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# Statistical Distribution of Crack Widths in Reinforced Concrete Bars

## Distribution statistique de l'ouverture des fissures des barres en béton armé

## Statistische Verteilung der Rissbreiten an Stahlbetonstäben

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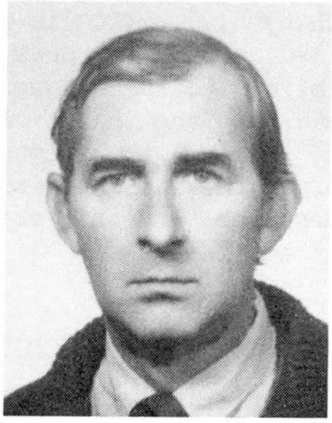

Born 1945, István Hamza obtained his engineer-architect degree at the Faculty of Architecture at the Technical University of Budapest. Since 1968 he has been working as a scientific research associate at the Department of Strength of Materials and Structures of the same University. His research field is the serviceability of structures.

**SUMMARY**

The characteristics of the statistical distribution of crack widths, and crack spacing were determined from a large amount of data obtained by simulating the crack formation on reinforced concrete bars under simple tension. The tensile and bond strengths were generated as random variables. The numerical results obtained were compared with experimental data and with the crack width calculated according to several standards or codes.

**RESUME**

En simulant par ordinateur la fissuration des barres sollicitées en traction pure, on a déterminé les caractéristiques des distributions statistiques de l'ouverture, et de la distance des fissures. Dans les essais, la résistance à la traction ainsi que celle à l'adhérence du béton ont été prises en compte comme des variables aléatoires. Les résultats numériques ont été comparés avec des données expérimentales et avec l'ouverture des fissures calculée selon différents codes.

**ZUSAMMENFASSUNG**

An zugbeanspruchten Stahlbetonstäben wurden die Charakteristiken der statischen Verteilung der Rissbreiten und der Rissabstände ermittelt, und zwar mittels Computersimulation der Rissbildung aus zahlreichen Daten. In den Untersuchungen wurden die Verbund- und Zugfestigkeit des Betons als zufallsverteilt betrachtet. Die ermittelten numerischen Ergebnisse wurden mit Versuchsdaten und mit gerechneten Werten der Rissbreite nach verschiedenen Normen verglichen.

## 1. INTRODUCTION

Recently one can notice a trend applying probability methods to determine the characteristics of service conditions, like the design value of crack widths, just as it became a practice in calculating loadbearing properties. In accordance with international recommendations the 95%-fractile of the distribution of crack widths, developing along intervals under constant stress-condition, will be considered as the characteristic value of crack width.

In the CEB-FIP Model Code [1] the 95%-fractile is obtained by multiplying the mean value of crack width by 1,7. (It is the 0,4 value of the variation coefficient of crack width, that is presumed in the multiplier.) The new CEB-FIP Model Code [2], similary to the Hungarian standard [3], calculates the characteristic crack width on the basis of maximum crack spacing. Those calculations, although they involve the elements of probability calculus, are not suitable for determining the characteristics of the statistical distribution of crack widths. Moreover, it is a question, how the calculated value relates to the actual 95%-fractile.

## 2. THE METHOD OF THE DISTRIBUTION CALCULATION

The characteristics of the statistical distribution of crack widths (the type of distribution, the mean value, the standard deviation, the 95%-fractile, etc.) can not be determined analytically, because of the accidental nature of cracking, thus the Monte Carlo method is applied instead. The tensile strength and the bond strength of the concrete were generated at random on axially tensed members and the development of cracks was simulated by computer. Whenever a crack has "appeared" the width of the cracks was determined at the end of the crack formation as well as at a higher and a lower level of service load. Having examined a sufficient number of members, conclusions can be drawn regarding the statistical distribution of crack spacings and crack widths.

## 3. THE BASIC ASSUMPTIONS OF THE CALCULATION

In the course of the examinations the tensile strength of the concrete was treated as a probability variable of normal distribution. The tensile strength - considring its mean value and standard deviation - was generated at random for each examined model. The change of tensile strength along the length of the element was disregarded.

