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Some Examples of Lower-Bound Design of Reinforcement in Plane Stress Problems

Dimensionnement de l'armature et problèmes d'états plans de contrainte, en appliquant la méthode statique

Beispiele zur Bemessung der Bewehrung nach der statischen Methode bei Problemen mit ebenen Spannungszuständen

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SUMMARY
The paper demonstrates how simple statically admissible stress fields can be used for the determination of reinforcement in plane stress problems, and presents some standard formulae for the necessary amount of reinforcement as a function of stresses. The problems treated are shear, torsion and combined bending, torsion and shear in beams.

RESUME
On montre comment on peut utiliser des champs de contraintes statiquement admissibles et simples pour l'analyse des problèmes avec états plans de contrainte. Des formules sont données pour dimensionner l'armature en fonction des contraintes. On traite les problèmes du cisaillement et de la torsion dans une poutre ainsi que l'action combinée de la flexion, de la torsion et du cisaillement.

ZUSAMMENFASSUNG
Es wird gezeigt, wie für Probleme mit ebenen Spannungszuständen einfache statisch zulässige Spannungsfelder angewendet werden können. Zur Ermittlung der erforderlichen Bewehrung in Abhängigkeit der Spannungen werden Formeln angegeben. Behandelt werden die Beanspruchung von Balken durch Querkraft und Torsion sowie die kombinierte Beanspruchung durch Biegung, Torsion und Querkraft.

1. INTRODUCTION

It is the purpose of this paper to demonstrate that reinforcement design according to the theory of plasticity in many plane stress problems can be done with advantage by using simple lower bound solutions in connection with the author's reinforcement formulas, [1],[4],[5] and [6]. The methods were originally proposed in [6], in an internal report. Alternative methods dealing with the same problems have been developed by Thürlimann and his associates,[8] and [9].

We shall begin by giving a short summary of the reinforcement formulas and the theory on which they are based. The basic assumption are that both concrete and steel can be treated as rigid, perfectly plastic materials. The yield condition for the concrete is assumed to be the usual square yield locus $-\sigma_c \leqq \sigma_1 \leqq 0$, $-\sigma_c \leqq \sigma_2 \leqq 0$, where σ_1 and σ_2 are principal stresses (tension positive) and σ_c the compressive strength. The tensile strength is thus assumed to be zero. The reinforcement bars are assumed to be able to carry only forces in their longitudinal direction. While the assumption for the reinforcement can be easily justified, the assumption concerning the concrete has a more doubtful connection to reality. The basic reason for this is that although the ductility of the concrete is rather high in compression, the stress falls drastically, when the peak of the stress-strain curve has been reached. Therefore, the theoretical results often have to be modified. One way of doing this is to introduce an effective compressive strength $\nu\sigma_c$, where ν is a so-called effectivity factor, a lower bound of which is representing some kind of average stress in the actual strain region. The effectivity factor, however, also has to take care of other defects of the theory, see [10].

Considering for simplicity only orthogonal reinforcement in the directions x and y , see fig.1.1, and letting A_{ax} represent the reinforcement area in the x-direction per unit length measured in the y-direction, σ_{fx} the yield stress of the reinforcement in the x-direction, σ_x, σ_y and τ_{xy} , the stresses which have to be carried, and finally letting t being the thickness, the reinforcement formulas can, using similar notation for the y-direction, be written, [6]:

<u>Case 1</u>: $\sigma_x \geqq -|\tau_{xy}|\sqrt{\alpha}$ $\qquad \sigma_y \geqq -|\tau_{xy}|/\sqrt{\alpha}$

$$\sigma_{tx} = \frac{A_{ax}\sigma_{fx}}{t} = \sigma_x + |\tau_{xy}|\sqrt{\alpha} \qquad \sigma_{ty} = \frac{A_{ay}\sigma_{fy}}{t} = \sigma_y + |\tau_{xy}|/\sqrt{\alpha} \qquad (1.1),\ (1.2)$$

$$\sigma_b = |\tau_{xy}|(\sqrt{\alpha} + 1/\sqrt{\alpha}) \qquad (1.3)$$

<u>Case 2</u>: $\sigma_x \leqq \sigma_y\alpha$ $\qquad \sigma_x < -|\tau_{xy}|\sqrt{\alpha}$

