Zeitschrift: IABSE reports = Rapports AIPC = IVBH Berichte

Band: 67 (1993)

Artikel: Planes of weakness in finite element analysis

Autor: Hohberg, Jörg-Martin

DOI: https://doi.org/10.5169/seals-51373

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 09.12.2025

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

Planes of Weakness in Finite Element Analysis

Surfaces de rupture en analyses aux éléments finis

Versagensflächen in Finite-Element-Berechnungen

Jörg-Martin HOHBERG
Dr. Eng.
IUB Engineering Ltd
Berne, Switzerland

Jörg-Martin Hohberg (born 1953) studied structural engineering at TU Berlin, Imperial College London and ETH Zürich. After two years with Dyckerhoff & Widmann in Munich on p/c LNG storage tanks, and research in nonlinear seismic arch dam analysis at ETH, he is presently engaged in numerical rock analysis and design of underground structures.

SUMMARY

Interface elements enable the modelling of existing cracks as well as potential failure mechanisms within conventional FEM computations. If used to investigate limit equilibrium states without knowing the cracking process in detail, several tacit assumptions are made as to the load redistribution capacity of the structure. This bears a certain similarity to plastic limit analysis, which also features kinematical discontinuities, albeit with more idealized constitutive models. The discussion is followed by two simple applications involving a voussoir arch and a beam.

RÉSUMÉ

Les éléments joints permettent la modélisation des fissures existantes aussi bien que des mécanismes de rupture potentiels dans la méthode des éléments finis (FEM) conventionelle. Dans le cas où on les utilise dans la recherche des états limites d'équilibre sans connaître le processus de fissuration en détail, certaines hypothèses tacites sont faites concernant la capacité de la structure à redistribuer les charges. On retrouve ainsi certains aspects de la méthode de la charge ultime de plasticité, qui elle aussi, considère des discontinuités cinématiques, avec toutefois des modèles constitutifs plus idéalisés. La discussion est suivie par deux applications simples aux arcs et poutres en voussoirs.

ZUSAMMENFASSUNG

Trennflächenelemente gestatten die Modellierung bereits existierender Risse wie auch möglicher Versagensmechanismen innerhalb der herkömmlichen Finite-Element-Methode (FEM). Falls mit ihrer Hilfe Grenzgleichgewichtszustände ohne genaue Kenntnis des Rissprozesses untersucht werden, unterliegen sie einigen stillschweigenden Annahmen hinsichtlich der Fähigkeit des Tragwerks zur Kraftumlagerung. Darin ähnelt die Methode dem plastischen Traglastverfahren, das ebenfalls kinematische Diskontinuitätslinien kennt, allerdings mit weitergehender Idealisierung des Trennflächenverhaltens. Auf die Diskussion folgen als einfache Anwendungsbeispiele ein Bogen und ein Balken in Blockkonstruktion.

1. INTRODUCTION

The finite element method (FEM) is widely used for the assessment of material damage by following the gradual development of deterioration in structures in a step-by-step procedure. Usual material models are based on incremental plasticity, damage theory or smeared cracking, where for monotonic loading the anisotropy of damage is often neglected to avoid overstiff numerical results; such overstiff behaviour is absent in *discrete* crack models [1]. Apart from distributed ageing phenomena as continuum deterioration, the inspection of deficient structures may reveal a number of existing fractures, which are possibly oriented oblique to the present stress regime and need be modelled as to their effect on the stress redistribution and the failure mode of the cracked structure.

Through the joining of finite elements at their common nodes, the conventional FEM is basically a continuum method. At least with nodal displacements as primary variables, equilibrium is only satisfied in an integral sense: Although the displacement fields are compatible along the element sides, the stress fields exhibit finite jumps at interelement boundaries, thus precluding the computation of strict lower bound limit loads [2]. However, lines or planes of displacement discontinuity can be introduced via double nodes with suitable constraint conditions and used to investigate the ultimate bearing capacity of structures by means of *postulated* failure planes, a concept which is akin to the kinematic approach of limit analysis in that one looks for the mechanism giving the smallest failure load as *upper* bound to the true limit load.[1] The presence of an elastic compliance below the onset of plastic deformation does not invalidate the limit analysis theorems as long as displacements remain small [4]. It is rather the behaviour of the weak planes which infringes on certain vital hypotheses.

