode statique avec des techniques de programmation linéaire qui a été améliorée concernant la formulation du 'simplex method'. Le théorème de la dualité et un algorithme ont été adoptés pour traiter les mécanismes tridimensionnels résultant de combinaisons de glissements et rotations à l'interface des blocs. Les résultats ont été comparés avec ceux d'essais réalisés sur un modèle de dôme, afin d'étudier le modèle de rupture de ce type de structure.

ZUSAMMENFASSUNG

Dieser Beitrag beschreibt die Grenzwertanalyse dreidimensionaler Mauerwerkstrukturen, insbesondere Kuppeln. Eine statische Methode wurde mit linearer Programmierungstechnik entwickelt und in bezug auf die Formulierung der Simplex-Methode verbessert. Unter Verwendung des Dualitätstheorems wurde ein Algorithmus entwickelt, um die dreidimensionalen Mechanismen zu behandeln, die durch Gleiten und Rotation oder beides kombiniert an der Innenflächen der Quadersteine entstehen. Die gewonnenen Erkenntnisse werden mit denjenigen aus Experimenten an einem Gewölbemodell verglichen, um das Bruchverhalten dieser Strukturen zu untersuchen.

1. INTRODUCTION

The limit-state analysis method was first applied to the collapse analysis of masonry structures by Heyman [1] [2]. The scheme for the material consist of rigid blocks and joints incapable of carrying tension stress, with friction coefficients high enough to prevent sliding. Subsequently Livesley [3] developed a numerical procedure for the analysis of plane single span arches with in-plane loading, which he recently extended to multi-span arch structures, considering both hinging and sliding [4].
The more recent applications directly involved in the masonry dome analysis have followed two different approaches: the one mentioned above and the finite-element method, which considers the structure in either its uncracked or cracked state. The works of Bridle-Hughes [5] are based on the first approach and propose an energy method in which the arch geometry and stiffness are modified according to the fracture evolution; Oppenehim et al. [6] also deals with an analytical approach that leads to a closed form solution of the fundamental differential equilibrium equations in the case of axisymmetric loads under the assumption of zero hoop stress. This greatly simplifies the problem reducing it to a monodimensional case. The assumption is correct for the lower part of the dome, but not for the upper part, this being more extensive as the ratio between rise and span of the dome is bigger, therefore it can be assumed to be within the lower bound of the possible solutions.
The necessity of the solution of a full three-dimensional problem is first assessed by Melbourne [7] in 1991. He isolates the barrel vaults that constitute the structure of masonry bridges in order to define through a number of experimental test the collapse mechanism due to variable restraint conditions along the abutments and non symmetric loads.
The limit-state analysis therefore appears as a fundamental tool for the assessment of the safety levels of such structures, and in the following sections a fully three-dimensional analytical method is presented.

2. DEFINITION OF CONSTITUTIVE LAWS

The Coulomb failure criteria is assumed to rule the mechanical behaviour of a structure of material whose tensile imaginary strength is due to friction. The criteria states that there will be sliding in every point of the material where the applied shear (τ) overcomes the friction strength (defined by the coeff. μ and cohesion c_0). It also states that the two sides of the sliding surface will separate if compressive stress (σ) is not applied on the surface:

$$\tau \leq c_0 + \mu \sigma \qquad (1)$$

In the block-work masonry sliding movements can occur at the joint surfaces only, and therefore the sliding layers are a known of the problem; other possible mechanisms are rotations around axes parallel or orthogonal to the sliding surface, or around an edge or a vertex of the block, or a superposition of any of these primary mechanisms.
The choice of whether the contact surfaces should be thought slightly convex or concave (i.e. 1 point or 4 points contact in space; 1 or 2 in the plane), does not affects the collapse load factor in two-dimensional problems, but is fundamental when the kinematic chain and the applied action that produces it are not in the same plane. The hypothesis of a convex surface (a simple contact point) that is usually assumed in two-dimensional problems, no longer applies here; actually with this type of surface the collapse load factor will be always zero, unless it is somehow coupled with an additional torsional constraint. It is therefore evident that in those cases the contact should be realised on a concave surface, that can eventually degenerate in a line or a point. The least number of nodes are then 3, if the surface is flat, and 4 in the case that a finite curvature is present. In each of these points orthogonal and tangential stresses develop, the tangential ones oriented in any direction but retaining equilibrium.
Because in real structures blocks are quadrangular parallelepipeds and the contact surfaces will not in general be flat, the present formulation is written for a 4 contact points with normal to the surface not necessarily parallels.
For structures with simple curvature Livesley [4] showed how the elimination of one of the contact points only slightly reduces the collapse load factor while it does not affect the mechanism, so that the contact surface can be well enough approximated by a plane. The same cannot be said for double curvature structures, as will be shown later, this involving a strong constraint for the collapse thrust surfaces.

