

**Zeitschrift:** IABSE reports = Rapports AIPC = IVBH Berichte  
**Band:** 46 (1983)  
  
**Artikel:** Reinforced concrete members with subsequently bonded steel sheets  
**Autor:** Ladner, Marc  
**DOI:** <https://doi.org/10.5169/seals-35851>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 05.02.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

## Reinforced Concrete Members with Subsequently Bonded Steel Sheets

Eléments en béton armé renforcés ultérieurement par collage d'armature

Stahlbetonbauteile mit nachträglich aufgeklebter Bewehrung

### Marc LADNER

Dr. sc.techn.

EMPA

Dübendorf, Switzerland



Born in 1939, Marc Ladner received his diploma as a civil engineer in 1962 and his Dr. sc.techn. degree in 1968 from the Swiss Federal Institute of Technology (ETH) in Zürich. Since 1970, he is working at the Swiss Federal Laboratories for Materials Testing and Research where he is the head of the reinforced concrete structure department.

### SUMMARY

The paper deals with the design of reinforced concrete members, strengthened by subsequently bonded steel sheets. The amount of additional reinforcement is limited for several reasons which are discussed. Moreover, two methods for determining the stress distribution between concrete and external reinforcing steel are summarized, from which practical design rules are drawn.

### RESUME

L'exposé traite du dimensionnement d'éléments en béton armé dont le renforcement ultérieur fait appel à la méthode d'armature collée. Les considérants portent tout d'abord sur la détermination du taux d'armature extérieure pouvant être appliqué à titre de renforcement sur une section existante de béton armé. Il est ensuite fait état de deux procédés pour le calcul des contraintes entre l'acier et le béton. Des conclusions sont tirées pour la pratique.

### ZUSAMMENFASSUNG

Der vorliegende Artikel befasst sich mit der Bemessung von Stahlbetonbauteilen mit nachträglich aufgeklebter Bewehrung. Dabei wird zunächst auf die Ermittlung der Grösse des Bewehrungsanteiles eingegangen, der als äussere Bewehrung auf einen bestehenden Stahlbetonquerschnitt aufgebracht werden kann. Sodann werden kurz zwei Verfahren zur Berechnung der Spannungen zwischen Stahl und Beton gestreift und daraus Folgerungen für die Bemessungspraxis gezogen.



## 1. ASSESSMENT OF THE QUANTITY OF EXTERNALLY BONDED STEEL

### 1.1 Brittleness of Cross Section

In order to avoid a brittle failure of a strengthened reinforced concrete member with externally bonded steel sheets, the total area of reinforcement (Fig. 1) should not exceed an upper limit of

$$\rho_{\text{tot,max}} = \frac{A_{s,\text{tot}}}{A_c} \quad (1)$$

where  $A_{s,\text{tot}} = A_{si} + A_{sa}$  = total area of reinforcement  
 $A_c = b d_i$  = area of concrete.

This limiting value has to be chosen in such a way, that both the inside reinforcement and the externally bonded steel come to yield before the concrete fails by crushing in the compression zone.

On the other hand, neither of the two reinforcements should reach its rupture strain, before the concrete fails in compression. Therefore, the total ratio of the reinforcing steel should be higher than

$$\rho_{\text{tot,min}} = \frac{A_{s,\text{tot}}}{A_c} \quad (2)$$

as a lower limit.

In the following, the expressions for  $\rho_{\text{tot,max}}$  and  $\rho_{\text{tot,min}}$  will be derived under the assumption, that the cross section of a rectangular beam remains plane during bending.

With the notations of Fig. 1, the depth  $x$  of the compression zone can be expressed by

$$x = \frac{f_{si,y} A_{si} + f_{sa,y} A_{sa}}{f_c b k_1} \quad (3)$$

The factor  $k_1$  describes the stress distribution in the concrete compression zone.

