

**Zeitschrift:** IABSE reports = Rapports AIPC = IVBH Berichte  
**Band:** 62 (1991)

**Artikel:** Tension stiffening in reinforced concrete elements  
**Autor:** Creazza, Giuseppe / Marco, Roberto di  
**DOI:** <https://doi.org/10.5169/seals-47633>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 01.04.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

## Tension Stiffening in Reinforced Concrete Elements

Contribution du béton tendu dans des éléments en béton armé fissurés

«Tension-Stiffening» in Stahlbeton-Strukturen

### Giuseppe CREAZZA

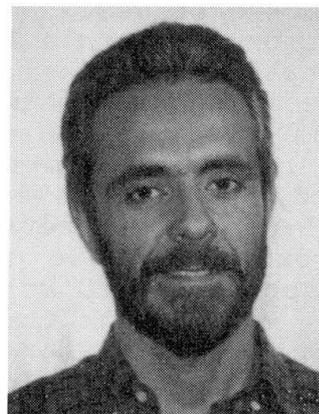
Professor  
Arch. Univ. of Venice  
Venice, Italy



Giuseppe Creazza, born 1927, is Professor of Structural Mechanics. Currently he is carrying out research on «one and two way» reinforced concrete structures in the cracking phase.

### Roberto DI MARCO

Assist. Professor  
Arch. Univ. of Venice  
Venice, Italy



Roberto Di Marco, born 1948, is Assist. Professor of Civil Engineering. At present, he is carrying out research on the non-linear analysis of reinforced concrete structures.

### SUMMARY

The methods to evaluate tension stiffening effects in reinforced concrete structures, are generally based on empirical formulations derived from experimental tests. Recently, an analytical study was presented for modelling the post-cracking behaviour of reinforced concrete bars and membrane elements in a general manner, without a priori hypotheses. The aim of the present paper is to develop an analytical model for reinforced concrete beams subject to bending and axial load, whose formulation is based solely on the classical hypothesis of cross sections remaining plane using suitable constitutive laws for materials and bond characteristics.

### RÉSUMÉ

L'évaluation de la contribution du béton tendu dans des éléments en béton armé fissurés se base généralement sur des formules empiriques dérivées d'essais expérimentaux. Une étude analytique a été récemment présentée afin de modéliser le comportement après fissuration de tirants, ainsi que d'éléments-membrane dans la phase suivant la fissuration, mais sans introduire des hypothèses à priori. Le modèle analytique est donc développé pour des poutres en béton armé soumises à des charges axiales et flexionnelles; leur formulation repose uniquement sur l'hypothèse classique de la conservation des sections planes, tout en adoptant certaines lois constitutives ainsi que des mécanismes d'adhérence appropriés.

### ZUSAMMENFASSUNG

Die Beurteilung der «tension stiffening»-Auswirkungen in Stahlbeton-Strukturen stützt sich im allgemeinen auf empirische Formulierungen, die aus Versuchen abgeleitet wurden. Vor kurzem wurde eine analytische Untersuchung zur allgemeinen Bestimmung des Verhaltens von Zugstangen und Membranstrukturen nach dem Auftreten von Rissen bekannt, die sich nicht von vornherein auf irgendwelche Annahmen stützt. Das Ziel dieser Arbeit ist, ein analytisches Modell für Stahlbeton-Träger zu entwickeln, die Biege- und Normalkraftbeanspruchung unterworfen sind und dessen Formulierung auf der klassischen Hypothese des Ebenbleibens der Querschnitte und der Verwendung geeigneter Material- und Verbundgesetze basiert.



## 1. INTRODUCTION

Generally the studies to evaluate the tension stiffening effects in reinforced concrete cracked elements are mainly based on semiempirical formulations established to fit test data. [1+9]

These studies, even if show a good agreement between the theoretical model and experimental results, don't fulfil the requirement of a rational approach to the phenomena [10].

Recently Gupta and al. [11+12], referring to reinforced concrete bars and membrane elements, presented a theoretical model for the tension stiffening, without resorting to any empirical hypothesis.

The aim of the present paper is similar to the Gupta's one.