Smooth steel bars were considered for reinforcement. Experiments showed that in the case of smooth steel the magnitude of the relative slip is irrevelant to the efficiency of the bond, it is only the direction of the bond stresses, which depends on the direction of the slip. Under these circumstances crack widths can be expressed in an explicit form. (Fig.1.; the true behavior of $f_{cb}$ - denoted with a dotted line in the figure - can be neglected.) There was a stochastic relationship assumed between the bond strength and the tensile resistance of the concrete and its value was supposed to be $f_{cb} = \alpha * f_{ct,i}$. In this relation $\alpha$ is a probability variable of normal distribution the mean value of which is 1 and its variation coefficient was taken as independent of that of the tensile strength. The effects of geometrical features (the cross-section of the concrete, the diameter of the steel bar) as

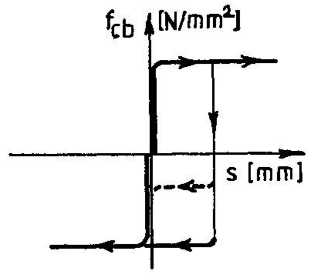


Fig.1 Bond-slip low

well as that of the elastic modulus of the concrete were neglected as being insignificant with regard to the deviation of crack widths.

In the analysis elastic stress condition was assumed and all the time-dependent effects were neglected.

## 4. CALCULATION OF CRACK WIDTHS

If the tensile stress reaches the value $f_{ct,i}$, the tensile strength of the concrete, Crack 1 is equally probable to appear anywhere along the length of the axially tensed member (Fig.2). Since there is no crack to the right of this one yet, the stress, strain and deformation conditions, being in differential-integral relationship with each-other, can develop freely on the bar. (This state is marked with a continuous line in the figure.) This condition can be called a primary, undisturbed crack width state, and the part of the width of Crack 1 which can be calculated from the relative slip between the concrete and the steel on its right is $s_{1r} = \epsilon_{sr,i} * l_b/2$.

In the case of the next crack (Crack 2), occuring at random with in a distance $l_b \leq l_r \leq 2*l_b$ from Crack 1, the stress and deformation conditions on its left side (see the figure) will not be independent from Crack 1. This situation can be termed as secondary, disturbed crack width state. On the left of Crack 2 - because of the direction of the displacement - there are bond stresses acting in the opposite direction, and it follows from equilibrum equation that the change of signes takes place at $l_r/2$. In this place and on the left of it (towards Crack 1) the conditions of stress and deformation do not change; on the right side of this place (towards Crack 2) they develop as indicated by the dotted line in the figure 2. It is clear from the figure, that along this distance there is that point where the relative slip between concrete and steel is zero. The relative displacement on the left side of Crack 2 depends on the parameter $\lambda = l_r/l_b$; $s_{2l} = s_{1r} * [1 - 2*(1 - \lambda/2)^2]$.

On the left side of the first and of the right side of the second crack there may occur either a primary or a secondary state of crack formation. According to these versions the width of a certain crack can be determined - using the relations above - as the sum of the right-side and the left-side relative slips.

Between two existing cracks a new one can occur if $l_r > 2*l_b$. The new crack can appear anywhere along $l_r - 2*l_b$ with the same probability. Since the tensile strength of the concrete was considered to be constant along the bar, the place of the new crack was determined by deviding the $l_r - 2*l_b$ length in proportion of a random number between 0-1 of equal distribution.

On the bar with $f_{ct,i}$ tensile strength the crack formation is final if the distance between the cracks is less than twice the bond length ($l_b$). If the fictitious tensile stress, which can be calculated from the external load, exceeds the tensile strength of the concrete - $\sigma_c = N_r/(A_c + A_s*E_s/E_c) > f_{ct,i}$ -, the distance between the cracks does not change, but their width increases. The change in crack width can be expressed by the increment of strain in the reinforcement : $\Delta s = \lambda * \Delta\epsilon_s/\epsilon_{sr,i}$. (See the part of the figure 2 which is drawn with a dot-dash line.) It is clear from the figure that the point which is free of relative slips moves - depending on the magnitude of the stress - towards the midpoint of the distance between the cracks. After all mentioned above it is possible to examine the distribution of the crack distance and crack widths at the end of the crack formation and at any level of the service load.