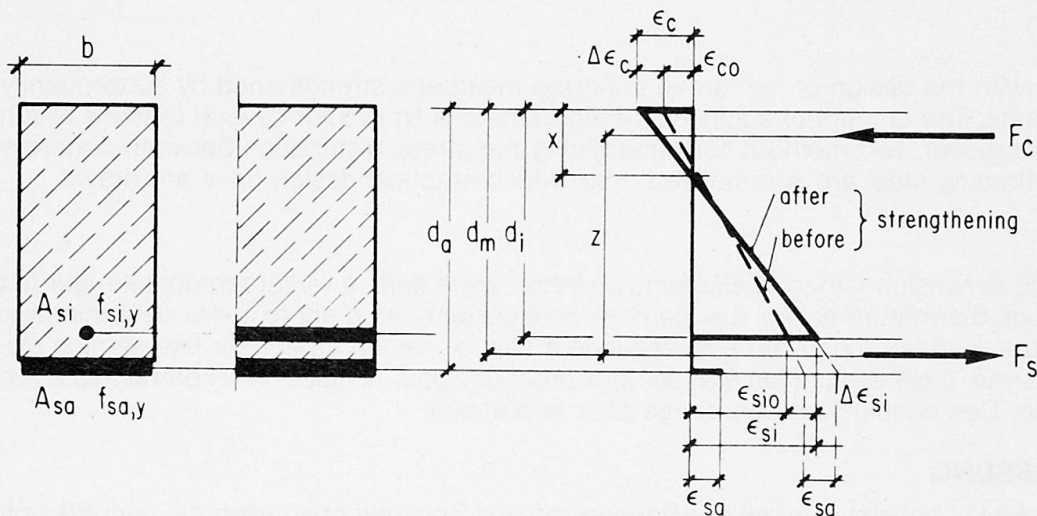


Fig. 1: Notations and dimensions

Using the mean value of the static depth of the cross section

$$d_m = \frac{f_{si,y} A_{si} d_i + f_{sa,y} A_{sa} d_a}{f_{si,y} A_{si} + f_{sa,y} A_{sa}} \quad (4)$$

the ultimate bending moment of the cross section can then be written as

$$M_u = z F_s = (d_m - k_2 x) F_s \quad (5)$$

where again  $k_2$  depends on the stress distribution in the concrete compression zone.

With  $x = \xi d_m$  equation (5) can be rewritten as

$$M_u = d_m (1 - k_2 \xi) F_s \quad (6)$$

or, after some calculations,

$$M_u \cong d_i \left\{ f_{si,y} A_{si} + \frac{d_a}{d_i} f_{sa,y} A_{sa} \right\} \left\{ 1 - \frac{k_2}{k_1} (\bar{\rho}_i + \bar{\rho}_a) \right\} \quad (7)$$

with the notations

$$\bar{\rho}_i = \frac{f_{si,y} A_{si}}{f_c b d_i} \quad (8)$$

and

$$\bar{\rho}_a = \frac{f_{sa,y} A_{sa}}{f_c b d_i} \quad (9)$$

When a linear strain distribution is assumed over the cross section, an additional equation for  $x$  can be found

$$x = \frac{d_i (\Delta \epsilon_c + \epsilon_{co})}{\Delta \epsilon_c + \epsilon_{co} + \epsilon_{sio} + \Delta \epsilon_{si}} = \frac{d_i}{1 + \epsilon_{si}/\epsilon_c} \quad (10)$$

The combination of the equations (3) and (10) leads to an expression for the sum of  $\bar{\rho}_i$  and  $\bar{\rho}_a$  as a function of  $\epsilon_{si}/\epsilon_c$

$$\bar{\rho}_i + \bar{\rho}_a = \frac{k_1}{1 + \epsilon_{si}/\epsilon_c} \quad (11)$$

