Tension Stiffening in Reinforced Concrete Elements

Contribution du béton tendu dans des éléments en béton armé fissurés

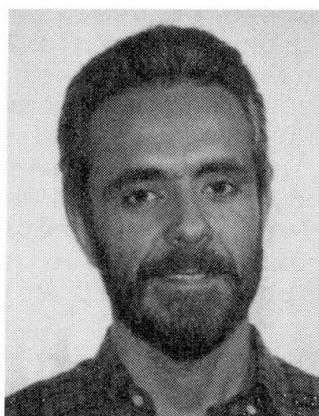## «Tension-Stiffening» in Stahlbeton-Strukturen

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SUMMARY

The methods to evaluate tension stiffening effects in reinforced concrete structures, are generally based on empirical formulations derived from experimental tests. Recently, an analytical study was presented for modelling the post-cracking behaviour of reinforced concrete bars and membrane elements in a general manner, without a priori hypotheses. The aim of the present paper is to develop an analytical model for reinforced concrete beams subject to bending and axial load, whose formulation is based solely on the classical hypothesis of cross sections remaining plane using suitable constitutive laws for materials and bond characteristics.

RÉSUMÉ

L'évaluation de la contribution du béton tendu dans des éléments en béton armé fissurés se base généralement sur des formules empiriques dérivées d'essais expérimentaux. Une étude analytique a été récemment présentée afin de modéliser le comportement après fissuration de tirants, ainsi que d'éléments-membrane dans la phase suivant la fissuration, mais sans introduire des hypothèses à priori. Le modèle analytique est donc développé pour des poutres en béton armé soumises à des charges axiales et flexionnelles; leur formulation repose uniquement sur l'hypothése classique de la conservation des sections planes, tout en adoptant certaines lois constitutives ainsi que des mécanismes d'adhérence appropriés.

ZUSAMMENFASSUNG

Die Beurteilung der «tension stiffening»-Auswirkungen in Stahlbeton-Strukturen stützt sich im allgemeinen auf empirische Formulierungen, die aus Versuchen abgeleitet wurden. Vor kurzem wurde eine analytische Untersuchung zur allgemeinen Bestimmung des Verhaltens von Zugstangen und Membranstrukturen nach dem Auftreten von Rissen bekannt, die sich nicht von vornherein auf irgendwelche Annahmen stützt. Das Ziel dieser Arbeit ist, ein analytisches Modell für Stahlbeton-Träger zu entwickeln, die Biege- und Normalkraftbeanspruchung unterworfen sind und dessen Formulierung auf der klassischen Hypothese des Ebenbleibens der Querschnitte und der Verwendung geeigneter Material- und Verbundgesetze basiert.

1. INTRODUCTION

Generally the studies to evaluate the tension stiffening effects in reinforced concrete cracked elements are mainly based on semiempirical formulations established to fit test data.[1÷9]
These studies, even if show a good agreement between the theoretical model and experimental results, don't fulfil the requirement of a rational approach to the phenomena [10].
Recently Gupta and al. [11÷12], referring to reinforced concrete bars and membrane elements, presented a theoretical model for the tension stiffening, without resorting to any empirical hypotesis.
The aim of the present paper is similar to the Gupta's one.
A mathematical model for r.c. beams subject to bending with and without axial load is developed, on the basis of the hypotheses generally agreed in studying the behaviour of r.c. cross sections in uncracked and cracked stages.
The basic assumptions concern a linear relationship between concrete compressive strains and steel tensile strains and a linear behaviour of concrete in tension between cracks; no further restrictive hypothesis, of empiric type, is introduced.
In the paper a linear stress-strain behaviour for concrete and steel was adopted, as well as a linear bond stress-slip relationship at the steel-concrete interface.
These crude approximations are used only to reduce the complexity that would be associated with more general relationships, but still keeping a general validity to the formulation.
The proposed model gives reasonably good results, allowing to improve the mathematical formulation by adopting non linear constitutive laws for materials and bond behaviour.
The paper is mainly intended to run an unified approach to describe the post-cracking behaviour of elements in bending in more general way, according to a comprehensive theoretical treatment of the complex phenomena.