Taking the centre of gravity as origin of the coordinates, in the case of contact along only one direction and assuming the block has not curvature of its own, the equilibrium equations and the constraint expressions for a single block, are as follows:

$$\mathbf{H}\,\mathbf{r} = \mathbf{p}$$

$$\sqrt{t_i^2 + s_i^2} \leq c_0 + \mu q_i \qquad i = 1,4 \tag{2}$$

where : **r** is a vector whose elements are s, t, q (generalised stresses),
H is a matrix of linear geometrical relations,
p is the vector of dead and live loads

In the case of a single block the matrix **H** is a 6 x 12 (3 generalised stress components for each contact point) for one contact surface only, 6 x 24 if the contact surfaces are two.
The constraints in (2), which are quadratic, represents a conical limit surfaces with vertex $\mathbf{q} = \mathbf{0}$ and circular section on the plane s-t.

3. OUTLINES OF THE PROPOSED METHOD

To be able to formulate an algorithm in linear programming terms the constraints (2) have to be linearised, i.e. the conic surface has to be approximated by a pyramid. Different approximating polygonals on the plane s-t are proposed in literature, from a minimum of 3 edges to a maximum of 12. In the present case an 8 edges polygonal has been chosen, obtained by introducing two auxiliary variables, u and v, in order to define the two directions at an angle of 45° with respect to s and t, and 8 new constrained variables ω_i (while s, t, u, v are unconstrained), one for each orientation of the four primary directions.
With the introduction of these new variables, the dimension of the global system becomes 46 x 52 for each contact surface. The system has 6 redundant variables.
The position of the optimisation problem takes then the canonic form:

$$\max \{ \lambda \ : \ \lambda\,\mathbf{p} + \mathbf{w} = \mathbf{H}\,\mathbf{r}\,,\ -\mathbf{r}_L \leq \mathbf{r}^-\,,\ \mathbf{r}^+ \leq \mathbf{r}_U \} \tag{3}$$

The linear programming algorithm used works extracting the maximum rank sub-matrix of the system, choosing the variable to be maximised in relation with the correspondent known vector and the active constraints, computing the increment in the load factor associated with the maximisation of the variable, and then operating a substitution between the column of this variable and one, appropriately chosen, external of the present sub-matrix.
The first problem that arises in a limit-state analysis approach is the fact that the normality rule does not apply when sliding is involved, because the generalised stress limit surface and the generalised strain limit surface do not coincide: the first one is a cone and the limit value of the shear linearly depends on the normal stress, while the corresponding sliding of the associate mechanism will take place without any change in volume. Alternatively applying the normality rule to the conical limit surface, the generalised strain vector ε_{pl}, assumed to be normal to it, will show a component along the **q** direction that implies separation of the facing nodes; but this is not possible if $\mathbf{q} \neq \mathbf{0}$.
The numerical methods and their optimising strategies can be different according to the object and size of the analytical problem. It is worthwhile noting that the adopted algorithm, a series of Gauss-Jordan reductions, with forward and backward substitutions, has the advantage of operating only on the meaningful portion of the matrix (all the equations but only the primary variables, so that the matrix to be stored is a 46 x 12 instead of 46 x 52 for each block) and of exploiting the dual nature of the problem so as to obtain from the same solution the collapse load factor and the associated mechanism. The back substitution is actually carried out at the end of the forward process to obtain the sliding displacement component, associated with the variables **s, t, u, v** that are unconstrained, as independent from **q** and therefore observing the normality rule.