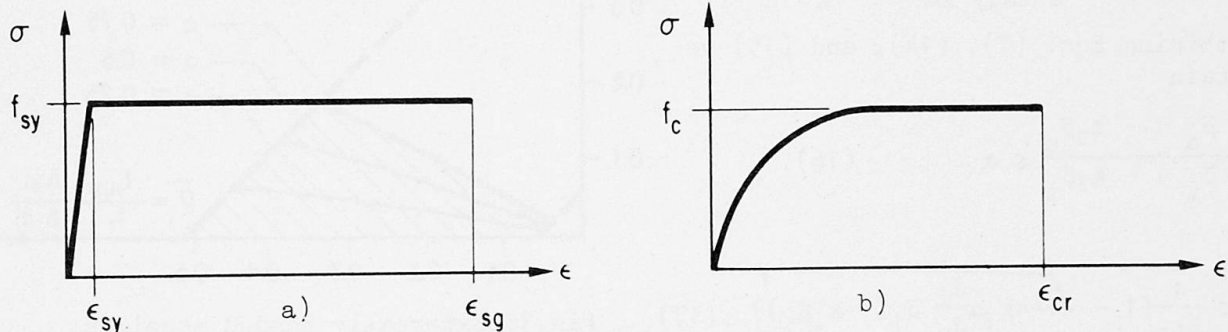


Fig. 2: Idealized  $\sigma$ - $\epsilon$ -diagram for steel (a) and concrete (b)





The  $\sigma$ - $\epsilon$ -diagram for steel can be simplified in a bi-linear form (Fig. 2a). Here,  $\epsilon_{sy}$  denotes the strain, when the yield stress is reached for the first time, and  $\epsilon_{sg}$  is the strain at rupture. On the other hand, the stress-strain curve for concrete is given in Fig. 2b with  $\epsilon_{cr}$  being the strain at rupture.

From the condition

$$\epsilon_{si,y} \leq \epsilon_{si} \leq \epsilon_{si,g}$$

a first set of lower and upper limits for the reinforcement ratio can be obtained

$$\frac{k_1}{1 + \epsilon_{si,g}/\epsilon_{cr}} \leq \bar{\rho}_i + \bar{\rho}_a \leq \frac{k_1}{1 + \epsilon_{si,y}/\epsilon_{cr}} \quad (12)$$

In a similar way, the deformation of the externally bonded steel can also be checked. When  $\epsilon_{sio}$  and  $\epsilon_{co}$  denote the strains of the inner reinforcing steel and of the concrete before strengthening, an additional set of lower and upper limits for the total reinforcing steel area can be established (Eq. 13):

$$\frac{\frac{d_a}{d_i} k_1}{1 + \frac{\epsilon_{sio}}{\epsilon_{cr}} \{1 + (1 + \frac{\epsilon_{co}}{\epsilon_{sio}})(\frac{d_a}{d_i} - 1)\} + \frac{\epsilon_{sa,g}}{\epsilon_{cr}}} \leq \bar{\rho}_i + \bar{\rho}_a \leq \frac{\frac{d_a}{d_i} k_1}{1 + \frac{\epsilon_{sio}}{\epsilon_{cr}} \{1 + (1 + \frac{\epsilon_{co}}{\epsilon_{sio}})(\frac{d_a}{d_i} - 1)\} + \frac{\epsilon_{sa,y}}{\epsilon_{cr}}}$$

Equations (12) and (13) are graphically represented in Fig. 3. For any combination of  $\bar{\rho}_i$  and  $\bar{\rho}_a$  which lays between the limiting straight lines, it is assured that both inner and outer steel will reach the yield stress before the concrete will fail in the compression zone.

### 1.2 Limitation by $\Delta M_u$

Some codes or regulations suggest that the increase of the ultimate bending moment  $\Delta M_u$ , due to strengthening, should be limited

$$\frac{\Delta M_u}{M_{uo}} \leq \alpha \quad (14)$$

where  $M_{uo}$  = ultimate bending moment before strengthening (Eq.7,  $A_{sa}=0$ )

$M_u$  = ultimate bending moment after strengthening (Eq.7)

$$\Delta M_u = d_a f_{sa,y} A_{sa} (1 - k_3 \bar{\rho}_a) \quad (15)$$

Combining Eqs. (7), (14), and (15) we obtain

$$\frac{d_a}{d_i} \frac{\bar{\rho}_a}{\bar{\rho}_i} \frac{1 - k_3 \bar{\rho}_a}{1 - k_3 \bar{\rho}_i} \leq \alpha \quad (16)$$

or

$$\bar{\rho}_a \leq \frac{1}{2k_3} \{1 - \sqrt{1 - 4k_3 \alpha \frac{d_i}{d_a} \bar{\rho}_i (1 - k_3 \bar{\rho}_i)}\} \quad (17)$$