A mathematical model for r.c. beams subject to bending with and without axial load is developed, on the basis of the hypotheses generally agreed in studying the behaviour of r.c. cross sections in uncracked and cracked stages.

The basic assumptions concern a linear relationship between concrete compressive strains and steel tensile strains and a linear behaviour of concrete in tension between cracks; no further restrictive hypothesis, of empiric type, is introduced.

In the paper a linear stress-strain behaviour for concrete and steel was adopted, as well as a linear bond stress-slip relationship at the steel-concrete interface.

These crude approximations are used only to reduce the complexity that would be associated with more general relationships, but still keeping a general validity to the formulation.

The proposed model gives reasonably good results, allowing to improve the mathematical formulation by adopting non linear constitutive laws for materials and bond behaviour.

The paper is mainly intended to run an unified approach to describe the post-cracking behaviour of elements in bending in more general way, according to a comprehensive theoretical treatment of the complex phenomena.

## 2. THE ANALYTICAL MODEL

Fig. 1 shows a part of a reinforced concrete beam between two cracks. The total length of this part is the same as crack spacing  $2a$  and the origin of the  $x$  axis is taken midway between the cracks.

The beam has a rectangular, singly reinforced cross section and is subjected to combined axial force and bending moment.

Let us consider an infinitesimal element of length  $dx$ , at distance  $x$  from the origin. Fig. 2 shows the stress distributions on the opposite sides of this element. The force  $F_{bx}$  is the resultant of the steel-concrete interface stresses in the element and, under the hypothesis of linear bond relationship, is given as

$$F_{bx} = E_p \cdot s \cdot p \cdot dx$$

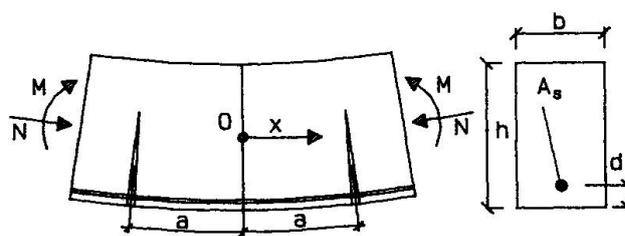


Fig. 1 A part of r.c. beam between cracks and cross section

(1)

where  $E_b$  = the slope of the linear bond stress-slip curve, generally called the slip modulus and having unit of  $FL^{-3}$ ,  $s = u_s - u_c$  = the slip between steel and concrete, defined by axial displacements  $u$ ,  $p$  = the sum of the perimeters of the steel bars.

The governing parameters of the problem are the stress state and the slip. If the numerical values of these parameters on one side section of the element are fixed, the unknown quantities are the corresponding values on the other side or their variations:

e.g. of the concrete stress in compression  $d\sigma_{cc}$  and in tension  $d\sigma_{ct}$ , of the neutral axis depth  $dy$  and of the slip  $ds$ . So a set of four equations, relating these four variations of governing parameters, are necessary to solve the problem.

The equilibrium of axial forces leads to the first equation:

$$(\sigma_{ct} - \sigma_{cc}) \frac{dy}{dx} - (h-y) \frac{d\sigma_{ct}}{dx} - y \frac{d\sigma_{cc}}{dx} - \frac{[2E_b p]}{[b]} s = 0 \quad (2)$$

The equilibrium of moments of the same forces, referred to the steel in tension, leads to the second equation:

$$\left[ -\sigma_{cc} \left( -h + \frac{2}{3}y + d \right) - \sigma_{ct} \left[ \frac{2(h-y)}{3} - d \right] \right] \frac{dy}{dx} + (h-y) \left[ \frac{h-y-d}{3} \right] \frac{d\sigma_{ct}}{dx} + y \left[ \frac{h-y-d}{3} \right] \frac{d\sigma_{cc}}{dx} = 0 \quad (3)$$

The third equation is related to the hypothesis of a linear relationship between concrete compressive strains and steel tensile ones.