4. APPLICATION TO SHELLS AND DOMES

Two versions, applicable to structure of double curvature will be proposed: the first aims to define the collapse of the material medium from a local approach, the second is intended to

define the collapse load factor on a global scale. In the former the equations are written for the individual block constituting the masonry work. Each block is in contact with 6 adjacent ones and, if each interface is defined by 4 contact points, the number of generalised primary stresses will be 6 x 4 x 3 = 72. Some of the points (vertices of the block), being common to two normal surfaces, can be regarded as a single one with six applied generalised stress components, fig. 1. The middle points, related with the staggering, can be thought as common to the considered block and the two above and below, fig. 2, so to have a single node. In this way the increment in matrix dimension is only due to the increased number of stresses in each node, while the bandwidth of the matrix is considerably increased according to the complex contact scheme. Therefore the coefficient matrix **H** of the equilibrium equations takes a three-diagonal form, while the limit constraints duplicate to take into account the friction constraint onto the parallel surface of the block as well as onto the meridian ones.
As far as the equilibrium equations are concerned, dimensions of each block matrix are 6 x 24, plus 80 equations for the auxiliary constraints. As it can be seen, the size of the problem quickly increases even if a limited number of elements are treated (i.e. hundreds in order to simulate a part of a real structure), CPU time growing with the power of 2 of the matrix dimension.
In order to reduce the problem dimensions, a simplification can be introduced by extending the shear reciprocity rule ($t_{12} = t_{21}$) to the plastic field; it is worthwhile noticing that this assumption provides approximated values, on the conservative side, of the collapse load factor λ.
If no external load is applied to the block, the equilibrium equation along the direction tangent to a generic parallel becomes :

$$\frac{\partial(t_{12}R_2\sin\varphi)}{\partial\varphi} + R_1\frac{\partial q_2}{\partial\vartheta} - t_{21}R_1\cos\varphi = 0 \qquad (3)$$

being R_1 and R_2 the two local radii of curvature, φ and θ the spherical coordinates. If $q_2 = 0$ then also $t_{21} \leq c_o$ because of the friction relation, and locally approximating the surface with a sphere:

$$t_{12}(R_2\cos\varphi - R_1\cos\varphi) + \frac{\partial t_{12}}{\partial\varphi} = 0 \qquad (4)$$

The first element is 0 and thus the variation of t_{12} along the meridian.
The simplification $t_{12} = t_{21}$ therefore reduces of 1 the number of variables. The other assumption is that, the flexural stiffness along the parallel being much greater than the one along the meridian, generalised stress **s** normal to the surface will develop only on the parallel faces of the element (of normal q_1) while their variation along the meridian line is equilibrated by the normal component of q_2. This reduces the number of generalised stresses for each node to 4 and only one conical surface is needed for the constraints, checking the **t** variable as function of the lesser between q_1 and q_2, so that the block matrix is 46 x 16.
This formulation has been used to study the local behaviour and collapse mechanism of 1/8 of the experimental dome for an outward horizontal displacement. The result has also been compared with the one obtained for a spherical dome (fig. 3).
It is evident that the present formulation is not only unsuitable for analysing global mechanisms but is also unnecessary at the global scale. Therefore a slightly different algorithm has been prepared that derives from the observation of the most common structural layout and associated failure patterns of domes. The shell may be divided into not less than 8 slices with straight sliding surfaces along a discrete number of meridian and parallel curves, fig. 4. The meridian surfaces represent the meridian cracks, while the parallels are the lines where an hinge along the meridian arch is most likely to occur.
In this way the entire structure can be modelled with a small number of macro-elements. For each one of these the equilibrium equations must take into account the relation between q_1 and q_2 due to the finite double curvature of the element and the discontinuity conditions at the sliding interfaces. Equilibrium equations are as follows:

$$2\pi R_2 q_1(\sin\varphi)^2 + 2\pi R_2 s\cdot\sin\varphi\cdot\cos\varphi = \gamma\int 2\pi R_2\sin\varphi\cdot d\varphi$$

$$R_2q_1(\sin\varphi) + R_1q_2(\sin\varphi) + \frac{\partial(R_2s\cdot\sin\varphi)}{\partial\varphi} = \gamma R_2R_1\sin\varphi\cdot\cos\varphi \qquad (5)$$

$$\frac{\partial(R_2q_1\sin\varphi)}{\partial\varphi} - R_1q_2(\cos\varphi) - R_2t\cdot\sin\varphi = \gamma R_2R_1(\sin\varphi)^2$$

While the discontinuity condition for a material (with φ = polar coordinate in the vertical plane and $\alpha = \pi/2 - \varphi$) are:

$$q_n = \frac{q_1+q_2}{2} - \frac{3}{2}[(q_1-q_2)\sin\alpha] \qquad n = 1, 2$$

$$(6)$$

$$\tau = \sqrt{s^2+t^2} = \frac{(q_1-q_2)\cos\alpha}{2}$$

The equilibrium differential equations can not straight be integrated and linearised in the general form. Once a curve has been chosen to approximate the real curve that generate the dome, however, is then possible to define an analytical expression for R_1 and R_2 and therefore the equation that represents the curve can be written in the Hessian normal form choosing φ as the parameter. Substituting these in the system (5) and applying the hypothesis of a uniform stress state in the block when the ultimate state is reached, leads now to linearised equations. The method has been applied to structure whose approximating curve are sections of circles, cycloids and ellipses.