2. KINEMATICAL DISCONTINUITIES

2.1 The Concept in Limit Analysis

The general idea is that arbitrary velocity fields can be introduced, which do not need to satisfy equilibrium and may be discontinuous as long as they are kinematically compatible. For instance it is permissible to assume that large parts of the structure move as rigid blocks, separated by narrow plastic regions of thickness t. These are characterized by a high homogenous strain rate, which is the relative velocity between blocks per thickness, $\dot{\delta}/t$. The discontinuities are supposed to consist of a thin layer of material, which obeys a modified Mohr-Coulomb yield criterion (with associated flow rule) and behaves just like a solid, except that the in-plane 'stretching' strain rate $\dot{\varepsilon}_{ss}$ is zero because of the adjacent rigid bodies. Computing principal strain rates with $\dot{\varepsilon}_{ns} = \frac{1}{2}\dot{\gamma}_{ns}$ (Fig. 1),

$$\dot{\varepsilon}_{1,2} = \tfrac{1}{2}\dot{\varepsilon}_{nn} \pm \tfrac{1}{2}\sqrt{\dot{\varepsilon}_{nn} + \dot{\gamma}_{ns}} = \frac{\dot{\delta}}{2\,t}\,(\sin\alpha \pm 1) \tag{1}$$

their directions are found to bisect the angle between the n-direction and the velocity vector, resp. between the s-direction and the normal to the velocity. While $\dot{\varepsilon}_1$ denotes a volume increase due to shear dilatancy or opening, $\dot{\varepsilon}_2$ corresponds to a compression field in the adjacent block [5]. The latter would only disappear for a pure cleavage at $\alpha = 90°$, i.e. if the discontinuity were to coincide with a mode-I crack ($\dot{\varepsilon}_1 \geq \dot{\varepsilon}_2 = 0$). Principal directions at $45° \pm \frac{\alpha}{2}$ (with respect to s) characterize slip lines in a state of pure shear.

The internally dissipated work per unit area is that of a ductile homogenous material, the band thickness droping out during integration:

$$\dot{W}_l\,(\sigma_1\dot{\varepsilon}_1 + \sigma_2\dot{\varepsilon}_2)\,t = \tfrac{1}{2}\dot{\delta}\,\sigma_1\,(\sin\alpha + 1) + \tfrac{1}{2}\dot{\delta}\,\sigma_2\,(\sin\alpha - 1) \tag{2}$$

For a general α one obtains [6]

$$\dot{W}_l = \dot{\delta}\left[f_c\,\frac{1-\sin\alpha}{2} + f_t\,\frac{\sin\alpha - \sin\phi}{1-\sin\phi}\right] \tag{3}$$

[1] As stated in [3]: "The structure will collapse if there is any compatible pattern of plastic deformation for which the rate of work of the external loads exceeds the rate of internal dissipation."

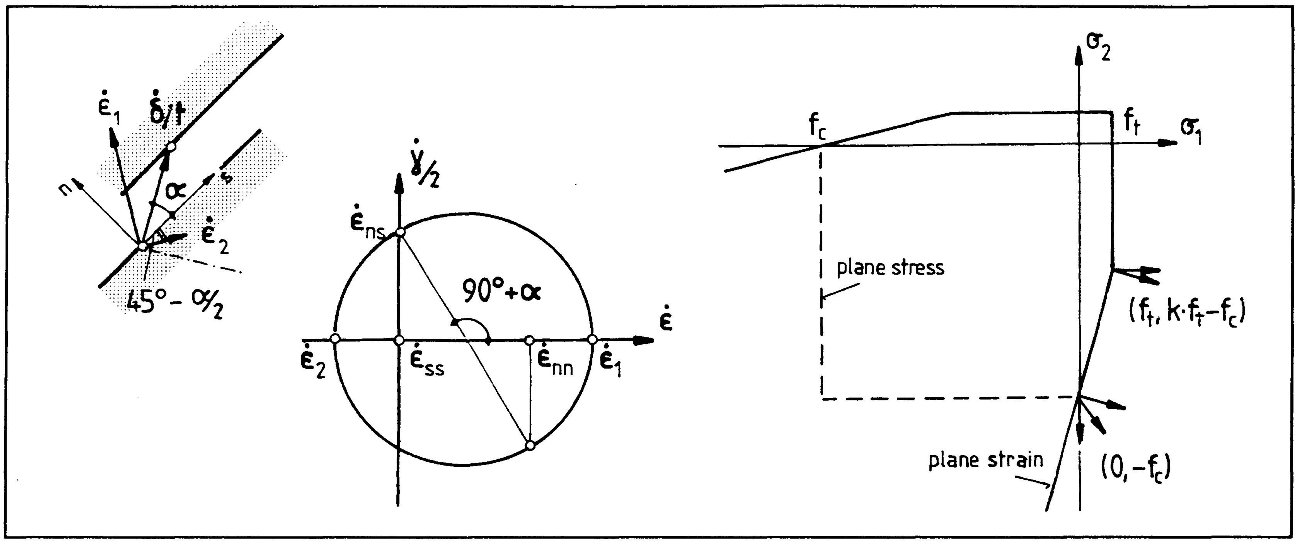

Figure 1: Strain rates and failure surface in band discontinuity

with the pure failure modi of shear and opening, using $k = (1+\sin\phi)/(1-\sin\phi)$:

$$\text{shear}\,(\alpha=\phi): \quad \dot{W}_l = \tfrac{1}{2}\dot{\delta}\, f_c\,(1-\sin\phi) \equiv \dot{\delta}\, c\cos\phi \quad , \qquad \text{opening}\,(\alpha=\tfrac{\pi}{2}): \quad \dot{W}_l = \dot{\delta} f_t \tag{4}$$

From letting $t \to 0$ in eq. (1) – for which $\dot{\varepsilon}_{1,2}$ grows to ± infinity – it is concluded that the joint material needs to be formulated for plane-strain conditions [7]. Together with the associated flow rule arising from von-Mises' postulate of maximum dissipation, this implies that $\alpha < \phi$ is not permitted by this kind of model; it would become feasible only in *plane stress* where another corner stress state allows for simultaneous shear and compression failure [8].

2.2 Interface Element Formulations

The FEM knows a similar concept of degenerating a solid to a layer of finite thickness t, assuming a strain-formulated layer material model for a constant strain gradient across the thickness [9]:

$$\{d\sigma\} = [\mathbf{D}^e - \mathbf{D}^p]\,\frac{1}{t}\,\{d\boldsymbol{\delta}\} \tag{5}$$

The stretching strain component ε_{ss} is again assumed to vanish, because of the assumption $t \ll L$, the length of the layer element [10]. In view of the fact that also $\frac{1}{t}\mathbf{D}^e$ grows to infinity with $t \to 0$, a very thin layer would infact behave rigid-plastic if $\mathbf{D}^e$ were not corrected for the layer element aspect ratio t/L [11]. With nodal displacements as primary unknowns, this is required by the finite-precision arithmetic of the equation solver. Note that only the plastic strain components correspond to the 'kinematic strains' in limit analysis and dissipate energy on the stress state.

For a vanishing layer thickness the interface can directly be formulated in relative displacements $d\boldsymbol{\delta} = d\boldsymbol{\delta}^e + d\boldsymbol{\delta}^p$. The elastic stiffness of the bonded state is given by local penalty parameters $\kappa_s = G/t$ and $\kappa_n = E/t$, and the stress-strain constitutive model is just converted into a relationship between tractions and relative displacements, the factor $1/t$ being virtually incorporated into $\mathbf{D}^e$ and $\mathbf{D}^p$. Because of the traction formulation plain-stress and plane-strain states can no longer be distinguished in the joint. Whether or not a thin-layer element approaches indeed the slip behaviour of a zero-thickness interface, depends on the form of stress evaluation: Unless the information of the interface orientation is passed on to the constitutive routine, an ordinary principal stress criterion will result in premature failure of the interface compared to Coulomb friction, because the shear planes in the layer material are predicted according to eq. 1 as being inclined relative to the interface orientation [12].