Reinforcement is only necessary if $\sigma_x\sigma_y \leqq \tau_{xy}^2$

$$\sigma_{tx} = 0 \qquad \sigma_{ty} = \sigma_y + \frac{\tau_{xy}^2}{|\sigma_x|} \qquad (1.4),\ (1.5)$$

$$\sigma_b = |\sigma_x|\left(1 + \left(\frac{\tau_{xy}}{\sigma_x}\right)^2\right) \qquad (1.6)$$

<u>Case 3</u>: $\sigma_x \geqq \sigma_y\alpha$ $\qquad \sigma_y < -|\tau_{xy}|/\sqrt{\alpha}$

Reinforcement is only necessary if $\sigma_x\sigma_y \leqq \tau_{xy}^2$

$$\sigma_{tx} = \sigma_x + \frac{\tau_{xy}^2}{|\sigma_y|} \qquad \sigma_{ty} = 0 \qquad (1.7), (1.8)$$

$$\sigma_b = |\sigma_y|(1 + (\frac{\tau_{xy}}{\sigma_y})^2) \qquad (1.9)$$

The quantities σ_{tx} and σ_{ty} defined by (1.1) and (1.2) are the yield forces of the reinforcement bars per unit area of the concrete (equivalent reinforcement stresses). They represent the tensile strength of the reinforced material in the x- and y-directions, respectively. The quantity α is given by

$$\alpha = \frac{\sigma_{fx}}{\sigma_{fy}} \qquad (1.10)$$

and σ_b is the compression stress in the concrete. The above formulas determine the optimum value of $A_{ax} + A_{ay}$.

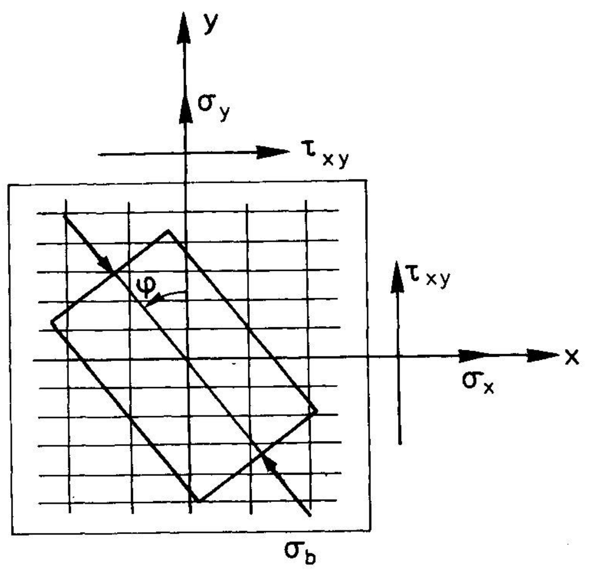

Figure 1.1

If for some reason the reinforcement in one direction is known, or if the optimum value of the total reinforcement of the structure is found for other inclinations of the compressive stresses than found above, see section 2, the following formulas determine the necessary amount of reinforcement:

$$\sigma_{tx} = \sigma_x + \gamma|\tau_{xy}| \qquad (1.11)$$

$$\sigma_{ty} = \sigma_y + \frac{1}{\gamma}|\tau_{xy}| \qquad (1.12)$$

$$\sigma_b = |\tau_{xy}|(\gamma + \frac{1}{\gamma}) \qquad (1.13)$$

The quantity $\gamma > 0$ can be determined if σ_{tx} or σ_{ty} is known. The relation between γ and the angle φ determining the direction of the concrete stress is $\gamma = \mathrm{tg}\,\varphi$, see fig.1.1. The formulas (1.11) - (1.13) can also be used instead of (1.1) - (1.9) if minimum reinforcement is not looked for. In this case, γ can theoretically be arbitrarily chosen. However, in order to avoid yielding of the reinforcement for service loads, limitations should be put on the choice of γ, see the following.

Having determined the necessary reinforcement by means of the above formulas, it is often found advantageous to use another, perhaps more economical or practical distribution of the reinforcement. In the case of homogeneous stress fields, it should be noted that the reinforcement theoretically might be distributed in any other way resulting in the same statical equivalence of the reinforcement forces. Sometimes, for instance in slabs and shells, such a transformation changes the compression forces in the concrete and, if so, the concrete stresses of course have to be calculated taking account of these extra forces. Also the complete equilibrium of the whole transformed stress field at the boundaries should be considered. Examples of reinforcement transformations are given in the following.