Load factor and collapse mechanism, for a given c_0 and μ, depend on several parameter as: radius, variable thickness, ratio between rise and span of the generating curve, and relative position of the discontinuity surfaces.

As it can be seen from fig. 5, where the relation between the load factor and one of these parameters has been drawn, normalising the load factor value with respect to the dead load value, the relation is not monotone but reaches in general a maximum for a certain value and then decrease exponentially for the others. This depends on the fact that the geometrical parameters are not independent.

Other peculiar characteristic of dome structure is the fact that, the behaviour being mainly ruled by the shear strength, the mechanism shows an antisymmetric pattern even if the action is symmetric. More in general once the first crack has open the symmetry is lost and the final collapse mechanism evolves toward an highly non symmetric pattern.

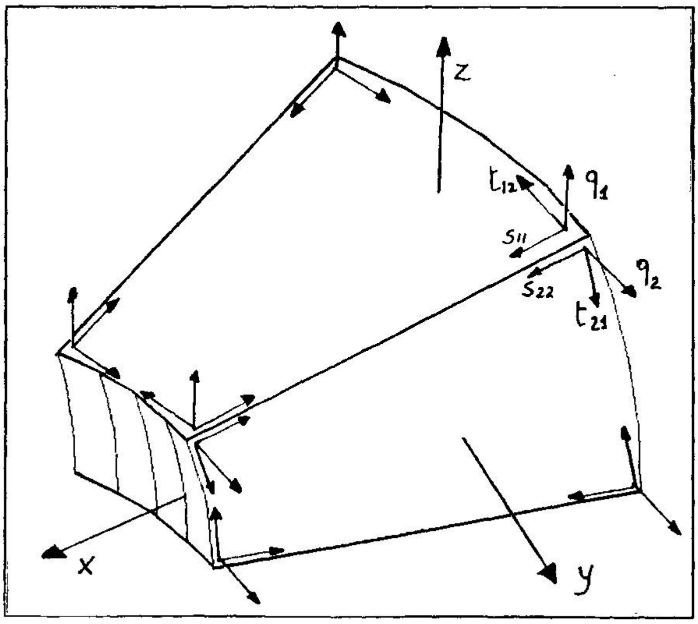


fig. 1 - Coordinate system and generalised nodal stresses for a block

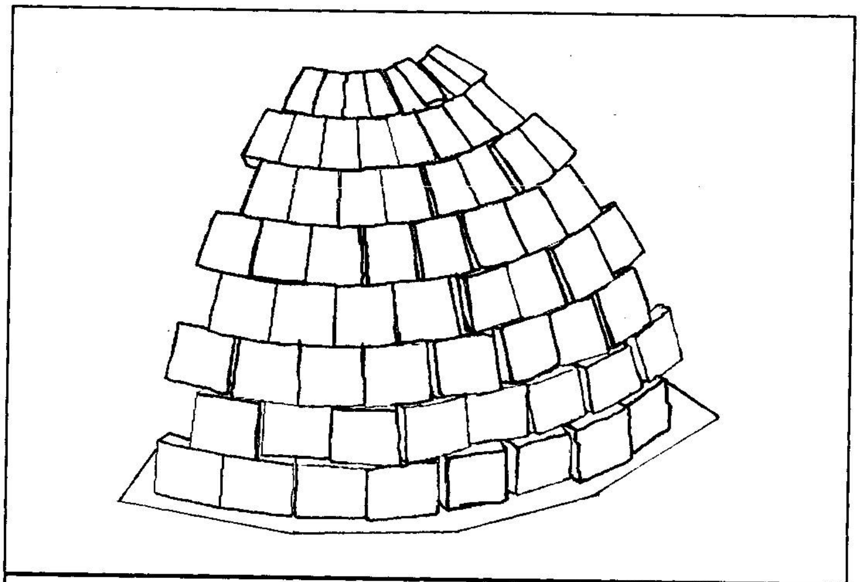


fig.2-A 90° sector of a spherical dome subjected to horizontal outward action on the second row