2. THE ANALYTICAL MODEL

Fig. 1 shows a part of a reinforced concrete beam between two cracks. The total length of this part is the same as crack spacing 2a and the origin of the x axis is taken midway between the cracks.
The beam has a rectangular, singly reinforced cross section and is subjected to combined axial force and bending moment.

Fig. 1 A part of r.c. beam between cracks and cross section

Let us consider an infinitesimal element of length dx, at distance x from the origin. Fig. 2 shows the stress distributions on the opposite sides of this element. The force F_{bx} is the resultant of the steel-concrete interface stresses in the element and, under the hypothesis of linear bond relationship, is given as

$$F_{bx} = E_b \cdot s \cdot p \cdot dx \qquad (1)$$
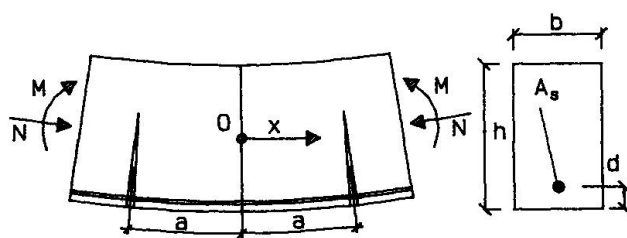
where E_b = the slope of the linear bond stress-slip curve, generally called the slip modulus and having unit of FL^{-3}, $s = u_s-u_c$ = the slip between steel and concrete, defined by axial displacements u, p = the sum of the perimeters of the steel bars.

The governing parameters of the problem are the stress state and the slip. If the numerical values of these parameters on one side section of the element are fixed, the unknown quantities are the corresponding values on the other side or their variations :

Fig. 2 Free body diagrams for cracked element

e.g. of the concrete stress in compression $d\sigma_{cc}$ and in tension $d\sigma_{ct}$, of the neutral axis depth dy and of the slip ds. So a set of four equations, relating these four variations of governing parameters, are necessary to solve the problem.

The equilibrium of axial forces leads to the first equation:

$$(\sigma_{ct}-\sigma_{cc})\frac{dy}{dx} - (h-y)\frac{d\sigma_{ct}}{dx} - y\frac{d\sigma_{cc}}{dx} - \left\lceil\frac{2E_b p}{b}\right\rceil s = 0 \qquad (2)$$

The equilibrium of moments of the same forces, referred to the steel in tension, leads to the second equation:

$$\left[-\sigma_{cc}(-h+\frac{2}{3}y+d)-\sigma_{ct}\left[2\frac{(h-y)}{3}-d\right]\right]\frac{dy}{dx}+(h-y)\left[\frac{h-y}{3}-d\right]\frac{d\sigma_{ct}}{dx}+y\left[h-\frac{y}{3}-d\right]\frac{d\sigma_{cc}}{dx}=0 \qquad (3)$$

The third equation is related to the hypothesis of a linear relationship between concrete compressive strains and steel tensile ones.

From the geometric relation $\epsilon_s = -\epsilon_{cc}\frac{(h-y-d)}{y}$ it follows

$$\sigma_s = -\sigma_{cc}\frac{E_s}{E_c}\frac{(h-y-d)}{y} = -n\sigma_{cc}\frac{(h-y-d)}{y} \text{ and by differentiating}$$

$$\frac{d\sigma_s}{dx} = -n\frac{(h-y-d)}{y}\frac{d\sigma_{cc}}{dx} + n\sigma_{cc}\frac{(h-d)}{y^2}\frac{dy}{dx} \qquad (4)$$

The equilibrium for the steel axial forces gives:

$$\frac{d\sigma_s}{dx} = \frac{(E_b p)}{A_s} s \qquad (5)$$

By introducing eq. (5) in eqs. (2),(4), the final form of third equation can be worked out:

$$\left[\sigma_{ct}-\left[1+\frac{2nA_s}{b}\frac{(h-d)}{y^2}\right]\sigma_{cc}\right]\frac{dy}{dx} - (h-y)\frac{d\sigma_{ct}}{dx} + \left[\frac{2nA_s}{b}\frac{(h-y-d)}{y}-y\right]\frac{d\sigma_{cc}}{dx} = 0 \qquad (6)$$