From the geometric relation  $\epsilon_s = -\epsilon_{cc} \frac{(h-y-d)}{y}$  it follows

$$\sigma_s = -\sigma_{cc} \frac{E_s}{E_c} \frac{(h-y-d)}{y} = -n\sigma_{cc} \frac{(h-y-d)}{y} \text{ and by differentiating}$$

$$\frac{d\sigma_s}{dx} = -n \frac{(h-y-d)}{y} \frac{d\sigma_{cc}}{dx} + n\sigma_{cc} \frac{(h-d)}{y^2} \frac{dy}{dx} \quad (4)$$

The equilibrium for the steel axial forces gives:

$$\frac{d\sigma_s}{dx} = \frac{(E_b p)}{A_s} s \quad (5)$$

By introducing eq. (5) in eqs. (2), (4), the final form of third equation can be worked out:

$$\left[ \sigma_{ct} - \left[ 1 + \frac{2nA_s(h-d)}{b} \frac{1}{y^2} \right] \sigma_{cc} \right] \frac{dy}{dx} - (h-y) \frac{d\sigma_{ct}}{dx} + \left[ \frac{2nA_s(h-y-d)}{b} \frac{1}{y} - y \right] \frac{d\sigma_{cc}}{dx} = 0 \quad (6)$$

The last equation refers to the slip between concrete and steel, both in tension. The basic differential equation is:

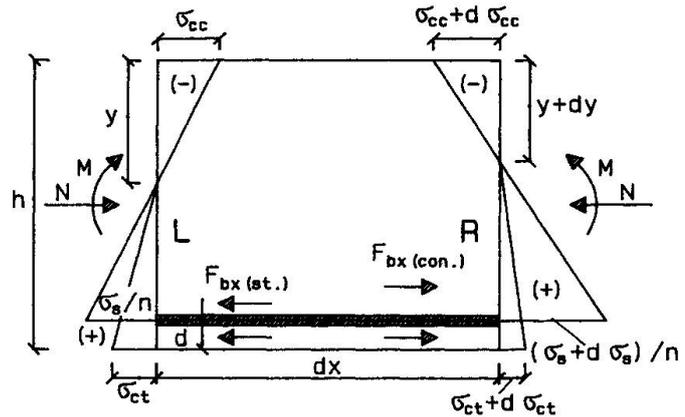


Fig. 2 Free body diagrams for cracked element



$$\frac{d^2 u_s}{dx^2} - \frac{d^2 \bar{u}_{ct}}{dx^2} = \frac{d^2 s}{dx^2} = \frac{1}{E_s} \frac{d\sigma_s}{dx} - \frac{1}{E_c} \frac{d\bar{\sigma}_{ct}}{dx} \quad (7)$$

where  $d\bar{\sigma}_{ct}$  and  $d^2 \bar{u}_{ct}$  are referred to concrete at steel level. From the relation between tensile stresses at different levels

$$\bar{\sigma}_{ct} = \sigma_{ct} \frac{(h-y-d)}{(h-y)}, \text{ by differentiating it follows:}$$

$$\frac{d\bar{\sigma}_{ct}}{dx} = \left[ \frac{h-y-d}{h-y} \right] \frac{d\sigma_{ct}}{dx} - \sigma_{ct} \frac{d}{(h-y)^2} \frac{dy}{dx} \quad (8)$$

By introducing the relationships (5), (8) in the slip equation (7) the last equation required to define the problem becomes:

$$-\left[ \frac{\sigma_{ct}}{E_c} \frac{d}{(h-y)^2} \right] \frac{dy}{dx} + \left[ \frac{(h-y-d)}{E_c (h-y)} \right] \frac{d\sigma_{ct}}{dx} - \left[ \frac{E_b p}{E_s A_s} \right] s + \frac{d^2 s}{dx^2} = 0 \quad (9)$$

### 3. BOUNDARY CONDITIONS AND NUMERICAL SOLUTIONS

At the onset of a new crack between the existing ones of Fig. 1 the section at  $x=0$  (midway of the element) is still uncracked and the end section at  $x=a$  is fully cracked, as in the classical theory for r.c. elements in the II stage.

The boundary conditions at  $x=0$  are quite known because  $\sigma_{ct} = f_{ct}$ , being  $f_{ct}$  = concrete strength in tension,  $s=0$  for symmetry condition,  $\sigma_{cc}$ ,  $\sigma_s$  and  $y$  have to fulfil equilibrium conditions for fixed values of axial load  $N$  and bending moment  $M$ .