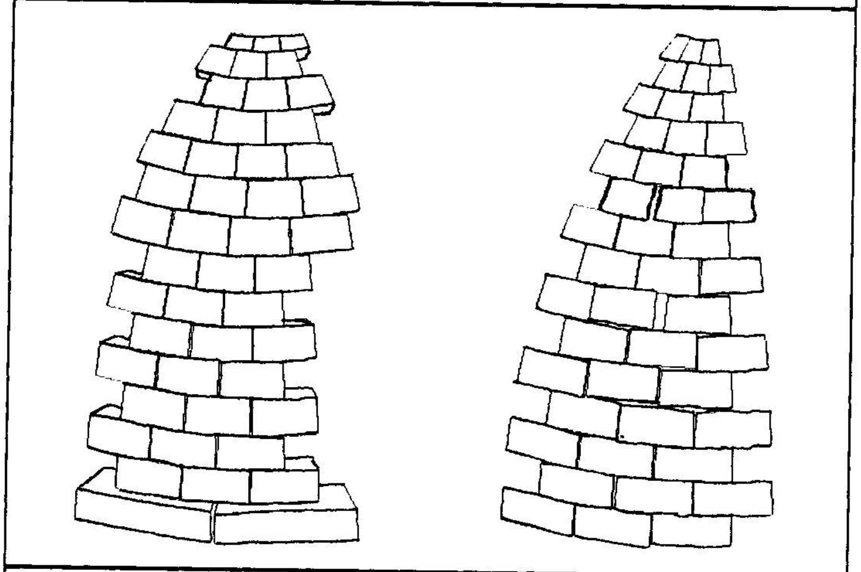


fig. 3 - Collapse mechanism for a 45° sector of the experimental dome (λ = 0.55) and of a spherical dome (λ = 0.325) for outward horizontal sliding of the base

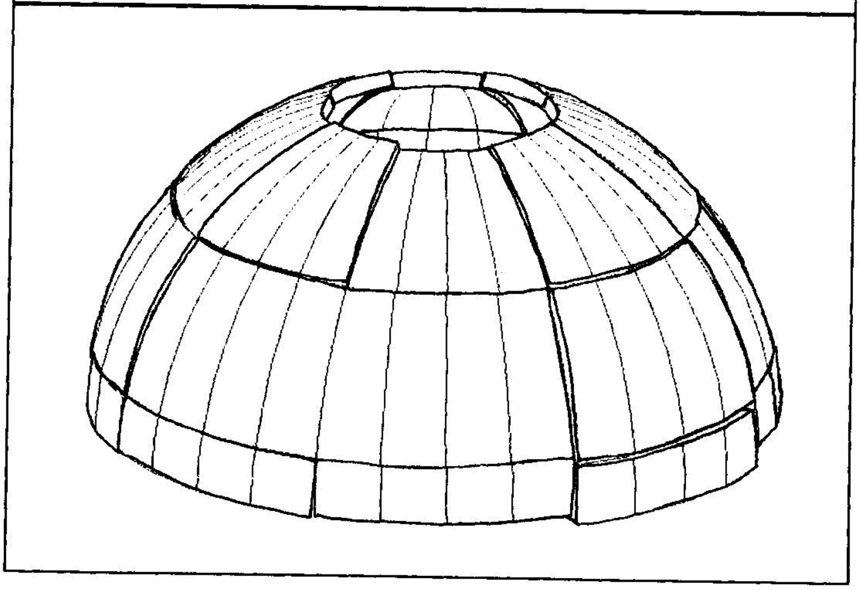


fig.4 - Global collapse mechanism for a spherical dome with hinges positioned at 45° and 80° for the settlement of one basis.

5. CONCLUSIONS

The analytical method described above is part of a Ph.D. research program which also investigates the behaviour of masonry domes affected by foundation movements, through laboratory testing. For this purpose a model dome (fig. 6) has been built in the Laboratory of the Cambridge University Engineering Department, U.K. [8]. One of the most interesting tests involves the sliding out of one or two of the sectors of the base ring which simulate the lower structure. The change in shape of the dome and the relative movement between blocks are recorded by means of special transducers. Results show that the damping behaviour for elastic shells still applies, even if the thickness cannot be thought of as significantly smaller than the radius (fig. 7). Therefore the crack pattern only develops in a region close to the point of application of the action. When the action is further increased, the crack pattern tends to extend and flow in the one caused by an upper load, simulating the lantern effect, and a series of local hinging mecanisms take place at the single block level (fig. 8). The load factor has been deduced from the decrease in the level of bearing capacity of the dome (for the upper imposed load) when the base is moved out. The values and shape of the mechanism show good correspondence with the results obtained by the analytical model in fig. 4.