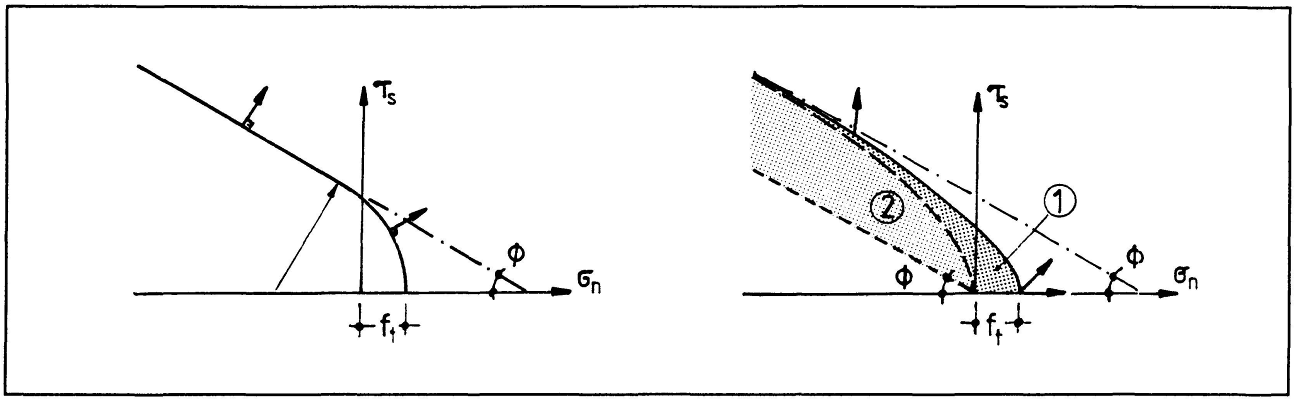

Figure 2: Modified Coulomb models in strains and relative displacements

A popular failure surface for combined slip and opening is the hyperboloid, which differs not too much from the cone with spherical cusp in limit analysis (Fig. 2). The continuous curvature simulates the added geometrical strength component resulting from the inclination of asperities with a mean roughness angle ψ, which are overridden under small compression and become progressively sheared off under high compression. With increasing $|\sigma_n|$ the surface approaches the asymptotic friction cone of a plane interface with a 'basic' friction angle ϕ_μ and zero dilatancy. Since the 'mobilised' angle of friction is of the form $\phi_{mob} = \phi_\mu + \psi(\sigma_n, \delta_s)$, the flow (or slip) rule can *never be associated* [13]. For a rough surface the truncated friction cone is but a linearization, where the geometric stiffness component is simplified to an apparent cohesion intercept.

Angles of δ^p larger than ψ must contain an opening component. There is no reason why the flow potential should display a smooth transition from shear to opening. More likely, the potential surface for shear dilatancy forms a corner singularity with the n-axis. This allows to distinguish irreversible opening due to override in shear from reversible gap displacements. If the interface is initially cemented, a retractable tension cap extends into the tension/shear domain, furnishing a tensile strength and a true cohesion. Both quantitites are destroyed together in any arbitrary combination of tension and shear (area '1' in Fig. 2 right) [14]. The roughness or apparent cohesion is treated separately: As continued override wears the asperities down, the failure surface will shrink in function of the accumulated sliding distance δ_s or, alternatively, of the plastic slip work W_s^p (softening of area '2').

3. LIMIT ANALYSIS VS. LIMIT EQUILIBRIUM METHODS

The theorems of limit analysis offer the great avantage that neither the initial conditions in the structure nor the exact load path to failure need be known, provided the material is sufficiently ductile and stable in Drucker's sense. Concrete departs from the assumption of unlimited deformability already in compression, such that a hypothetical plateau need be fitted at a reduced average stress over a particular strain range [3], this reduction being commonly termed the 'effectiveness factor' [15]. Since the strain history differs for each particular problem setting (bending, shear, etc.), this factor varies and accounts for different influences in a global manner. To confuse the matter further, also the effect of construction joints is sometimes subsumed in there [8] even though it could be accounted for by reduced material parameters in an explicitely modelled weak plane.

The definition of kinematical discontinuities ignores any dependence of the ductility on the angle α, which would be valid only for the assumption of 'unlimited ductility' at *zero* tensile strength. Even then a strain-capacity problem is present in the crack width across which shear stresses can still be carried by aggregate interlock. Because the kinematical discontinuities are usually not identified with cracks – except for socalled collapse cracks in pure tension [5] – plasticity theory tacitly assumes that the compression struts ($\dot{\varepsilon}_2$ in Fig. 1) are not restrained by the crack pattern in their ability to

For a joint inclined under the angle β to the horizontal, the external work due to a uniaxial compressive stress σ_y is given (per width B and unit thickness of the specimen) by

$$\dot{W}_E = \dot{\delta}\, \sigma_y \sin(\beta - \alpha)\, B$$

and the internal plastic work in the line of discontinuity

$$\dot{W}_I = \dot{\delta}\, f_c \frac{(1 - \sin\alpha)\, B}{2\cos\beta}$$

Equating external and internal work and minimizing with respect to β, one finds simply $\sigma_{y,min} = f_c$, just as for the associated case of $\alpha = \phi$ [8], only at a different critical orientation $\beta = \pi/4 + \alpha/2$.

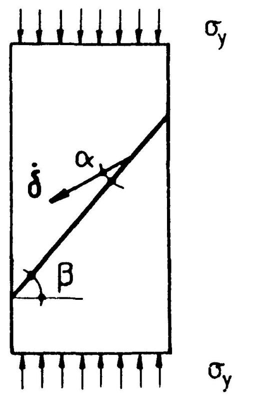

Figure 3: Analysis of a prism test with construction joint

adapt themselves to principal stress rotations during loading, cf. [16]. Interestingly, FEM interface models may be liable to the same pitfall if the limitation of dilatancy by the height of asperities is not incorporated in the constitutive model. This information need be supplied explicitely to force the stress point during continued plastic shear flow to the apex of the failure surface (Fig. 2), where alone subsequent gap can take place [17].