2. SHEAR IN BEAMS

Consider a stringer beam, i.e. a beam with the tensile and the compression zone concentrated in stringers, see fig.2.1. The distance between the stringers is h. Let us determine the shear reinforcement in a zone with constant shear force Q. If the shear zone has the thickness b, the reinforcement may be determined using the homogeneous statically admissible stress field

$$\sigma_x = \sigma_y = 0 \qquad \tau_{xy} = \tau = \frac{Q}{bh} \tag{2.1}$$

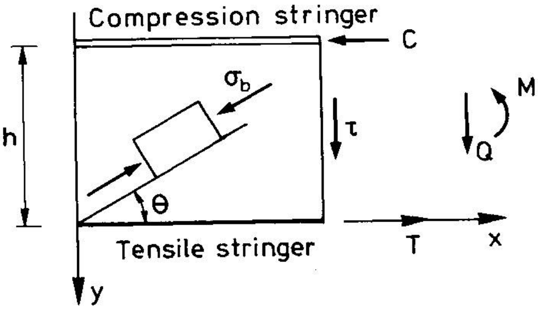

Figure 2.1

The necessary reinforcement in the x- and the y-direction and the concrete stress σ_b is determined by (1.11)-(1.13). We find

$$\sigma_{tx} = \tau \cot\theta \tag{2.2}$$

$$\sigma_{ty} = \tau \operatorname{tg}\theta \tag{2.3}$$

$$\sigma_b = \tau(\cot\theta + \operatorname{tg}\theta) \tag{2.4}$$

where the meaning of θ is shown in fig.2.1. If the shear zone is reinforced accordingly, i.e. reinforced in both directions x and y, the stringers have to carry the forces

$$T = C = \frac{M}{h} \tag{2.5}$$

However, such a shear reinforcement is more expensive than necessary, although it has the advantage of giving zero stringer force at a simple support, a fact which may facilitate the design of this part of the beam because of the small anchorage length of the tensile reinforcement required. The shear reinforcement in the x-direction can be avoided since the total force in this direction $Q\cot\theta$ can be carried by the stringers. This means that the stringers have to carry the forces

$$T = \frac{M}{h} + \frac{1}{2}Q\cot\theta \qquad C = \frac{M}{h} - \frac{1}{2}Q\cot\theta \tag{2.6), (2.7}$$

The result implies that we have to reinforce for a tensile stress in the y-direction determined by (2.3), to secure that the stringers can carry the forces (2.6) and (2.7) and that the concrete stress σ_b determined by (2.4) can be carried by the concrete. Formula (2.6) shows that at a simple support, the tensile stringer must be able to carry half the reaction times $\cot\theta$, i.e. proper care must be taken to anchorage the reinforcement at an end section.

The same result would, of course, be found if the bending moment was assumed to be carried by the stringers and a uniformly distributed compressive stress $\tau\cot\theta$ in the shear zone. The above results can be obtained even in other ways, see for instance [10].

The total amount of reinforcement can, of course, be minimized with respect to θ. The optimum value θ is different from that corresponding to the formulas in section 1 because of the special way, in which the x-reinforcement is arranged. The result of the optimization shall not be given here since other requirements often determine the most practical θ-value. The reader is referred to [3].

If reinforcement in other orthogonal directions is preferred for som reason, the stress field (2.1) in the shear zone just have to be transformed to these directions. An extremely simple result is found when the shear zone is reinforced in the principal directions where only reinforcement in one direction is required.