6. REFERENCES

[1] HEYMAN J., "On shell solution for masonry domes" - Int. J. Solid Struct. 3, 1967.
[2] HEYMAN J., "Equilibrium of shell structures" - Oxford Engineering Science Series - 1977.
[3] LIVESLEY R.K., "Limit analysis of structures formed from rigid blocks" - Int. Journ. for Numer. Methods in Engineering - 12 1978
[4] LIVESLEY R.K., A Computational Model for the Limit Analysis of Three-Dimensional Masonry Structures, Meccanica 27-3,1992
[5] BRIDLE R.J., HUGHES T.G., An energy method for arch bridge analysis, Proc. Inst. of Civ. Engrs., Part 2, 89, 1990
[6] OPPENHEIM I.J., GUNARATMAN D.J.,ALLEN R.H., "Limit State Analysis of Masonry Domes", J. of Structural Engineering ASCE n.4 april 1989
[7]MELBOURNE C., The Assessment of Masonry Arch Bridges, 9th International Brick/Block Masonry Conference, Berlin 1991

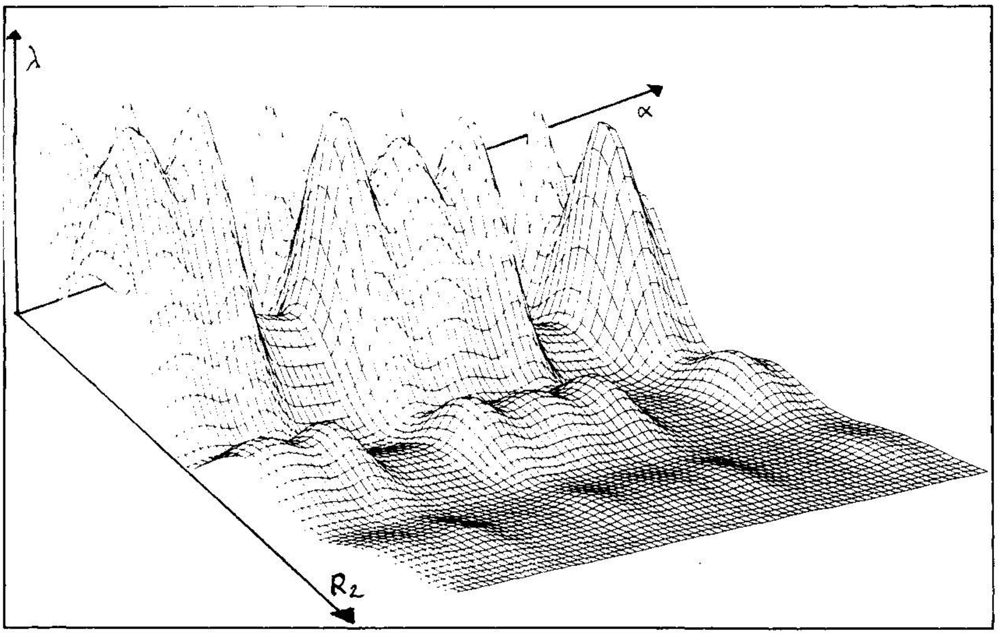


fig.5 - Load factor variation for different dimension of the base blocks

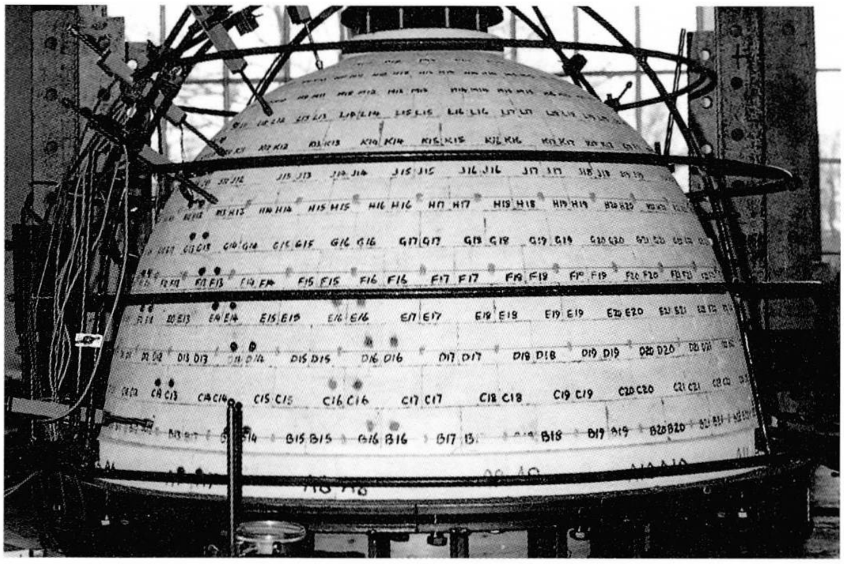


fig.6- The model dome is realised with 380 small concret blocks casted in situ. To simulate the friction caractheristiques of the stone sandpaper has been interposed between surfaces.

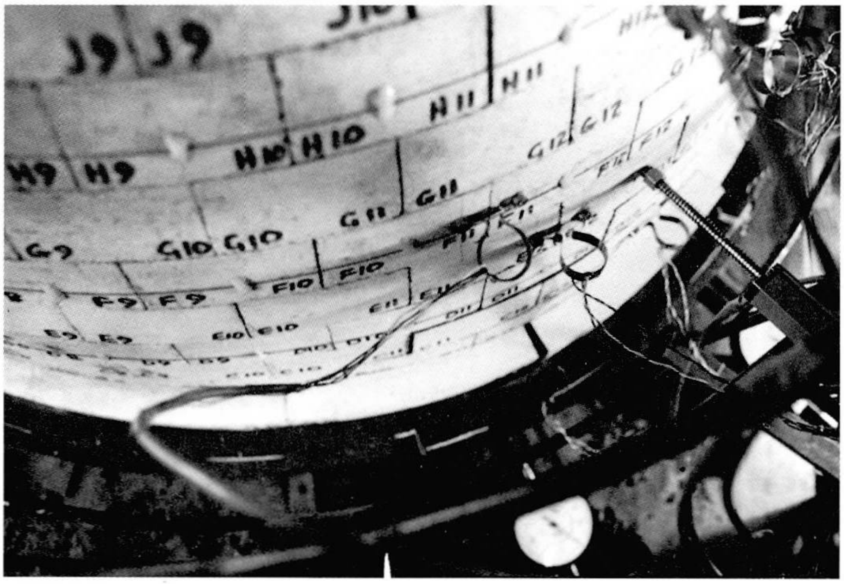


fig. 7 - Crack pattern for outward horizontal sliding of the base

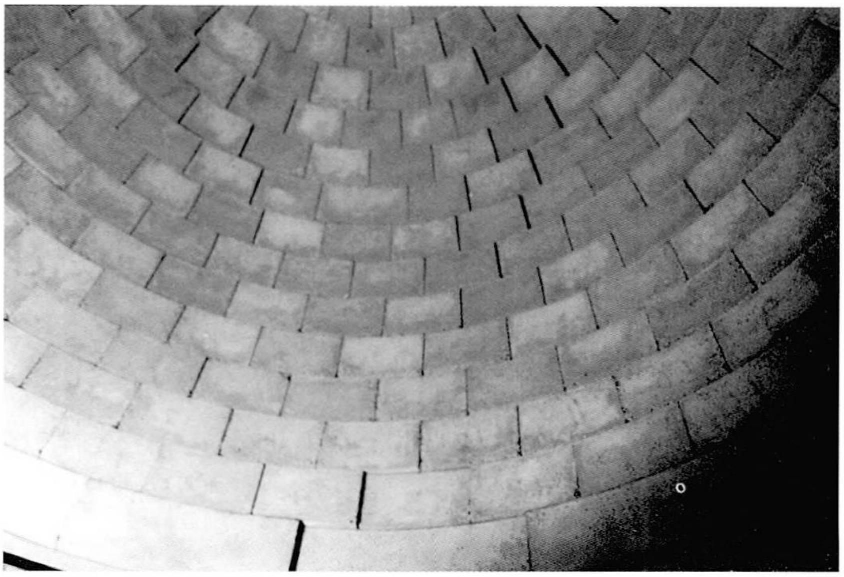


fig.8 - An inside view of the collapse mechanism.