In the section at  $x=a$   $\sigma_{ct}=0$ , because of cracking, the slip value  $s(a)$  is unknown and is determined together with the distance "a" from the origin.

For solving equations (2), (3), (6), (9) a finite differences technique is used. By substituting the derivatives with the corresponding increment ratios, the set of four differential equations is transformed into multiple sets of four linear algebraic equations.

Each set allows to determine the values at section  $x=x^{i+1}=x^i+\Delta x$  if the corresponding ones are known at section  $x=x^i$ . So if we start from section  $x=0$  (section L in Fig. 2), where all the boundary conditions are known, the stress state and slip in the opposite section (R) can be easily calculated and from these, in turn, the values in other sections, by proceeding in the same manner.

The distance "a" is unknown, so a value has to be guessed and an iterative numerical procedure is used which stops when in a section the fully cracked condition  $\sigma_{ct}=0$  is fulfilled with a fixed degree of accuracy.

The sum of  $\Delta x$  gives half-length between the first cracks and the slip value in this section is the final one:  $s=s(a)$ .

From the stress state it is possible to calculate  $u_{cc}$  and  $u_s$  displacements, the rotations and from these the mean cross section curvature in the element between two cracks.

If one starts the procedure from a value of  $M=M_{cr}$ , being  $M_{cr}$  = the cracking moment, and continues until steel yields, a complete bending moment versus mean curvature diagram can be evaluate and, as a consequence, the bending deformability changes for increasing ap-



plied moments.

#### 4. NUMERICAL RESULTS

To test the model a beam is examined having the following characteristics :  $h=500$  mm,  $b=300$  mm,  $d=30$  mm,  $\sigma_{ct}=1.6$  MPa,  $E_b=20,40,80$  MN/mm<sup>3</sup>,  $E_c=2000$  MPa,  $E_s=20000$  MPa. The beam is subjected to bending moment only, without axial load and the steel reinforcement consists of n. 3 bars, placed in tension zone; three different diameters were considered  $\phi=10,14,20$  mm, leading to the following:

Case A	3 $\phi$ 10	$A_s=235.6$ mm <sup>2</sup>	$p=94.25$ mm	$A_s/A_c = 0.157\%$
B	3 $\phi$ 14	$A_s=461.8$ mm <sup>2</sup>	$p=131.95$ mm	$A_s/A_c = 0.308\%$
C	3 $\phi$ 20	$A_s=942.5$ mm <sup>2</sup>	$p=188.50$ mm	$A_s/A_c = 0.628\%$

The "a" values when  $E_b=40$  MN/mm<sup>3</sup> and cracking moments are applied, result

A  $M_{cr}=21$  MN·m  $a=269$  mm  
 B  $M_{cr}=22$  MN·m  $a=235$  mm  
 C  $M_{cr}=24$  MN·m  $a=193$  mm  
 and well agree, within the crude approximations adopted, with existing formulae.

In Fig. 3 the moment-mean curvature relationships for the examined cases in uncracked and cracked stage are plotted. When tension stiffening effects are considered, according to the hypotheses of linear constitutive laws for materials and bond, the curves shift parallel from the fully cracked curves (II stage) at a distance depending on the geometrical percentage of steel reinforcement.

The model shows that the bond parameter doesn't affect the slope of the curve but only the crack spacing, just as in the most common formulations. If  $E_b$  is halved or doubled the following values are obtained:

$E_b=20$ MN/mm <sup>3</sup>	case A $a=380$ mm	$E_b=80$ MN/mm <sup>3</sup>	case A $a=190$ mm
	B $a=332$ mm		B $a=166$ mm
	C $a=272$ mm		C $a=136$ mm

while the moment-curvature diagrams remain the same, within the numerical approximations.