$$\text{with } k_3 = k_2 / k_1 \quad (18)$$

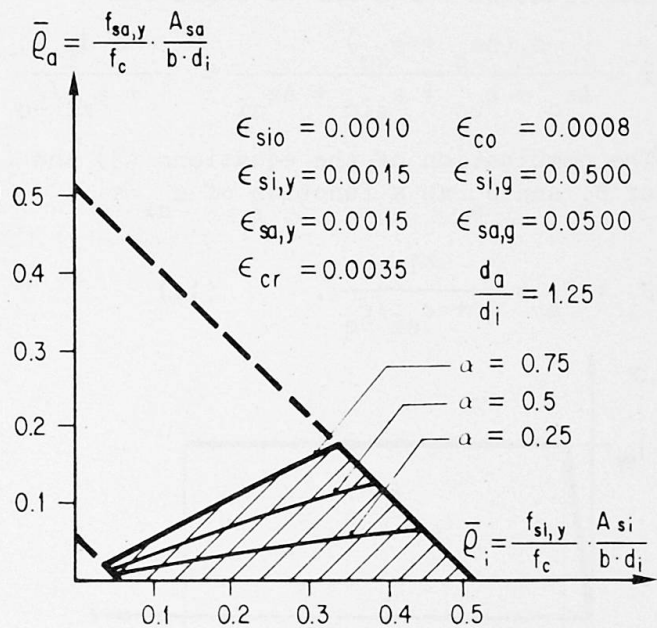


Fig.3: Externally bonded steel vs. internal reinforcement

## 2. LOAD TRANSFER BETWEEN EXTERNALLY BONDED STEEL AND CONCRETE

### 2.1 General Remarks

Although the mathematical models used for describing the load transfer from one material to another throughout an adhesive are of simple nature, the formulas given in the next paragraphs do not show a simple form and may therefore not be applicable in practice. In spite of this fact, the results obtained this way may help to understand some of this process and, together with test results, may lead to some advices for dimensioning and design of the anchorage zone of an additional reinforcement.

### 2.2 Elastic Solution for the Bond Stress Distribution

Let us consider first a concrete prism with bonded steel sheets on two opposite sides (Fig. 4). Assuming that

- the materials concrete, steel, and resin follow Hook's law;
- the resin takes shear stresses only;
- the thicknesses  $t$  and  $d$  as well as the width  $b$  are constant over the total bond length  $\ell_a$ ;
- the stresses and strains are distributed uniformly over the total cross section (bending effects are neglected)

then, according to [1], the bond stress distribution is given by

$$\tau(x) = \sigma_{so} t \omega \frac{\text{Ch}(\omega x)}{\text{Sh}(\omega \ell_a)} \quad (19)$$

with

$$\omega^2 = \frac{G}{s} \left( \frac{1}{E_s t} + \frac{2}{E_c d} \right) \quad (20)$$