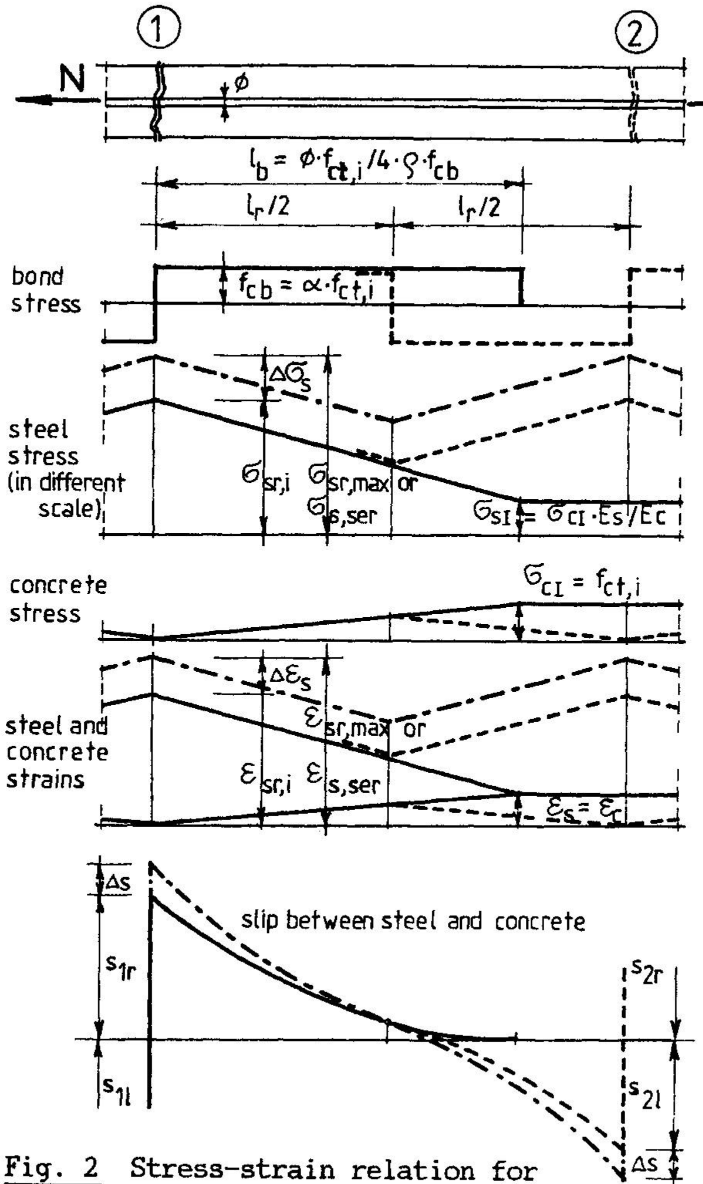


Fig. 2 Stress-strain relation for the calculation of crack width

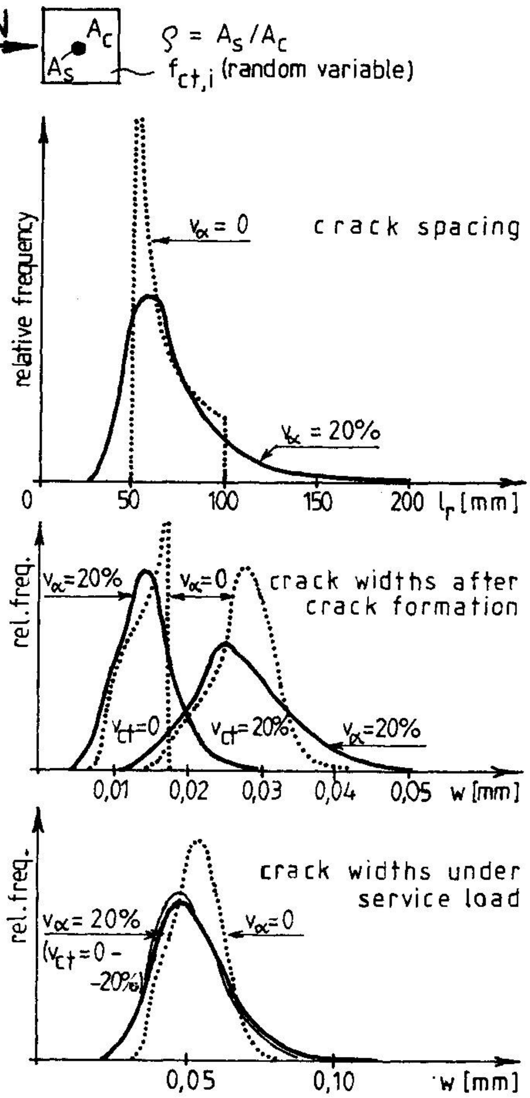


Fig. 3 The types of the distributions

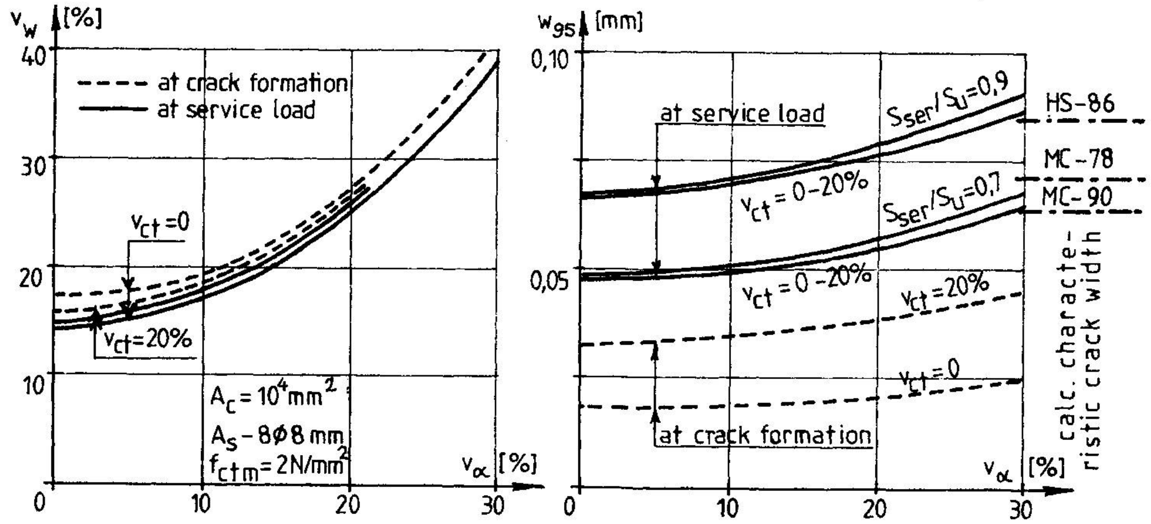


Fig. 4 The coefficient of variation and the 95%-fractile of crack widths related to the variation factors $v_{ct}$ and $v_\alpha$

The limit state of crack formation is definied by the tension determined by the maximum tensile strength ($f_{ct,max}$) that occured during the examinations, and the strain of the reinforcement ($\epsilon_{sr,max}$) corresponding to this tension. In the serviceability limit state a 0,7 ‰ and a 0,9 ‰ elongation of the steel was considered. The first value reflects the level of the service load according to the CEB-FIP Model Code, the second one reflects it according to the Hungarian standard.

## 5. RESULTS AND THEIR INTERPRETATION

The distribution were determined on axially tensed members of $f_{ct,m}$ = 2 N/mm² average tensile strength and $A_c$ = 100*100 mm² cross-section, from the data obtained from cca. fifty thousand cracks.In the computer examinations the variation factor of the concrete tensile ($v_{ct}$) and bond strengths ($v_\alpha$), just like the reinforcement of the bar were variable parameters. (The ratio between the moduli of elasticity of the steel and concrete was regarded as constant with a value of 10.)