#### 5. CONCLUDING REMARKS

The proposed analytical model, which is based only on the classical hypothesis of cross section remaining plane, seems to be a good

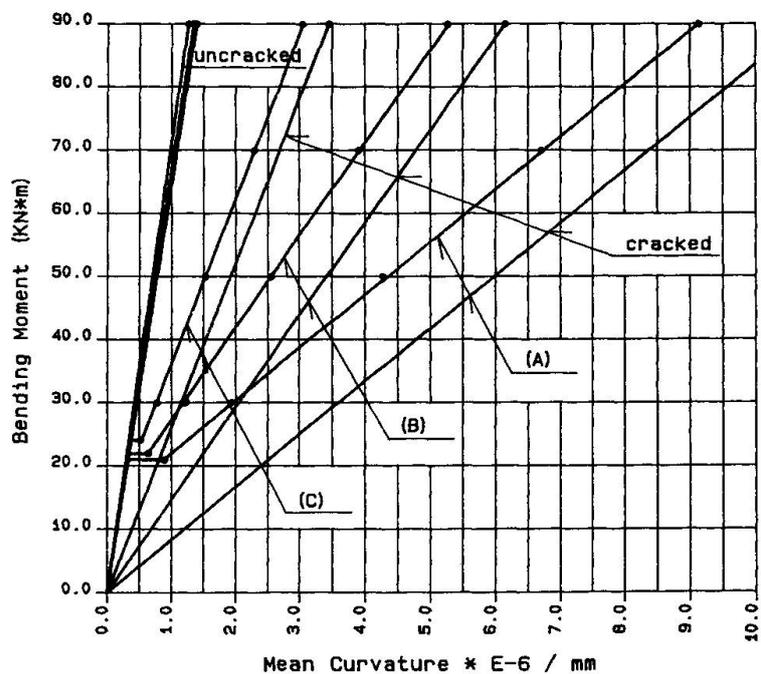


Fig. 3 moment-mean curvature relationships



starting-point for a rational approach to tension stiffening phenomena. Owing to its quite general formulation the model may be improved by adopting more realistic non linear constitutive laws for materials and bond behaviour. From this point of view it's well suited for studying the tension stiffening effects even after steel yields.

#### REFERENCES

1. GERGELY P., LUTZ L.A., Maximum Crackwidth in Reinforced Concrete Flexural Members. Causes, Mechanism and Control of Cracking in Concrete. ACI Special Publication, SP-20, American Concrete Institute, Detroit, Michigan 1968
2. RAO, S.P., SUBRAHMANYAN B. V., Trisegmental Moment-Curvature Relations for Reinforced Concrete Members. Proceedings of the American Concrete Institute, Vol. 70, No. 5, May 1973
3. CLARK L.A., SPEIRS D.M., Tension Stiffening in Reinforced Concrete Beams and Slabs under Short-term Load. Cement and Concrete Association, Rep. 42.521, 1978
4. GILBERT R.I., WARNER R.F., Tension Stiffening in Reinforced Concrete Slabs. J. of the Structural Division, ASCE, Vol.104, December 1978.
5. SCANLON A., MURRAY D.W., Discussion to paper of Gilbert, Warner. J. of the Structural Division, ASCE, Vol.104 January 1980.
6. FAVRE R., BEEBY A.W., FALKNER H., KOPRNA M., SCHIESSL M., Manuel du CEB Fissuration et Dèformations. École Polytechnique Fédérale de Lausanne, Suisse, 1983
7. DESAY P., GANESAN N., An Investigation on Spacing of Cracks and Maximum Crackwidth in Reinforced Concrete Flexural Members. RILEM, Materials and Structures, Paris Mars-Avril 1985
8. ESPION B., PROVOST M., HALLEUX P., Rigidité d'une Zone Tendue de Béton Armé. RILEM, Materials and Structures, Paris Mai-Juin 1985
9. CEB-FIP Model Code 1990. Bulletin d'information CEB No. 195-196, Mars 1990
10. HOLMBERG Å., Progress report of C.E.B. Task Group V/6 for a revision of CEB-FIP Model Code , April 1986
11. GUPTA A.K., MAESTRINI R.M., Post-cracking Behavior of Membrane Reinforced Concrete Elements Including Tension-Stiffning. J. of the Structural Division, ASCE, Vol.115, April 1989.
12. GUPTA A.K., MAESTRINI R.M., Tension-Stiffness Model for Reinforced Concrete Bars. J. of the Structural Division, ASCE, Vol.116, March 1990.