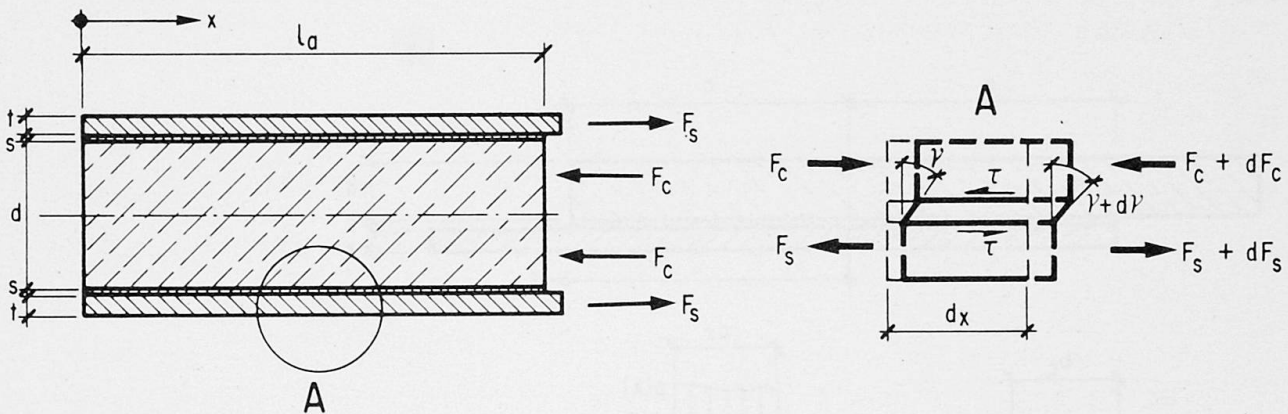


Fig. 4: Concrete prism with bonded steel sheets

The general shape of the bond stress distribution  $\tau(x)$  is shown in Fig. 5. From this, it can be seen clearly, that only a very small part of the total anchorage length  $\ell_a$  is needed for load transfer. To find the length  $\ell'$  which is needed for transferring  $k$  percent of the total tensile force  $F_s$  from the steel to the concrete ( $0 \leq k \leq 1$ ) the bond stress  $\tau(x)$  has to be integrated from  $x = x_0$  to  $x = \ell_a$ :

$$\int_{x_0}^{\ell_a} \tau b dx = k F_s = k \sigma_{so} b t \quad (21)$$

$$\sigma_{so} tb \left(1 - \frac{Sh(\omega x_o)}{Sh(\omega l_a)}\right) = k \sigma_{so} bt \quad (22)$$

For  $\omega x_o > 1$  this expression can be simplified to

$$\frac{e^{\omega x_o}}{e^{\omega l_a}} = 1 - k \quad (23)$$

or

$$x_o = l_a + \frac{1}{\omega} \ln(1-k). \quad (24)$$

Finally, the effective anchorage length  $l'$  can be written as

$$l' = l_a - x_o = \frac{-1}{\omega} \ln(1-k). \quad (25)$$

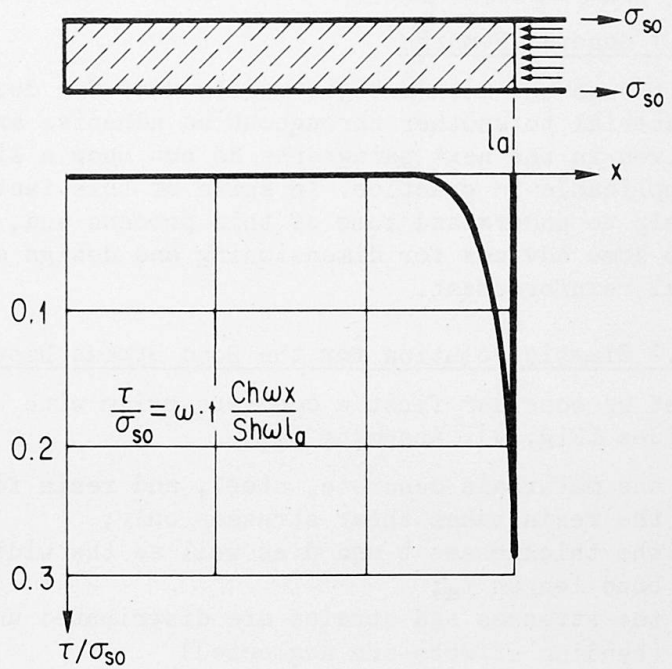


Fig. 5: Bond stress distribution between steel and concrete

### 2.3 Bond and Normal Stress Distribution in a Glued Joint

A very early study about the normal and bond stress distribution  $\sigma$  and  $\tau$  in a glued joint was performed by Goland and Reissner in 1944 [2]. Based on their investigation a more general form of the corresponding differential equations for  $\sigma$  and  $\tau$  will be given in the following (Fig. 6).