The last equation refers to the slip between concrete and steel, both in tension. The basic differential equation is:
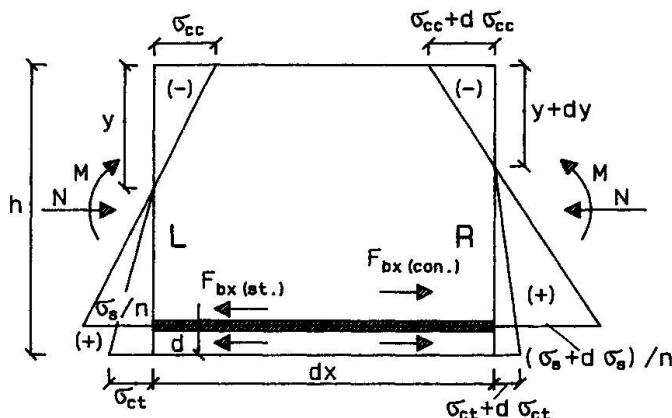
$$\frac{d^2 u_s}{dx^2} - \frac{d^2 \bar{u}_{ct}}{dx^2} = \frac{d^2 s}{dx^2} = \frac{1}{E_s}\frac{d\sigma_s}{dx} - \frac{1}{E_c}\frac{d\bar{\sigma}_{ct}}{dx} \qquad (7)$$

where $d\bar{\sigma}_{ct}$ and $d^2\bar{u}_{ct}$ are referred to concrete at steel level.
From the relation between tensile stresses at different levels

$$\bar{\sigma}_{ct} = \sigma_{ct}\frac{(h-y-d)}{(h-y)} \quad , \text{ by differentiating it follows:}$$

$$\frac{d\bar{\sigma}_{ct}}{dx} = \left[\frac{h-y-d}{h-y}\right]\frac{d\sigma_{ct}}{dx} - \sigma_{ct}\frac{d}{(h-y)^2}\frac{dy}{dx} \qquad (8)$$

By introducing the relationships (5),(8) in the slip equation (7)
the last equation required to define the problem becomes:

$$-\left[\sigma_{ct}\frac{d}{E_c(h-y)^2}\right]\frac{dy}{dx} + \left[\frac{(h-y-d)}{E_c(h-y)}\right]\frac{d\sigma_{ct}}{dx} - \left[\frac{E_b}{E_s}\frac{p}{As}\right]s + \frac{d^2 s}{dx^2} = 0 \qquad (9)$$

3. BOUNDARY CONDITIONS AND NUMERICAL SOLUTIONS

At the onset of a new crack between the existing ones of Fig. 1 the
section at x=0 (midway of the element) is still uncracked and the
end section at x=a is fully cracked, as in the classical theory for
r.c. elements in the II stage.
The boundary conditions at x=0 are quite known because $\sigma_{ct}=f_{ct}$,
being f_{ct}=concrete strength in tension, s=0 for symmetry condition,
σ_{cc}, σ_s and y have to fulfil equilibrium conditions for fixed values
of axial load N and bending moment M.
In the section at x=a $\sigma_{ct}=0$, because of cracking, the slip value
s(a) is unknown and is determined together with the distance "a"
from the origin.
For solving equations (2),(3),(6),(9) a finite differences techni-
que is used. By substituting the derivatives with the corresponding
increment ratios, the set of four differential equations is tran-
sformed into multiple sets of four linear algebraic equations.
Each set allows to determine the values at section $x=x^{i+1}=x^i+\Delta x$ if
the corrisponding ones are known at section $x=x^i$. So if we start
from section x=0 (section L in Fig. 2), where all the boundary con-
ditions are known, the stress state and slip in the opposite sec-
tion (R) can be easily calculated and from these, in turn, the va-
lues in other sections, by proceeding in the same manner.
The distance "a" is unknown, so a value has to be guessed and an i-
terative numerical procedure is used which stops when in a section
the fully cracked condition $\sigma_{ct}=0$ is fulfilled with a fixed degree
of accuracy.
The sum of Δx gives half-length between the first cracks and the
slip value in this section is the final one: s=s(a).
From the stress state it is possible to calculate u_{cc} and u_s displa-
cements, the rotations and from these the mean cross section curva-
ture in the element betweeen two cracks.
If one starts the procedure from a value of $M=M_{cr}$, being M_{cr}=the
cracking moment, and continues until steel yields, a complete ben-
ding moment versus mean curvature diagram can be evaluate and, as a
consequence, the bending deformability changes for increasing ap-

plied moments.