Neglecting the cementitious cohesion, the nonassociated slip rule and shear softening still violate the assumptions of limit analysis [13]. Only in statical determinate situations, where the amount of dilatancy does not play a role, certain limit load formulae remain valid (Fig. 3); but for highly confined situations as typical in geotechnics the limit load decreases substantially with $\psi < \phi$ [18]. Limit load theorems in their classical sense – i.e. the maximum lower bound and the minimum upper bound converging to a unique value – are no longer valid but need be recast in a weak form furnishing 'safer' lower and higher upper bound values [19,20]. On the basis of associated flow, solutions with finite element interfaces can still be obtained by optimization methods [21].

With the exception of blast loading and other energy-based design cases, upper-bound solutions are of little value in civil engineering practise anyhow. Through the prudent choice of material parameters one strives rather at obtaining conservative limit loads in spite of an underlying mechanism concept. Very good results have been obtained with interface elements for difficult limit load problems [22]. Pre-inserting planes of discontinuity without tracking their formation (i.e. strain localisation) means that part of the stress history is neglected in favour of a *limit equilibrium* analysis for a mechanism which is not necessarily the one that would actually develop. As with plastic limit theory one must therefore check also the yield criterion in the solid domains between the planes of weakness and the strain limits and transient strength components, which – depending on the unknown stress history – may undermine the full mobilisation of the mechanism's resistance [23]. It seems thus very helpful if interface element constitutive models dispose of an initial cementitious strength with the capability for mixed-mode decohesion, so that they can be inserted in a mesh as 'sleeping discontinuities' in the sense of Hillerborg's fictitious cracks.

4. EXAMPLES OF VOUSSOIR ARCHES AND BEAMS

To conclude this contribution, two simple applications to arches are given, which are either supposed to be constructed from independent blocks or to be radially cracked. Such voussoir arches are a classical application of rigid-plastic limit analysis even though the modification of the plastic-hinge concept to accommodate no-tension gaps between blocks seems rather bold [24]. The simplicity lies in the fact that slipping of blocks is excluded from the catalogue of allowable mechanisms – postulating a sufficiently steep orientation of the force resultant with respect to the interfaces –, that the stress range is supposed to be low enough to avoid crushing of edges and that the joints have no tensile strength. Therefore no energy will be dissipated in the mechanism, and the energy balance must be