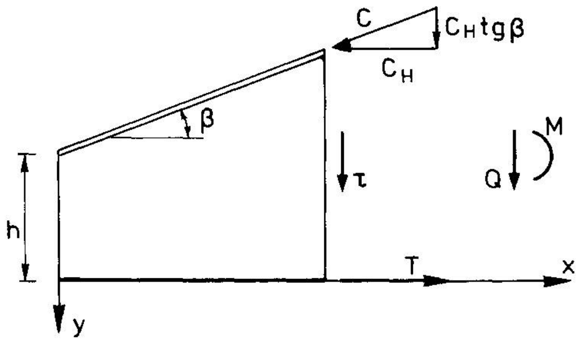

Figure 2.2

If the beam has variable depth, see fig.2.2, we may reinforce for the stress field:

$$\sigma_x = 0 \; ; \quad \sigma_y = 2\,\tau \frac{y}{h} tg\,\beta \; ;$$

$$\tau_{xy} = \tau = \frac{1}{bh}(Q - T\ tg\,\beta) \qquad (2.8)$$

which is easily seen to satisfy the equilibrium equations and the boundary conditions. The σ_y-stress generally is small and can be neglected. Doing so, the formulas (2.2)-(2.4) are still valid. If in (2.7) C is replaced by the horizontal component C_H, see fig.2.2, the formulas (2.6) and (2.7) are valid too.

If a beam has stronger flanges than necessary to carry the stringer forces, it would be natural to utilize the bending and the shearing strength of the flanges too. The most simple way of doing this, when applying a lower bound method, is to superimpose the above stress fields on a stress field, corresponding to ordinary beam action in a frame system composed by the flanges, the end sections and, if necessary, some compressive struts in the shear zone. Fig.2.3 illustrates how such a stress field can be determined in a simple example. The bending moments are here chosen in accordance with the yield mechanism shown in the figure. When the ratio between the bending moments in the hinges have been selected, the values of the bending moments can be determined by the work equation. Having done this, the normal forces and the shear forces can be determined by equilibrium equations. Superimposing the stress fields mentioned, the reinforcement in the flanges and the shear zone can be calculated.

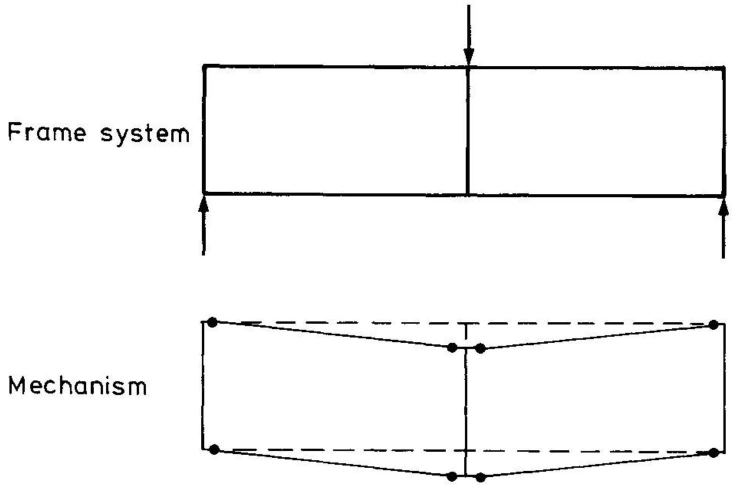

Figure 2.3

The effectivity factor ν of the concrete is rather well known in the case of beam shear. Also limitations to be put on cotθ in order to secure satisfactory behaviour for the working load have been proposed. The reader is referred to [10],[12], [8] and [14].

The stress fields treated in this section can as an approximation be used for other loading systems than those giving constant shear force, see [10]. The stress fields are not applicable for deep beams or beams with large concentrated loads near the supports, where arching action is capable of carrying all, or a significant part, of the load. A number of simple statically admissible stress fields for this case have been developed in [4] and [5]. An upper bound method is described in [10].

3. TORSION

An immediate application of the reinforcement formulas of section 1 to torsion problems is possible for a thin walled, closed section. For such a section a pure shear field

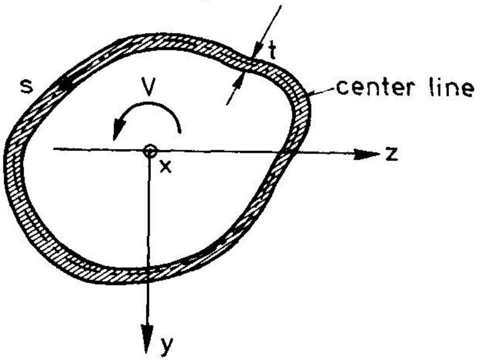

Figure 3.1

$$\tau_{xs} = \frac{V}{2\,A_0\,t} \qquad (3.1)$$

V being the torsional moment, A_0 the area within the center line of the section, s the arc length along the center line, and t the thickness, is statically admissible, see fig.3.1. The formula (3.1) is Bredt's formula. The area of the longitudinal bars and the area of the bars along the center line is determined by (1.1) and (1.2) or (1.11) and (1.12), the first mentioned formulas giving minimum amount of reinforcement. The concrete stress is determined by (1.3) or (1.13). The concrete stress of course has to satisfy the condition $\sigma_b \leqq \nu\,\sigma_c$.