The nature of the distribution of crack spacing and crack widths is shown in the figure 3. The distribution of crack spacing is practically independent from the variation factor of the tensile strength of the concrete, but it is considerably dependent on the bond strength, that is, on the α bond factor. When the bond factor is zero the ratio of the minimum and maximum crack spacing is - as it is well-known - one to two. The character of the distribution is asymmetrical and the variation factor of crack spacing is about 21%. If the variation factor of the bond factor is other then zero the character of the distribution is logarithmically normal, the mean value of the crack spacing is practically constant, but its variation factor - of course - increases. - The statistical distribution of crack widths in the limit state of crack formation depends on the variation factor of both the tensile and the bond strength. The distributions are asymmetrical. At service load the dispersion of tensile strength hardly influences the character of the crack width distribution. Here it is the dispersion of the bond factor again, what affects the nature of the distribution. When the variation factor of the bond factor is about zero the distribution is aproximately symmetrical, in other cases it is of logarithmically normal character - it stretches towards the big crack widths.

In the limit state of crack formation and in the serviceability limit state, the figure 4 shows the variation factor ($v_w$) and the 95%-fractile of crack width in relation to the variation of the bond factor and the concrete tensile strength in case of 8ø8 mm reinforcement. In case of service load the variation factor is smaller than during crack formation because of the equalization of crack widths, and its value is 14%, even though both the tensile and the bond strength were assumed to be constant. The variation factor of the bond has a great influence on the variation factor of the crack width, somewhat less on its 95%-fractile, whereas the effect of the tensile strength variation factor on the two above is negligible. The slight variations in the size of the concrete cross-section and/or in the steel-diameter does not modify the variation factor and the 95%-fractile of crack widths significantly. The ratio of reinforcement and the steel bar diameter influence neither the distribution type nor the variation factor of the crack width.

Experiments show that the variation factors of tensile and bond strength are nearly the same, about 15-25%, and they are generally higher than that of the compressive strength. Table 1 shows some characteristics of cracks on elements having 8ø8 mm reinforcement, obtained by computer examinations of cases when the variation factor of the tensile strength was 20% and that of the bond strength was 0,10,20 and 30%. (The strikingly high value of $w_{max}/w_m$ in brackets is from the fact that the distribution of tensile- and bond-strength is not limited.)

| Bond $v_\alpha$ [%] | Numerical results | | | Calc.values/num.res. $w_k/w_{95}$ | | | Exp. $v_w$ and $w_{max}/w_m \approx w_{95}/w_m$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | $S_{ser}/S_u$ | | $S_{ser}/S_u$ | |
| | $v_w$ [%] | $w_{95}/w_m$ | $w_{max}/w_m$ | MC-78 | MC-90 | HS-86 | ≈0,7 | ≈0,9 | ≈0.7 | ≈0.9 |
| 0 | 14,7 | 1,25 | 1,48 | 1,48 | 1,33 | 1,25 | on tensed bars | | on beams | |
| 10 | 17,5 | 1,33 | 1,83 | 1,39 | 1,25 | 1,20 | 38,3%- | 40,6%- | 27,6%- | 32,7%- |
| 20 | 25,3 | 1,46 | (3,15) | 1,24 | 1,14 | 1,10 | -1,89 | -2,11 | -1,35 | -1,76 |
| 30 | 38,2 | 1,66 | (6,15) | 1,04 | 0,94 | 0,91 | 21* | 35* | 20* | 27* |

Table 1 Comparison of the characteristics of crack widths (* number of cracks)

When comparing the above results with the data from our experiment on tensed and bent elements, relatively good correspondances can be observed. Hereby is presented the ratio of the characteristic value of crack width calculated according to [1],[2] and [3] to the 95%-fractile gained from computer examinations. From these values it is clear that if the deviation of bond strength rather big the 95%-fractile of crack width may exceed the calculated value, what goes to the expense of the security of the structures. It can also be observed that calculation according to [2] and [3] gives better estimations of the fractile than that according to [1].

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