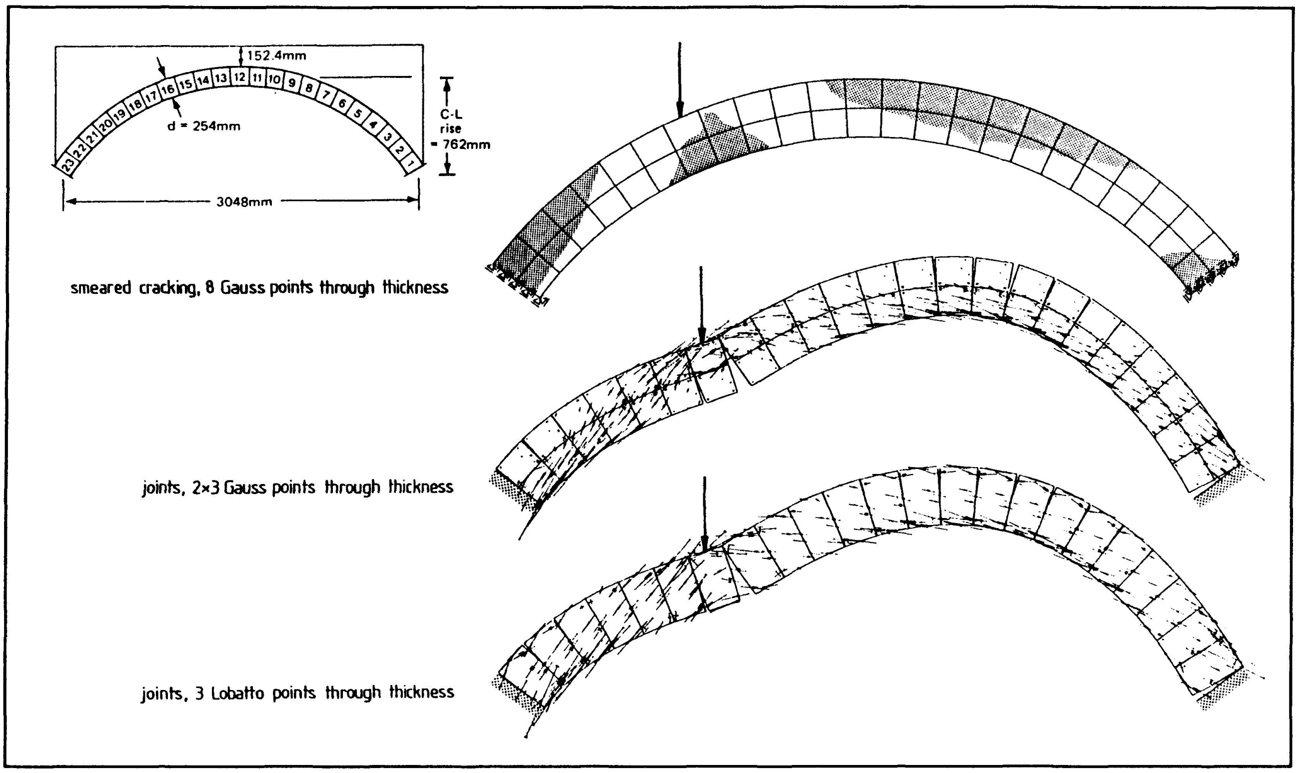

Figure 4: Discrete joint model of a circular arch

maintained by passive external work of parts of the structure moving against the direction of loading. According to limit analysis theory, any feasible thrust line which lies fully inside the arch would thus give a lower-bound limit load, whereas any collapse mechanisms would give an upper-bound limit load [25]. The added advantage of the FEM discretization of the joints is that the no-slip assumption is checked automatically, i.e. La Hire's vision of 1695 of arches as an assembly of wedges (viz. [26]) is 300 years later turned into a practical method.

The example in Fig. 4 shows a circular arch, which was tested in 1951 by Pippard & Chitty and previously analysed by mechanism and continuum FE methods [27]. Modelling every segment as a finite element separated by interfaces, it can be seen that the computed extent of joint opening – shortly before the fourth mechanism is formed – corresponds quite well to the prediction by the smared-crack model. This may surprise as it is often anticipated that the discrete model will automatically lead to a concentrated mechanism, but it finds an explanation in the tangential orientation of the thrust line and indicates that not all the joints would have to be included to catch the failure mechanism [25]. Observe also that the distance between the two outermost integration points determines the eccentricity of the pivot and hence the effective depth of the section in which the thrust line must reside. Other integration schemes – among them a socalled floating point scheme, which contracts the integration points into the remaining compression zone – have been tested [28], but the results for only one joint element across the thickness are seen to be rather satisfactory if a 3-pt. Lobatto rule (nodal integration) is used. Note also that the solids between partly open joints still exhibit tensile stresses, due to the coupling of equivalent nodal forces through the shape functions; this emphasizes once more the advantage of stress evaluation in discrete mechanisms.

The second case concerns the rather common problem of estimating the load carrying capacity of an unreinforced concrete beam by considering a hidden arch, even though in this particular case the 'beam' is a horizontal slice through a large concrete gravity dam under reservoir pressure [29]. According to the *lower*-bound theorem, any permissible stress field – i.e. not violating the yield limits of the material anywhere – would give a safe estimate of the load carrying capacity, irrespective of the