The same simple stress field is statically admissible in any solid section if (3.1) is applied to a thin walled closed section lying within the concrete area of the section. The thickness t of the thin walled section of course has to be so large as to render it possible to satisfy the condition $\sigma_b \leqq \nu\,\sigma_c$.

In many cases, the reinforcement does not have to be placed in the center line of the closed section. A statically equivalent reinforcement lay-out can be used if proper care is taken to design the end sections as in the case of shear in beams.

Consider, as an example, a rectangular section, see fig.3.2.

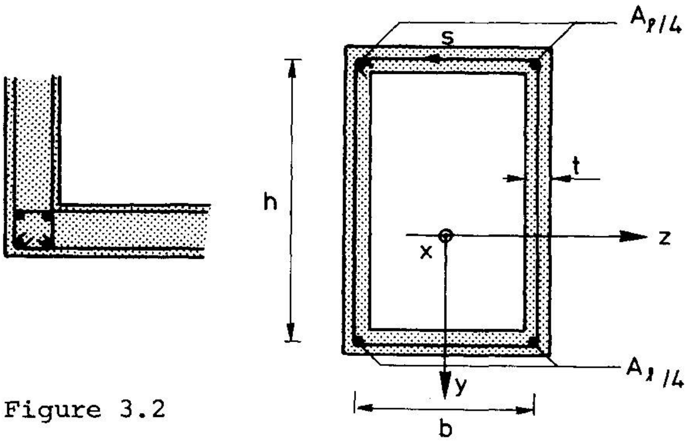

Figure 3.2

If the yield stresses in both reinforcement directions are equal to σ_f, we get by means of (1.1) and (1.2) the following reinforcement areas per unit length:

$$A_{ax} = A_{as} = \frac{V}{2h\,b\,\sigma_f} \qquad (3.2)$$

The total amount of longitudinal reinforcement is thus

$$A_\ell = \frac{V(h+b)}{hb\,\sigma_f} \qquad (3.3)$$

which can be concentrated in the corners, each of the corner bars having an area of $A_\ell/4$. The reinforcement along the center line can for small sections be chosen as closed stirrups as shown in fig.3.2 to the right. For large sections, closed stirrups in the individual wall sections can be used as illustrated in fig. 3.2 to the left.

For a long rectangular section, the bars in both reinforcement directions can be placed outside the thin walled section as long as both reinforcement layers are placed symmetrically with respect to the middle plane, see fig.3.3. In this case, we are in fact concerned with pure torsion in a slab, the action of which has been

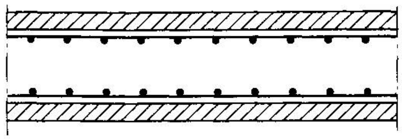

Figure 3.3

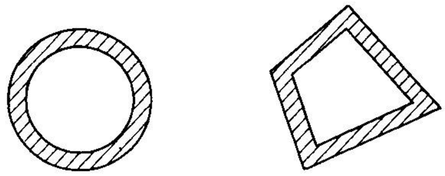

Figure 3.4

more deeply studied in [2], to which the reader is referred.

In fig.3.4, thin walled sections which can be used for the reinforcement calculation in some other cases of solid sections have been illustrated.

For circular sections, the concrete cover is in danger of spalling off, which has actually been observed in tests, see [7].

Only very little is known about the effectivity factor ν of the concrete in the case of torsion, but results are under way * . Until more refined results are available one has to rely upon crude but generally conservative code rules, see for instance [14]. Limits on γ have been studied in [11], see also [9] and [14].