4. NUMERICAL RESULTS

To test the model a beam is examined having the following characte-
ristics : h=500 mm, b=300 mm, d=30 mm, σ_{ct}=1.6 MPa, E_b=20,40,80
MN/mm^3, E_c=2000 MPa, E_s=20000 MPa. The beam is subjected to ben-
ding moment only, without axial load and the steel reinforcement
consists of n. 3 bars, placed in tension zone; three different dia-
meters were considered ϕ=10,14,20 mm, leading to the following:

Case A 3 ϕ 10 A_s=235.6 mm²   p= 94.25 mm A_s/A_c = 0.157%
     B 3 ϕ 14 A_s=461.8 mm²   p=131.95 mm A_s/A_c = 0.308%
     C 3 ϕ 20 A_s=942.5 mm²   p=188.50 mm A_s/A_c = 0.628%

The "a" values when
E_b=40 MN/mm^3 and crac-
king moments are ap-
plied, result
A M_{cr}=21 MN·m a=269 mm
B M_{cr}=22 MN·m a=235 mm
C M_{cr}=24 MN·m a=193 mm
and well agree, within
the crude approxima-
tions adopted, with e-
xisting formulae.
In Fig. 3 the moment-
mean curvature rela-
tionships for the exa-
mined cases in uncrac-
ked and cracked stage
are plotted. When ten-
sion stiffening effects
are considered, accor-
ding to the hypoteses
of linear constitutive
laws for materials
and bond, the curves
shift parallel from the
fully cracked curves
(II stage) at a distan-

Fig. 3 moment-mean curvature relationships

ce depending on the geometrical percentage of steel reinforcement.
The model shows that the bond parameter doesn't affect the slope of
the curve but only the crack spacing, just as in the most common
formulations. If E_b is halved or doubled the following values are
obtained:

E_b=20 MN/mm^3 case A a= 380 mm E_b=80 MN/mm^3 case A a= 190 mm
                 B a= 332 mm   B a= 166 mm
                 C a= 272 mm   C a= 136 mm
while the moment-curvature diagrams remain the same, whithin the
numerical approximations.

5. CONCLUDING REMARKS

The proposed analytical model, which is based only on the classical
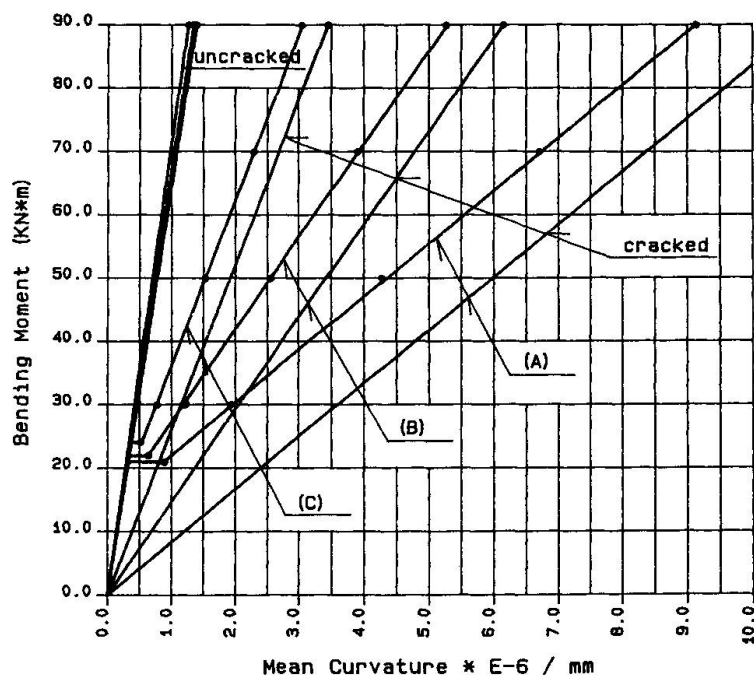hypothesis of cross section remaining plane, seems to be a good

starting-point for a rational approach to tension stiffening pheno-
mena. Owing to its quite general formulation the model may be im-
proved by adopting more realistic non linear constitutive laws for
materials and bond behaviour. From this point of wiew it's well
suited for studying the tension stiffening effects even after steel
yields.

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