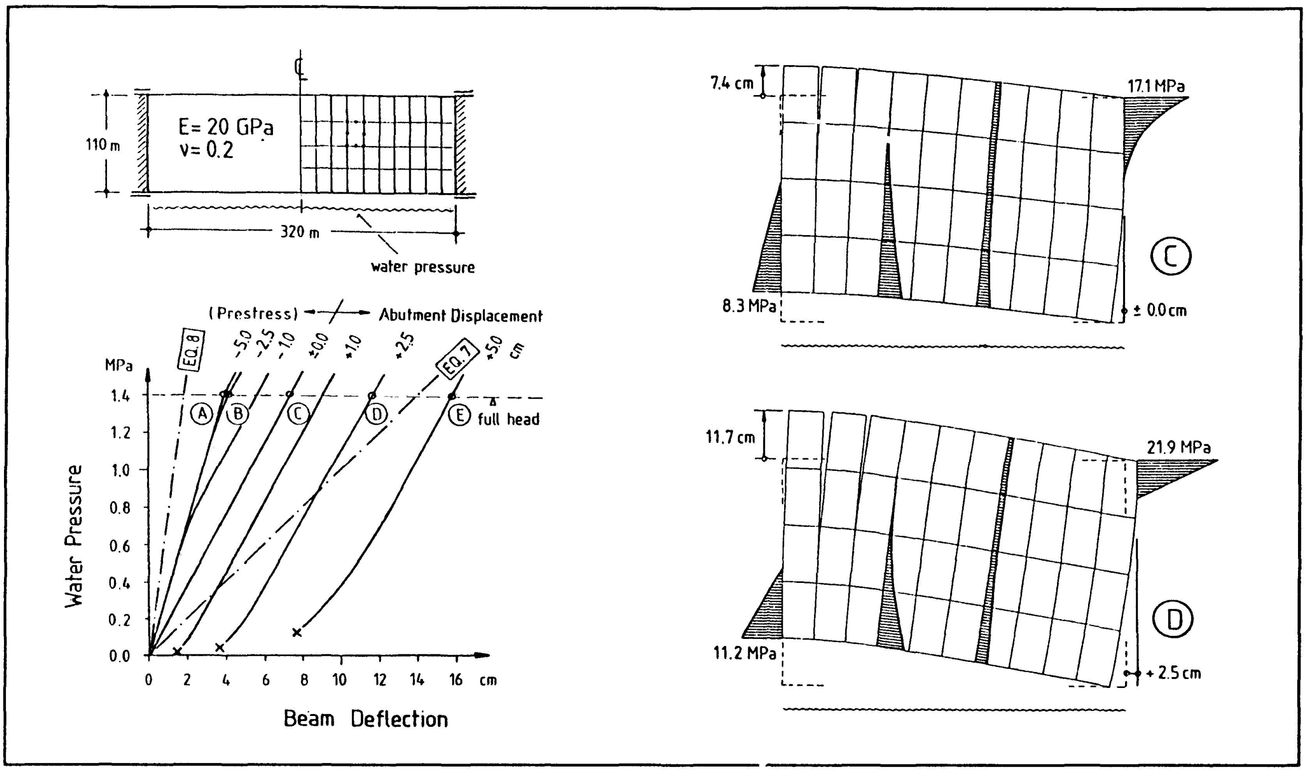

Figure 5: Hidden horizontal arch in a 'tension-free' beam

strain compatibility [3]. The maximum load will thus result from the arch with the largest camber, so that paradoxically the (elastic) beam seems the stiffer the deeper it is cracked in flexure. In terms of stress resultants, the arch is only stable if the bending moment does not exceed the normal force times half the depth of the cross-section, as otherwise the thrust line would pass outside the structure [25]. The problem with this particular loading is that the bending moment is already active *before* a sufficient normal force can build up. It would not arise if the segments were wedges [30], but as the joints are oriented parallel to the direction of loading, the thrust requires prying action in bending which is unstable under small pressure (points 'x' in the graph). If one does not count on residual prestress from joint grouting or cementitious cohesion – but takes rather some foregoing joint opening due to shrinkage or cold temperature into account –, the only way how such a voussoir beam could work without shear keys would be by means of considerable dilatancy developing during the relative slip between blocks [31]. Even then an absolute limit would be given by the height of asperities as previously mentioned.

5. CLOSING REMARKS

Interface elements to model weak planes or existing discontinuities are a very useful tool for limit equilibrium calculations. The conceptual similarity to upper-bound limit analysis lies in the need to perturbate prospective mechanisms for finding the most critical one, but fortunately there are many situations (like well-shaped arches) which are rather insensitive to the assumed location of discontinuities. However, phenomena of limited strain capacity and transient strength components need be regarded if they are not to defy the analysis results. The influence of more realistic interface constitutive models in the FEM may also be elucidating to limit analysis practise.

REFERENCES

1. Rots J.G. and Blaauwendraad J., Crack models of concrete: discrete or smeared? fixed, multi-directional or rotating? *Heron* **34** (1), 1989.