4. COMBINED BENDING, SHEAR AND TORSION

In the case of combined bending, shear and torsion in a thin walled closed section, the necessary amount of reinforcement can be determined by means of the reinforcement formulas of section 1 too. Considering as a simple example a box section acted upon by a bending moment M_z , a shear force Q_y and a torsional moment V, see fig.4.1, one statically admissible stress field can be found using the Navier distribution of the normal stresses σ_x from the bending moment, the corresponding Grashof distribution of the shear stresses τ_{xs} from the shear force Q_y and the Bredt distribution (3.1) of the shear stresses τ_{xs} from the torsional moment V . However, a more suitable statically admissible stress distribution is found by distributing the normal stresses from M_z uniformly, for instance, along the top and bottom flanges. The corresponding shear stress diagram then is linear in the individual walls. Having determined the stress distribution, the reinforcement formulas immediately give the necessary amount of reinforcement. Other thin walled closed sections can be treated in a similar way. For a solid section, the same method as described for pure torsion can be used, i.e. a thin walled section, lying within the concrete area, is selected for carrying the stresses. Consider as an example a solid rectangular section. If reinforcement is supplied in the longitudinal direction and a circumferential direction, perpendicular to this, and if the yield stress of the steel is the same in both directions, then for a section acted upon by a torsional moment V the reinforcement formulas (1.1) and (1.2) require the total longitudinal reinforcement to carry a force $P_\ell = A_\ell \sigma_f$, which can be calculated by means of (3.3). The corresponding reinforcement area can be placed as one fourth of the total area in each corner.

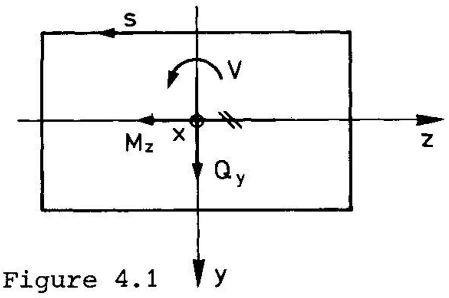

Figure 4.1

Small bending moments M_z , i.e. $M_z \leq \frac{1}{2} V (h+b)/b$ can be carried by moving a part of the reinforcement in the compression zone to the tensile zone, i.e. the force in the longitudinal reinforcement in the top flange can be reduced by M_z/h and the force in the longitudinal reinforcement in the bottom flange has to be increased by M_z/h.

* Current research seems to show that empirical formulas for ν can be given very similar forms for bending and torsion problems, see [13].

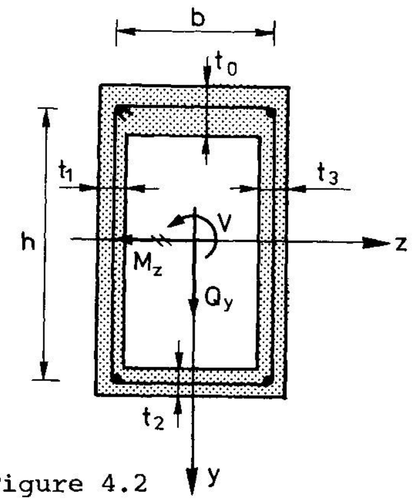

Figure 4.2

If $M_z > \frac{1}{2} V(h+b)/b$ the longitudinal reinforcement in the top flange can be chosen to be zero, while the force in the longitudinal reinforcement in the bottom flange still has to be increased by M_z/h .

If the section is acted upon by a shear force Q_y too, the reinforcement can be determined by adding the shear stress in the thin walled section from Q_y to the stresses from V and M_z . In one of the vertical walls, the shear stresses from Q_y add to the shear stresses from V and in the other one, they subtract to the shear stresses from V . The concrete stress in the individual walls can be determined by means of the formulas of section 1. The quantities t_o, t_1, t_2 and t_3, the meaning of which is shown in fig.4.2, of course have to be fixed at values making it possible to satisfy the condition $\sigma_b \leq \nu \sigma_c$ in each wall. Other solid sections may be treated in a similar way.

As in the case of pure torsion, reliable information about the effectivity factor ν of the concrete is still missing in the case of combined bending, shear and torsion.

5. OTHER PLANE STRESS PROBLEMS

The reinforcement formulas can be applied to several other plane stress problems. We shall, however, not be able to treat other cases in more detail here. A number of statically admissible stress fields for deep beams, to which the reinforcement formulas immediately apply, have been developed by the author, see [4] and [5]. The formulas also apply to the determination of reinforcement in slabs and shells, see [2],[4] and [5].

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