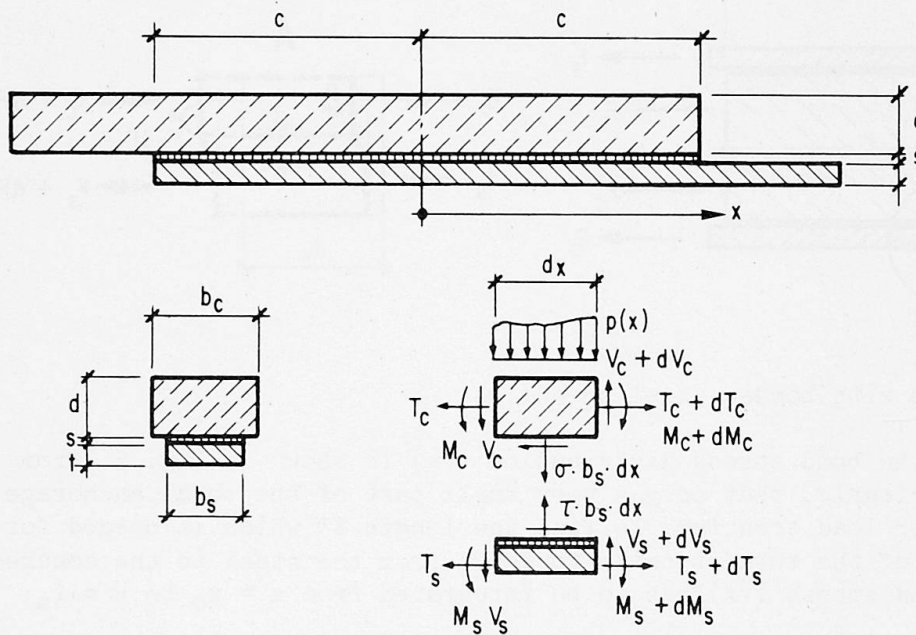


Fig. 6: Notations and dimensions





A first differential equation for  $\tau$  can be established from the conditions of equilibrium and compatibility

$$\frac{d^3\tau}{dx^3} - \frac{4G_a}{E_s} \frac{nb_s t + b_c d}{b_c dt} \frac{d\tau}{dx} + \frac{6G_a}{E_s} \left\{ \frac{nb_s t^2 - b_c d^2}{b_c d^2 t^2} \sigma + \frac{np(x)}{b_c d^2} \right\} = 0 \quad (26)$$

with  $E_s$  = Modulus of elasticity, steel  
 $E_c$  = Modulus of elasticity, concrete  
 $G_a$  = Shear modulus, adhesive  
 $n = E_s/E_c$

In a similar way, a differential equation for  $\sigma$  can be found

$$\frac{d^4\sigma}{dx^4} + \frac{12E_a}{E_s} \frac{b_c d^3 + nb_s t^3}{b_c d^3 t^3} \sigma + \frac{6E_a}{E_s} \frac{b_c d^2 - nb_s t^2}{b_c d^2 t^2} \frac{d\tau}{dx} + \frac{12E_a}{E_s b_c d^3} p(x) = 0 \quad (27)$$

with  $E_a$  = Modulus of elasticity, adhesive.

Combining the two differential equations (26) and (27) an equation for  $\sigma$  is obtained

$$\frac{d^6\sigma}{dx^6} - B' \frac{d^4\sigma}{dx^4} + C' \frac{d^2\sigma}{dx^2} - D'\sigma + F' = 0 \quad (28)$$

The constants  $B'$ ,  $C'$ ,  $D'$ , and  $F'$  contain all geometric and mechanical properties of the specific problem.

At the beginning of a joint, a compression stress  $\sigma$  is present which should be superimposed to the bond stress  $\tau$ .

For the case, where  $E_s = E_c = E$  ( $n = 1$ ),  $b_s = b_c = b$ ,  $d = t$ , and  $p(x) = 0$ , equation (26) becomes independent of  $\sigma$

$$\frac{d^3\tau}{dx^3} - \frac{8G_a}{E_s t} \frac{d\tau}{dx} = 0 \quad (29)$$

and equation (27) is then also independent of  $\tau$

$$\frac{d^4\sigma}{dx^4} + \frac{24E_a}{E_s t^3} \sigma = 0 \quad (30)$$

The differential equations (29) and (30) were found and solved by Goland and Reissner in [2].

### 3. CONCLUDING REMARKS

As it can be seen from equation (25), a relatively small amount of the total anchorage length  $\ell_a$  is effective for load transfer when all materials involved behave elastically. The total anchorage length  $\ell_a$  will only be needed when, due to local failure of the bond, the peak value of the bond stress moves towards the unloaded end of the steel sheet. This phenomenon could be seen clearly in test results as reported in [3]. Thus, a redistribution of the bond stresses takes place.