2. PIAN T.H.H. and TONG P., Basis of the finite element method for solid continua. *Int. J. Num. Meth. Eng.* **1**, 1969, pp. 3-28.
3. DRUCKER D.C., On structural concrete and the theorems of limit analysis. *IABSE Mémoires* **21**, 1961, pp. 49-59.
4. SALENÇON J., *Calcul à la rupture et analyse limite.* Presses ENPC, Paris 1983.
5. MÜLLER P., *Plastische Berechnung von Stahlbetonscheiben und -balken.* ETH Zurich IBK Rep. 83, Basle 1978.
6. CHEN W.-F. and DRUCKER D.C., Bearing capacity of concrete blocks or rock. *J. Eng. Mech. Div. ASCE* **95** (4), 1969, pp. 955-978.
7. MARTI P., Plastic analysis of reinforced concrete shear walls. IABSE Coll. *Plasticity of Reinforced Concrete*, Copenhagen 1979. Introd. Rep. AK-28, pp. 51-69.
8. JENSEN B.C., Lines of discontinuity for displacements in the theory of plasticity of plain and reinforced concrete. *Mag. Concrete Res.* **27** (92), 1975, pp. 143-150.
9. DESAI C.S., ZAMAN M.M. et al., Thin-layer element for interfaces and joints. *Int. J. Num. Anal. Meth. Geomech.* **8** (1), 1984, pp. 19-43.
10. HOHBERG J.-M., discussion of 'Modeling of cyclic normal and shear behavior of interfaces' (by C.S. Desai and B.K. Nagaraj). *J. Eng. Mech. ASCE* **116** (8), 1990, pp. 1870-1880.
11. HOHBERG J.-M. and SCHWEIGER H.F., On the penalty behaviour of thin-layer elements. *NUMOG IV*, Swansea 1992 (in print).
12. HOHBERG J.-M., A review of joint constitutive models. *2nd NUMEG*, Prague 1992 (in print).
13. DRUCKER D.C., Coulomb friction, plasticity and limit loads. *J. Appl. Mech. ASME* **21** (1), 1954, pp. 71-74.
14. HOHBERG J.-M., Multimechanism plasticity with coupled damage in tension and shear. *COMPLAS III*, Barcelona 1992. Proc. 2, pp. 1503-1514.
15. EXNER H., On the effectiveness factor in plastic analysis of concrete. IABSE Coll. *Plasticity in Reinforced Concrete*, Copenhagen 1979. Final Rep. AK-29, pp. 35-42 + disc. pp. 67, 132/133, 349, 353.
16. KUPFER H. and BULICEK H., Comparison of fixed and rotating crack models in shear design of slender concrete beams. Int. Workshop *Progress in Structural Engineering*, Brescia 1991 (typescript 10 p.)
17. HOHBERG J.-M, *A joint element for the nonlinear dynamic analysis of arch dams.* ETH Zurich IBK Rep. 186, Basle 1992.
18. DE BORST R. and VERMEER P.A., Possibilities and limitations of finite elements for limit analysis. *Géotechnique* **34** (2), 1984, pp. 199-210 + disc. **35** (1), 1985, pp. 90-94.
19. DE JOSSELIN DE JONG G., Lower bound collapse theorem and lack of normality of strain rate to yield surface for soils. IUTAM Symp. *Role of Plasticity in Soil Mechanics*, Grenoble 1964. Proc. (1966), pp. 69-75 + disc. pp. 75-78.
20. COLLINS I.F., The upper bound theorem for rigid/plastic solids generalised to include Coulomb friction. *J. Mech. Phys. Solids* **17**, 1969, pp. 323-338.
21. SLOAN S.W., Upper bound limit analysis using finite elements and linear programming. *Int. J. Num. Anal. Meth. Geomech.* **13** (3), 1989, pp. 263-282.
22. VAN LANGEN H. and VERMEER P.A., Interface elements for singular plasticity points. *Int. J. Num. Anal. Meth. Geomech.* **15** (5), 1991, pp. 301-315.
23. MANFREDINI G., MARTINETTI S. and RIBACCHI R., Inadequacy of limiting equilibrium methods for rock slope design. *16th US Symp. Rock Mechanics*, Minneapolis/MN 1975. Proc. (1977), pp. 35-43.
24. KOOHARIAN A., Limit analysis of voussoir (segmental) and concrete arches. *ACI Journal* **49**, 1952, pp. 317-328 + disc. pp. 328/1-4.
25. HEYMAN J., The stone skeleton. *Int. J. Solids Struct.* **2** (2), 1966, pp. 249-279.
26. KURRER K.-E., Zur Entstehung der Stützlinientheorie. *Bautechnik* **68** (4), 1991, pp. 109-117.
27. CRISFIELD M.A., *Finite element and mechanism methods for the analysis of masonry and brickwork arches.* TRRL Res. Rep. 19, Crowthorne 1985.
28. HOHBERG J.-M. and BACHMANN H., A macro joint element for nonlinear arch dam analysis. *6th ICONMIG*, Innsbruck 1988. Proc. 2, pp. 829-834.
29. HOHBERG J.-M., discussion of 'Spatial action of straight gravity dams in narrow valleys' (by M.A. Herzog). *J. Struct. Eng. ASCE* **117** (2), 1991, pp. 637-641.
30. LIVISLEY R.K., Limit analysis of structures formed from rigid blocks. *Int. J. Num. Meth. Eng.* **12** (12), 1978, pp. 1853-1871.
31. PENDER M.J., Prefailure joint dilatancy and the behaviour of a beam with vertical joints. *Rock Mech. Rock Eng.* **18** (4), 1985, pp. 253-266.